A GENTLE INTRODUCTION ON HOMOTOPY LIMITS AND COLIMITS

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Abstract. Limits and colimits are fundamental in theory of categories and in many areas of mathematics. In particular many useful constructions in topology are colimits (quotient space, space of orbits under a group action, attachment of cells, disjoint unions, wedges, attachment of a handle in surgery, realization of a simplicial set, etc...) or limits (product of spaces, fibre products, fixed point set of a group action, etc...). However these constructions are not homotopy invariant which is a serious flaw in homotopy theory. The notions of homotopy (co)limits is a homotopy-invariant versions of these classical (co)limits construction.
The goal of this primer is to explain these notion of homotopy (co)limits starting from the simple example of the (homotopy) pullback.

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1. A review of pullbacks

A very illustrative example of homotopy limits is given by homotopy pullbacks. Consider the following diagram of topological spaces and continuous maps

(1)
$$\mathbb{D} := \begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

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which can be represented in the flat form

$$\mathbb{D} := \left(X \xrightarrow{f} B \xleftarrow{g} Y \right).$$

The pullback of $\mathbb{D}$ is the space

$$P := \{(x, y) \in X \times Y | f(x) = g(y)\}.$$

The composition of the inclusion $P \hookrightarrow X \times Y$ with the projections give two canonical maps

$$\begin{aligned} p_X &: P \to X \ , \ (x, y) \mapsto x \\ p_Y &: P \to Y \ , \ (x, y) \mapsto y. \end{aligned}$$

Hence we have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd} .$$

Notice that we can also define a map $p_B \colon P \to B$ as the composite $p_B := fp_X = gp_Y$. In other words we get the following commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "p_X"'] \arrow[d, "p_B"] \arrow[dr, "p_Y"] & \\ X \arrow[r, "f"] & B & Y. \arrow[l, "g"'] \end{tikzcd}$$

Moreover the space P and the maps p_X, p_B, and p_Y are in a sense *universal* with respect to this diagram. More precisely, if we are given another space Q and maps q_X, q_B, q_Y making the following diagram commutative

$$\begin{tikzcd} & Q \arrow[dl, "q_X"'] \arrow[d, "q_B"] \arrow[dr, "q_Y"] & \\ X \arrow[r, "f"] & B & Y \arrow[l, "g"'] \end{tikzcd}$$

then there exists a unique map $q \colon Q \to P$ such that $q_X = p_X q$, $q_B = p_B q$ and $q_Y = p_Y q$.

Exercise 1.1. *Check that P and the projections p_X, p_B, and p_Y indeed have this universal property.*

Because the pullback has this universal property with respect to the diagram $\mathbb{D}$ we say that it is the *limit* of the diagram $\mathbb{D}$ (see Section 8) and we write $P = \lim \mathbb{D}$.

This limit construction can be seen as a functor. More precisely suppose given another diagram of topological space

$$\mathbb{D}' := \ X' \xrightarrow{f'} B' \xleftarrow{g'} Y'$$

of the same shape as $\mathbb{D}$. By a *morphism of diagrams* from $\mathbb{D}$ to $\mathbb{D}'$ we mean a natural transformation $\alpha\colon \mathbb{D} \to \mathbb{D}'$, that is a triple of maps $\alpha_X, \alpha_B, \alpha_Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"] & B \arrow[d, "\alpha_B"] & Y \arrow[l, "g"'] \arrow[d, "\alpha_Y"] \\ X' \arrow[r, "f'"] & B' & Y'. \arrow[l, "g'"'] \end{tikzcd} \tag{2}$$

Therefore we have a category whose objects are diagrams of the same shape as $\mathbb{D}$ and morphisms are natural transformations α as above. The morphism α induces an obvious map

$$\lim \alpha\colon \lim \mathbb{D} \to \lim \mathbb{D}'$$

so that lim defines a functor from the category of diagrams of the same shape as $\mathbb{D}$ to the category Top of topological spaces.

2. Homotopy equivalences and weak equivalences of topological spaces

Two continuous maps $u, v\colon X \to Y$ are *homotopic* if there is a continuous map

$$H\colon X \times [0,1] \to Y$$

such that $H(x,0) = u(x)$ and $H(x,1) = v(x)$, for all $x \in X$. We then write

$$u \simeq v.$$

A map $f\colon X \to X'$ is a *homotopy equivalence* (in the sense of Hurewicz, or sometimes *strong homotopy equivalence*) if there exists a map $g\colon Y \to X$ such that

$$gf \simeq \mathrm{id}_X \mathrm{and} fg \simeq \mathrm{id}_Y.$$

We then say that g is a *homotopy inverse* of f.

There is a weaker notion of homotopy equivalence between topological spaces which is usefull to work with. Namely we say that a map

$$f\colon X \to X'$$

is a *weak homotopy equivalence* (or an equivalence in the sense of Serre) if for each base point $x_0 \in X$ and each integer $n \geq 0$ the induced map

$$\pi_n(f)\colon \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X', f(x_0)) \tag{3}$$

are isomorphism of groups. When the spaces are simpy-connected (that is arcwise connected and with trivial fundamental groups), this is equivalent to ask that the induced maps in homology

$$H_n(f)\colon H_n(X) \xrightarrow{\cong} H_n(X') \tag{4}$$

are isomorphisms.

