Option Pricing with Long Memory Stochastic Volatility Models

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September 27, 2012
Stochastic Differential Equation

Let $X(t)$ be a stochastic process and suppose that there exists a real number $x(0)$ and two processes $\mu(t)$ and $\sigma(t)$ such that the following relation holds for all $t \geq 0$.

$$X(t) = x(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s),$$

where $B(t)$ is a Brownian motion. We write alternatively

$$dX(t) = \mu(t)dt + \sigma(t)dB(t),$$

$$X(0) = x(0).$$

The above equation is called stochastic differential equation (SDE) for $X(t)$. 
Important Stochastic Processes in Finance

- Geometric Brownian Motion

\[ dX(t) = \mu X(t) dt + \sigma X(t) dB(t), \]  

for given constants \( \mu \in \mathbb{R} \) and \( \sigma > 0 \). The initial value for the process is assumed to be positive, \( X(0) > 0 \).

- Ornstein-Uhlenbeck Process

\[ dX(t) = \kappa (\theta - X(t)) dt + \sigma dB(t), \]  

where \( \kappa, \theta \) and \( \sigma \) are constants with \( \kappa > 0 \) and \( \sigma > 0 \).

- Square Root Process

\[ dX(t) = \kappa (\theta - X(t)) dt + \sigma \sqrt{X(t)} dB(t), \]  

where \( \kappa, \theta \) and \( \sigma \) are constants with \( \kappa > 0 \) and \( \sigma > 0 \).
European Call Option

A European call with exercise price (or strike price) $K$ and time of maturity $T$ on the underlying asset $S$ is a contract defined by the following clauses:

- The holder of the option has, at time $T$, the right to buy one share of the underlying stock at the price $K$ from the underwriter of the option;
- The holder of the option is in no way obliged to buy the underlying stock;
- The right to buy the underlying stock at the price $K$ can only be exercised at the precise time $T$. 
Denote the time $t$ price of a European call with exercise price $K$ and time of maturity $T$ on the underlying asset $S(t)$ by $C(t; K, T, S(t))$.

Let $\{\Omega, \mathcal{F}, P\}$ denote the physical probability space.

Fundamental Theorem of Finance: The market is arbitrage free if and only if there exists a risk neutral measure $Q$, under which

$$
E^Q \left[ \frac{C(t; K, T, S(t))}{\exp(rt)} \bigg| \mathcal{F}_s \right] = \frac{C(s; K, T, S(s))}{\exp(st)}, \ s \leq t.
$$
Black-Scholes Model

- Assume under the measure $Q$ the stock price $S$ follows the dynamics

$$dS(t) = rS(t)dt + \sigma S(t)dB^Q(t),$$

where $r$ and $\sigma$ are non-negative constants. $B^Q(t)$ is a Brownian motion under measure $Q$.

- Denote $C^{BS}(t; r, K, T, \sigma, S(t))$ the time $t$ price of a European call with exercise price $K$ and time of maturity $T$ on the underlying asset $S(t)$ calculated based on Black-Scholes model.

- Black-Scholes Formula

$$C^{BS}(0; r, K, T, \sigma, S(0)) = S(0)\Phi(d_1) - \exp(-rT)K\Phi(d_2),$$

where $\Phi(x)$ is the cumulated normal distribution function,

$$d_1 = \frac{\ln \left( \frac{S(0)}{K} \right) + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}.$$
We define implied volatility $\sigma_{impl}$ by

$$C^{BS}(0; r, K, T, \sigma_{impl}, S(0)) = C^{obs}$$

Theoretically, options whose underlying is governed by the geometric Brownian motion should have a flat implied volatility surface, since volatility is a constant.

In practice, the implied volatility surface is not flat and $\sigma_{impl}$ varies with $K$ and $T$. This disparity is known as the volatility skew.

There are several patterns for the volatility skew:
Figure: Patterns of volatility skew.
Term Structure of Volatility Smiles Puzzle

- Stochastic volatility model:
  \[ dS(t) = \mu S(t)dt + \sigma(t)S(t)dB_1(t), \]
  \[ \sigma^2(t) = f(Y(t)), \]
  where \( Y(t) \) is another stochastic process.

- Under the assumption of short memory for the volatility processes, by a simple application of the law of large numbers to volatility process, the effects of the randomness of the volatility should vanish when the time to maturity of the option increases and therefore the volatility skew should be erased.

