MAT5171
Assignment 2

Assignment 2 is based on material from Sections 22 and 25. Do Q1, and choose two from Q2-Q5.

Due date: February 10.

Exercise 1 Assume that $\mathrm{E}[|X|] < \infty$. Show that $\lim_{x\to\infty} xP(|X| > x) = 0$.

Solution for Exercise 1 Note that $n1\{|X| > n\} \leq |X|1\{|X| > n\}$. Define $X_n = |X|1\{|X| > n\}$. Since $X_n \leq |X|$, then by the dominated convergence theorem we have

$$nP(|X| > n) = \mathrm{E}[n1\{|X| > n\}] \leq \mathrm{E}[|X|1\{|X| > n\}] \to 0 \ .$$

This question will not be marked

Exercise 2 Problem 25.5 in the textbook.

Solution for Exercise 2 Recall that $A\triangle B(A\setminus B)\cup(B\setminus)$ and the latter events are disjoint. In order to prove that

$$\lim_{x\to\infty} P(\{X \leq x\}\triangle\{X_n \leq x\}) = 0$$

we need to prove

$$\lim_{n\to\infty} P(\{X \leq x\} \setminus \{X_n \leq x\}) = 0 \ , \tag{1}$$

$$\lim_{n\to\infty} P(\{X_n \leq x\} \setminus \{X \leq x\}) = 0 \ . \tag{2}$$

We will do a proof of (1) only. It is equivalent to

$$\lim_{n\to\infty} P(\{X \leq x\} \setminus \{X_n \leq x\}) = \lim_{n\to\infty} P(X \leq x) - P(X_n \leq x, X \leq x) \ .$$

Thus it suffices to prove that

$$\lim_{n\to\infty} P(X_n \leq x, X \leq x) = P(X \leq x) \ . \tag{3}$$

We assumed convergence in probability, which implies convergence in distribution, that is

$$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x) \ .$$

Note that for any $\epsilon > 0$,

$$\{X_n \leq x\} \subset \{|X_n - X| > \epsilon\} \cup \{X \leq x + \epsilon\} \ . \tag{4}$$

We intersect the both sides with $\{X \leq x\}$ to get:

$$\{X_n \leq x, X \leq x\} \subset \{|X_n - X| > \epsilon, X \leq x\} \cup \{X \leq x + \epsilon, X \leq x\}$$

and thus

$$\{X_n \le x, X \le x\} \subset \{|X_n - X| > \epsilon\} \cup \{X \le x\} .$$

Since the probability is subadditive, we obtain

$$P\left(X_n \le x, X \le x\right) \le P\left(|X_n - X| > \epsilon\right) + P(X \le x) .$$

Using the assumed convergence in probability we have

$$\lim_{n\to\infty} P\left(X_n \le x, X \le x\right) \le P(X \le x) . \tag{5}$$

Similarly to (4) we have

$$\{X_n \le x - \epsilon\} \subset \{|X_n - X| > \epsilon\} \cup \{X_n \le x\} .$$

Taking intersection with $\{X \le x\}$ yields

$$\{X \le x - \epsilon, X \le x\} \subset \{|X_n - X| > \epsilon, X \le x\} \cup \{X_n \le x, X \le x\}$$

and hence

$$\{X \le x - \epsilon\} \subset \{|X_n - X| > \epsilon\} \cup \{X_n \le x, X \le x\} .$$

Therefore

$$P\left(X \le x - \epsilon\right) \le P\left(|X_n - X| > \epsilon\right) + P(X_n \le x, X \le x)$$

and

$$P(X \le x - \epsilon) \le \lim_{n\to\infty} P\left(X_n \le x, X \le x\right) . \tag{6}$$

Combining (5) and (6) and since ϵ is arbitrary, we finish the proof of (1).

Marking: 4 points. Note: you cannot write $P(X \le x) = P(X \le x - \epsilon)$ even if x is continuity point!

<u>Exercise</u> 3 Problem 25.7 in the textbook.

Solution for Exercise 3 We first prove that $X_n \Rightarrow X$ and $D_n \to_P 0$ yields $D_n X_n \Rightarrow 0$. We have for each $\epsilon > 0$ and $x > 0$,

$$\begin{aligned}
\{|D_n X_n| > \epsilon\} &= \{|D_n X_n| > \epsilon, |D_n| < \epsilon/x\} \cup \{|D_n X_n| > \epsilon, |D_n| \ge \epsilon/x\} \\
&\subseteq \{|X_n| > x\} \cup \{|D_n| \ge \epsilon/x\} \\
&= \{X_n > x\} \cup \{X_n < -x\} \cup \{|D_n| \ge \epsilon/x\} .
\end{aligned}$$

From $D_n \to_P 0$ we obtain $\lim_{n\to\infty} P(|D_n| \ge \epsilon/x) = 0$. Now, in the above calculation x can be chosen arbitrarily. Thus, we can choose it as a continuity point of the distribution of X. Moreover, we can choose X so large so that

$$P(X > x) < \eta/2 , \quad P(X < -x) < \eta/2 .$$

Thus,

$$P(X_n > x) = 1 - P(X_n \le x) \to 1 - P(X \le x) = P(X > x) < \eta/2$$

and

$$P(X_n < -x) \to P(X < -x) < \eta/2$$

and hence

$$\limsup_{n\to\infty} P(|D_n X_n| > 0) \le \eta \ .$$

Since η is arbitrary, we have $D_n X_n \to_P 0$ and hence $D_n X_n \Rightarrow 0$.

