MAT 5171

Final exam

Rafal Kulik

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Student Number: ______________________________

Family Name: ______________________________

First Name: ______________________________

- Do all questions.
- Write one solution only.
- Quote carefully any theorems or results you are using.
- The passing grade for the comprehensive examination is 60%.
- Note that the course grade for Probability II is treated separately from the PASS or FAIL decision in the comprehensive.
- Please write in such the way that I can read your solutions.
- Do the following questions: Q1, Q2, Q3, Q4, Q5 or Q6, Q7, Q8, Q9 or Q10. (total: 8 questions).
- Bring this exam to your oral exam.

Good luck !!!!

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QUESTIONS

Question 1 (10 points) (a) Show that if (X_n) and (Y_n) are both uniformly integrable sequences, then (X_n+Y_n) is also uniformly integrable. **(5 points)**

(b) Show that if $X_n \to_P X$ and (X_n) is uniformly integrable, then

$$\int_\Omega |X_n - X| dP \to 0$$

as $n \to \infty$, i.e. (X_n) converges to X in *mean* or in L_1. **(5 points)**

Question 2 (5 points) Assume that $X_1, X_2, \ldots$ is an infinite sequence of independent Poisson random variables with means $\lambda_1, \lambda_2, \ldots$, respectively. For $n \geq 1$, let $S_n = \sum_{i=1}^n X_i$. Prove that (S_n) converges in distribution if and only if $\sum_{i=1}^\infty \lambda_i < \infty$. Identify the limiting distribution.

If $n \to \infty$, then $S_n \to S_\infty = \sum_{i=1}^\infty X_i$

Question 3 (5 points) A sequence $\{\mu_n\}$ of measures on $\mathbb{R}$ is tight if there exists a compact set $K \subseteq \mathbb{R}$ such that $\mu_n(K) > 1 - \epsilon_0$ for all n. Show that if (μ_n) is a tight sequence of probability measures on $(R, \mathcal{B}(R))$, then the corresponding characteristic functions φ_n are uniformly equicontinuous (i.e., for each $\epsilon > 0$ there is a δ such that $|s - t| < \delta$ implies that $|\varphi_n(t) - \varphi_n(s)| < \epsilon$ for all n).

Question 4 (5 points) Let B_k, $k \geq 1$, be independent Bernoulli random variables such that $P(B_k = 1) = 1/k = 1 - P(B_k = 0)$. Define $R_n = B_1 + \cdots + B_n$.

1. Let $\sigma_n^2 = \mathrm{var}(R_n)$. Show that $\sigma_n^2 / \log(n) \to 1$ as $n \to \infty$;
2. Show that Lindeberg's CLT applies to $X_{n,k} = (\log(n))^{-1/2}(B_k - k^{-1})$;
3. Conclude that $(\log(n))^{-1/2}(R_n - \log(n))$ converges in distribution to a standard normal random variable.

Question 5 (10 points) (a) The δ-method: Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with mean μ and variance σ^2 and let $\bar{X}_n$ be the sample mean. Let $f(x)$ be a function with non-zero derivative at μ. Prove that as $n \to \infty$

$$\sqrt{n}\frac{f(\bar{X}_n) - f(\mu)}{\sigma|f'(\mu)|} \stackrel{\mathrm{d}}{\to} N(0,1)$$

where $N(0,1)$ has the standard normal distribution. **(5 points)**

(b) Assume that $X_1, \ldots, X_n$ are independent and identically distributed with an exponential density $f(x) = \beta\exp(-\beta x)$, $x \geq 0$. Find an estimator of β (method of moments, maximum likelihood method, whichever you prefer). Apply the delta method above to obtain a central limit theorem for your estimator. **(5 points)**

Question 6 (10 points) Assume that X_i, $i \geq 1$, are independent identically distributed random variables with mean 0 and variance σ^2. Assume further that $\mathrm{E}[X_1^3] = 0$ and $\mathrm{E}[X_1^4] < \infty$. Define $\bar{X} = (X_1 + \cdot + X_n)/n$ and $S^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - (\bar{X})^2$.

1. Show that $\bar{X}$ and S^2 are uncorrelated.
2. State central limit theorem for appropriately centered and normalized sample mean $\bar{X}$ (trivial).
3. State and prove central limit for appropriately centered and normalized sample variance S^2.
4. Prove that
$$\sqrt{n}(\bar{X}, S^2 - \sigma^2)^T \stackrel{\mathrm{d}}{\to} N(\mathbf{0}, \Sigma)\ ,$$
where $\mathbf{0} = (0,0)^T$ and Σ has to be determined by you. *Hint: Use Cramer-Wald device, see the textbook.*

Question 7 (10 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-field, and X, Y some random variables on this space.

(a) (5 points) Prove that if X and Y are bounded, then

$$E[XE(Y|\mathcal{G})] = E[YE(X|\mathcal{G})].$$

(b) (5 points) Define the conditional variance of X given $\mathcal{G}$ by:

$$\mathrm{Var}(X|\mathcal{G}) = E[(X - E(X|\mathcal{G}))^2|\mathcal{G}].$$

Prove that

$$E[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}[E(X|\mathcal{G})] = \mathrm{Var}(X).$$

Question 8 (5 points) *Wald's Lemma.* Let $Y_1, Y_2, \ldots$ be independent identically distributed integrable random variables with $\mu = E[Y_i]$ $\forall i$; and let $X_n = \sum_{i=1}^n Y_i$. An integer valued random variable τ if τ is $\sigma(Y_1, \ldots, Y_i)$-measurable, that is, its values depend on $Y_1, \ldots, Y_i$ only, but not on $Y_{i+1}, Y_{i+2}, \ldots$. Let τ be a finite (but not necessarily bounded) stopping time with respect to $(\mathcal{F}_n)$ where $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ and assume that $E[\tau] < \infty$. Prove that

$$E[X_\tau] = \mu E[\tau]. \tag{1}$$

Intuition: Let us start first with τ being independent of $\{Y_i\}$. Then

$$E\left[\sum_{i=1}^{\tau} Y_i\right] = \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{\tau} Y_i \mid \tau = n\right] P(\tau = n) = \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} Y_i\right] P(\tau = n)\ .$$

Hint:

- First prove the result when $Y_i \geq 0$ with probability 1.
- Show that $(X_n - n\mu)$ is a $(\mathcal{F}_n)$-martingale.
- Define $\tau_n = \min(\tau, n)$. Show that $(X_{\tau_n} - \tau_n\mu, \mathcal{F}_n)$ is a martingale and that (1) is satisfied for τ_n.
- Next, let $n \to \infty$.

Question 9 (5 points) Let P and Q be two probability measures on the same space $(\Omega; \mathcal{F}, P)$ and let $(\mathcal{F}_n)$ be a filtration. Assume that $Q \ll P$. Let X_n denote the Radon-Nikodym derivative of Q with respect to P on $\mathcal{F}_n$ i.e., X_n is $\mathcal{F}_n$-measurable and for any $A \in \mathcal{F}_n$, $Q(A) = \int_A X_n dP$. Show that $(X_n, \mathcal{F}_n)$ is a martingale with respect to P, that is $E_P[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$, where for any random variable X, $E_P[X] = \int X dP$.

Question 10 (5 points) Let $(Y_n)_{n\geq 1}$ be independent positive random variables such that $E(Y_n) = 1$ for all $n \geq 1$. Define

$$X_n = Y_1 \ldots Y_n \quad \text{for all } n \geq 1.$$

Show that $(X_n)_{n\geq 1}$ is a martingale and $(X_n)_{n\geq 1}$ converges almost surely to an integrable random variable X.