- In practice, volatility skew effects are significant for very long term options.
A long memory process is a stationary process with a hyperbolically decaying autocorrelation function,

\[ \rho(h) \sim C_\rho h^{2d-1}, \quad C_\rho \neq 0, \quad 0 < d < \frac{1}{2}, \quad \text{as } h \to \infty. \]

\( d \) is the so-called (long) memory parameter which controls the speed of decay of the autocorrelation.
Fractional Heston Model

We define the fractional Heston model by

\[ dX(t) = \left( r - \frac{1}{2} \sigma^2(t) - \frac{1}{2} \rho^2 \gamma^2 (\theta + \sigma_c^2(t)) \right) dt + \sigma(t) dB^Q_1(t) \]
\[ + \rho \gamma \sqrt{\theta + \sigma_c^2(t)} dB^Q_2(t), \]  
(4)

\[ \sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t), \]  
(5)

\[ (\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} \sigma_c^2(s) ds, \]  
(6)

\[ d\sigma_c^2(t) = -\kappa \sigma_c^2(t) dt + \gamma \sqrt{\theta + \sigma_c^2(t)} dB^Q_2(t), \]  
(7)

where \( X(t) = \ln(S(t)) \). 0 < \( d < \frac{1}{2} \), \( B^Q_1(t) \) and \( B^Q_2(t) \) are two independent Brownian motions under the risk neutral probability measure \( Q \). \( \rho \) is a constant to induce the correlation between the volatility and stock price processes.
Denote the time $t$ price of a European call with exercise price $K$ and time of maturity $T$ on the underlying asset $S(t)$ by $C(t; K, T, S(t))$.

$$C(t; K, T, S(t)) = S_t Q_1(X(T) > \ln K|\mathcal{F}_t) - \exp(-r(T - t)) K Q_2(X(T) > \ln K|\mathcal{F}_t),$$

$$Q_j(X(T) > \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( f_j(\phi) \frac{\exp(-i\phi \ln K)}{i\phi} \right) d\phi,$$

$j = 1, 2$.

$$f_1(\phi) = \exp(-r(T - t) - X(t)) f(-i + \phi),$$

$$f_2(\phi) = f(\phi).$$

$$f(\phi) = \mathbb{E}^Q[\exp(i\phi X(T))|\mathcal{F}_t].$$
Conditional Characteristic Function of Fractional Heston Model

We prove the conditional characteristic function $f(\phi)$ of $X(t + \tau)$ ($f(\phi) = E^Q[\exp(i\phi X(t + \tau))|\mathcal{F}_t]$) under the probability measure $Q$ can be computed as follows:

$$f(\phi) = \exp \left[ i\phi \tau r - \frac{1}{2} i\phi \rho^2 \gamma^2 \theta \tau - \frac{1}{2} \phi(i + \phi) \theta \tau - \frac{1}{2} \phi(i + \phi) \frac{1}{\Gamma(d + 1)} \int_0^t (t + \tau - s)^d 
- A(\tau) - (i\phi \rho + B(\tau)) \sigma_c^2(t) - i\phi X(t) \right],$$

where the functions $A(\tau)$ and $B(\tau)$ can be computed by numerically solving the following system of ODEs:

$$\dot{A}(\tau) = -\frac{1}{2} \gamma^2 \theta B^2(\tau),$$

$$\dot{B}(\tau) = -\kappa B(\tau) - \frac{1}{2} \gamma^2 B^2(\tau) + \frac{1}{2} i\phi \rho^2 \gamma^2 + \frac{1}{2} \phi(i + \phi) \frac{1}{\Gamma(d + 1)} \tau^d - i\phi \rho \kappa,$$

with boundary conditions $A(0) = 0$ and $B(0) = -i\phi \rho$. 
Numerical Results

- The greater $d$ is, the smoother the path of the volatility process is.
- Higher $d$ will decrease the kurtosis of stock returns and this has the effect of lowering far-in-the-money and far-out-of-the-money option prices and raising near-the-money prices.
- Higher $d$ results in a much less steep volatility skew at the short maturities. For each $d$, the volatility skew flattens out as the maturity is increased and the final steepness is roughly the same. It is clear that the flattening out is slower for the model with higher $d$ than the model with lower $d$.
- In fractional Heston model, zero-correlation between volatility and stock price processes only can produce a volatility smile. For more general volatility skew, we need to introduce non-zero correlation.
Figure: Option prices from the fractional Heston model with different long memory parameters minus that from Heston model.
**Figure**: Implied volatility plots from the fractional Heston model with different long memory parameters.
Figure: Implied volatility plots from the fractional Heston model with different correlation parameters.
Conclusion

- We propose two continuous time stochastic volatility models with long memory which haven’t been considered in the literature.
- We provide analytical formulae for the first time that allow us to study option prices numerically, rather than by means of simulation.
- We allow for the non-zero correlation between the stochastic volatility and stock price processes.
- We numerically study the effects of long memory on the option prices.