When X and X' are CW-complexes, Whitehead theorem states that $f\colon X \to X'$ is a weak homotopy equivalences (i.e. in the sense of Serre) if and only if f is a homotopy equivalence (i.e. in the sense of Hurewicz).

It is more convenient to work with weak homotopy equivalences because the definitions (3) and (4) give algebraic criteria to check that a map is an weak homotopy equivalences, with no need to find an explicit homotopy inverse.

In the rest of this paper when we talk of an equivalence of space $f\colon X \to X'$ (called a *weak equivalence* of spaces, or we, in the sequel) we always mean a weak homotopy equivalence and we denote this by

$$f\colon \xrightarrow{\simeq} X'.$$

By Whitehead theorem, when we deal with CW-complexes (which are the nice spaces for ageometer, or even to spaces which are homotopy equivalent (in the sense of Hurewics) to CW-complexes, then a map f is a (strong) homotopy equivalence as soon as it is a weak homotopy equivalence. In other words, for reasonable spaces, the two notions coincide.

3. Weak equivalences of diagrams

Given a morphism of diagram

$$\alpha\colon \mathbb{D} \to \mathbb{D}'$$

as in (2) we say that it is a *weak equivalence of diagrams* if the three maps $\alpha_X, \alpha_Y, \alpha_B$ are weak homotopy equivalences. If the spaces X', Y', B' are all CW-complexes, then by the Whitehead theorem each of these maps admit a homotopy inverse. However it is very important to note that it does **not** imply that there exists a map of diagrams $\beta\colon \mathbb{D}' \to \mathbb{D}$ such that β_X is a weak homotopy inverse of α_X, β_Y is a homotopy inverse of α_Y and β_Y is a homotopy inverse of α_Y.

Because of this notion of "weak equivalence" of diagrams is not as strong as we should expect: even if all spaces are CW-complexes, it is not true that a weak equivalence of diagrams α does admit an inverse up to homotopy β.

This weakness of the notion of weak equivalence of diagrams is also a big advantage. Indeed it is fairly easy to check that a map is a weak equivalence of diagrams, since it is enough to check it *objectwise*, i.e. at each object of the diagram, with no need to build an explicit homotopy inverse of a morphism of diagrams (which would require to check commutativities of the various squares).

4. Pullbacks do not preserve weak equivalences

The notion of pullbacks, or more generally of limits, is very important in mathematics and in particular in homotopy theory. However it has a serious weakness in the latter theory: It is not homotopy invariant. More precisely, suppose that a morphism of diagrams $\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{D}'$ is a weak equivvalence (that is, an *objectwise* weak homotopy equivalence, so each of the maps α_X, α_B, and α_Y are weak homotopy equivalences.) In general it is **not true** that the induced map $\lim \alpha\colon \lim \mathbb{D} \to \lim \mathbb{D}'$ is also a weak homotopy equivalence. This is illustrated by the following

Example 4.1. Let X be a space and $x_0 \in X$ be a base point. The *path space of* X is defined by

$$PX := \{\omega\colon [0,1] \to X : \omega(0) = x_0\}$$

equipped with the compact open topology of the mapping space $X^{[0,1]} = \mathrm{Map}([0,1], X)$. It is well known that the space PX is contractible (exercise!). Consider the one-point space $\{x_0\}$ and the map $\alpha_0\colon \{x_0\} \to PX$ that sends x_0 to the constant path at x_0. Consider also the continuous map $\mathrm{ev}_1\colon PX \to X, \omega \mapsto \omega(1)$. We have a commutative diagram

$$\begin{tikzcd}
\{x_0\} \arrow[r, hook] \arrow[d, "\alpha_0"'] & X \arrow[d, equal] & \{x_0\} \arrow[l, hook'] \arrow[d, equal] \\
PX \arrow[r, "ev_1"] & X & \{x_0\}. \arrow[l, hook']
\end{tikzcd}$$

Each of the vertical map is a homotopy equivalence. However the pullback of the top line and of the bottom line are not homotopy equivalent. Indeed the limit of the top line is a one-point space but the limit of the bottom line is the based loop space $\Omega X = \{\omega\colon [0,1] \to X | \omega(0) = \omega(1) = x_0\}$ which in general is not contractible.

To give a more specific example take $X = S^1$, the circle with a fixed base point $* \in S^1$. A continuous map $f\colon [0,1] \to S^1$ such that $f(0) = f(1) = *$, that is an element of ΩS^1, can be readily identified with a continuous map $\bar{f}\colon S^1 \to S^1$, since the interval $[0,1]$ with its extremities 0 and 1 identified is homeomorphic to the circle. This map $\bar{f}\colon S^1 \to S^1$ has a *degree*

$$\deg(\bar{(}f)) \in \mathbb{Z}$$

which counts how many times $\bar{f}$ winds the circle around itself. It is easy to convince ourself that this defines a continuous map

$$\deg\colon \Omega S^1 \to \mathbb{Z}$$

which is continuous. Moreover this maps is surjective because every degree can occurs. Therefore the space ΩS^1 is not connected and actually $\pi_0(\Omega S^1)$ is isomorphic to $\mathbb{Z}$ through this degree map. This implies that ΩS^1 is not weakly equivalent to a point, since the latter is path-connected.