Now, by the previous part, $(A_n - a)X_n \Rightarrow 0$ and hence

$$(A_n X_n + B_n) - (aX_n + b) = (A_n - a)X_n + (B_n - b) \Rightarrow 0 \ .$$

Since $h(x) = ax + b$ is continuous, the Continuous Mapping Theorem gives $aX_n + b \Rightarrow aX + b$. Using Theorem 25.4 we obtain the result.

Marking: 4 points.

<u>Exercise</u> 4 This is a modified Problem 25.1 from the textbook.

(a) Construct an example of a sequence of random variable such that the corresponding sequence of distributions converges, but densities do not converge. *Hint:* Consider densities $f_n(x) = 1 + \cos(2\pi n x)$, $x \in [0, 1]$.

(b) Show by an example that the distributions with densities can converge weakly to a limit that has no density. *Hint:* Consider X_n exponentially distributed with the parameter λ_n.

(c) Show by na example that the discrete distributions can converge weakly to a distribution that has a density. *Hint: We did one example in class, think about something else.*

Solution for Exercise 4 (a) Consider the densities $f_n(x) = 1 + \cos(2\pi n x)$, $x \in [0, 1]$. Note that $\int_0^1 f_n(x)dx = 1$. Then the corresponding distribution functions are $F_n(x) = x + \frac{\sin(2\pi n x)}{2\pi n}$, $x \in [0, 1]$. Clearly, for all $x \in [0, 1]$ we have $\lim_{n\to\infty} F_n(x) = x$, that is $F_n \Rightarrow F$, where F is the cumulative distribution of the standard uniform random variable with the density $f(x) = 1$, $x \in [0, 1]$. On the other hand, $f_n(0) = 2$ for each $n \ge 1$. Hence, $f_n(x)$ does not converge to $f(x)$ at $x = 0$.

(b) Let $X_n \sim Exp(\lambda_n)$, that is $P(X_n > x) = 1 - \lambda_n \exp(-\lambda_n x)$. Then for any $x > 0$,

$$P(X_n > x) \le \frac{\mathrm{var}(X_n)}{x} = \frac{1}{\lambda_n x}.$$

If $\lambda_n \to \infty$, then $\lim_{n\to\infty} P(X_n > x) = 0$. Hence, for all $x > 0$, $\lim_{n\to\infty} P(X_n \le x) = 1$. It follows that X_n converges weakly to X, that has point mass at zero. This argument works also for any sequence of positive random variables X_n such that $\mathrm{var}(X_n) \to 0$ as $n \to \infty$.

(c) Let X_i be i.i.d. sequence of any random variables with mean μ and variance σ^2 (for example, X_i can be Bernoulli random variables, or Poisson random variables). According to central limit theorem,

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\text{d}} N(0, \sigma^2)$$

and the limiting distribution has a density.

Marking: 4 points - 2 points for part a), 2 points for part b), additional points for a clever example in part c)

<u>Exercise</u> 5 This is a Problem 25.9 from the textbook:

Suppose that the distributions of random variables X_n and X have densities f_n and f. Show that if $f_n(x) \to f(x)$ for each x outside a set of Lebesgue measure 0, then X_n converges weakly to X.

Solution for Exercise 5

$$|P(X_n \leq x) - P(X \leq x)| \leq \int_{-\infty}^{x} |f_n(u) - f(u)| du \ .$$

Set $g_n(x) = |f_n(x) - f(x)|$. Let $A = \{x : f_n(x) \text{ does not converge to } f(x)\}$. We know that $\lambda(A) = 0$, where λ is the Lebesgue measure. We note that $g_n(x) \leq 2\max\{f_n(x), f(x)\}$ and

$$\begin{aligned}\int_{\infty}^{\infty} g_n(x)dx &\leq 2\int_{\{x:f_n(x)\geq f(x)\}} f_n(x)dx + 2\int_{\{x:f_n(x)\leq f(x)\}} f(x)dx \\ &\leq 2\int_{\infty}^{\infty} f_n(x)dx + 2\int_{-\infty}^{\infty} f(x)dx = 4.\end{aligned}$$

Since $g_n(x) \to 0$ as $n \to \infty$ for all $x \notin A$, we have $\int_{-\infty}^{\infty} g_n(x)dx \to 0$. The result follows.

You can also play with positive an negative parts.

Marking: This question will not be marked.

<u>Presentation</u> 1 Theorem 25.6 in the textbook.