5. Homotopy pullbacks

The notion of a **homotopy pullback** is introduced to correct the lack of homotopy invariance of the genuine pullback. More precisely the homotopy pullback of a diagram like $\mathbb{D}$, denoted by $\operatorname{holim}\mathbb{D}$, will have the following features:

(A) $\operatorname{holim}\mathbb{D}$ is a topological space[1] and if $\alpha\colon \mathbb{D} \to \mathbb{D}'$ is a morphism of diagrams we have an induced map $\operatorname{holim}\alpha\colon \operatorname{holim}\mathbb{D} \to \operatorname{holim}\mathbb{D}'$ that makes the obvious diagrams commutative. In other words holim is a functor from the category of diagrams with the shape of $\mathbb{D}$ to Top;

(B) holim is invariant under weak equivalences: If $\alpha\colon \mathbb{D} \to \mathbb{D}'$ is a weak equivalence of diagrams then $\operatorname{holim}\alpha\colon \operatorname{holim}\mathbb{D} \to \operatorname{holim}\mathbb{D}'$ is a weak equivalence of spaces;

(C) there is a natural transformation $\eta\colon \lim \to \operatorname{holim}$, hence for each diagram $\mathbb{D}$ we have a continuous map[2] $\eta_{\mathbb{D}}\colon \lim\mathbb{D} \to \operatorname{holim}\mathbb{D}$;

(D) there exists a class of diagrams $\mathbb{F}$ (called *fibrant diagrams*) for which the map $\eta_{\mathbb{F}}\colon \lim\mathbb{F} \to \operatorname{holim}\mathbb{F}$ is a weak equivalence. Moreover this class of diagrams is large in the sense that for any diagram $\mathbb{D}$, there exists a fibrant diagram $\mathbb{F}$ and a weak equivalence $\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{F}$: We say that $\mathbb{F}$ is a fibrant replacement of the diagram $\mathbb{D}$.

Properties (B) and (D) suggest that holim is the good homotopy invariant replacement of the genuine limit. Notice however that this homotopy limit does not have the universal property of lim, even "up to homotopy". Actually we even do not claim to have "projection" morphisms $p'_X\colon \operatorname{holim}\mathbb{D} \to X$, etc. as we had for the limit.

It is very important to note that even if what we will define is called a homotopy limit, all the diagrams we are considering are always commutative on the nose and not only up to homotopy. Working with diagrams commuting up to homotopy is asking for trouble...

We come to the precise construction of holim in the case of diagrams of the shape of (1), i.e. homotopy pullbacks

[1] It is important that the homotopy limit is an genuine space and not just a homotopy type.

[2] Note that $\eta_{\mathbb{D}}$ is going in the counter-intuitive direction since $\lim\mathbb{D}$ is its domain and not its codomain.

Definition 5.1. The *homotopy pullback* of the diagram $\mathbb{D} := X \xrightarrow{f} B \xleftarrow{g} Y$ is the space

$$\operatorname{holim}\mathbb{D} := \{(x,\beta,y) \in X \times B^{[0,1]} \times Y : \beta(0) = f(x), \beta(1) = g(y)\}.$$

Given a morphism of diagrams $\alpha\colon \mathbb{D} \to \mathbb{D}'$ we define the map $\operatorname{holim}\alpha\colon \operatorname{holim}\mathbb{D} \to \operatorname{holim}\mathbb{D}$ in the obvious way.

Exercise 5.2. *Let X be a space and let $x_0 \in X$ be a base point. Show that the homotopy pullback of $\{x_0\} \to X \leftarrow \{x_0\}$ is the based loop space ΩX.*

Exercise 5.3. *Let X be a space. Show that the homotopy pullback of the two diagonal maps $X \stackrel{\Delta}{\to} X \times X \stackrel{\Delta}{\leftarrow} X$ is the free loop space $X^{S^1} = \mathrm{Map}(S^1, X)$. What is the pullback of that diagram ?*

Exercise 5.4. *Show that the homotopy pullback is homotopy invariant in the sense of (B). (Non-hint: this exercise is not completely obvious.)*

The map

$$\eta_{\mathbb{D}}\colon \lim\mathbb{D} \to \operatorname{holim}\mathbb{D}$$

is defined by $\eta_{\mathbb{D}}(x,y) = (x,\beta,y)$ where β is the constant path at $f(x) = g(y)$. It is straightforward to check that η is a natural transformation.

Thus the homotopy pullback of Definition 5.1 satisfies the properties (A)-(C). To establish property (D) we recall some facts about fibrations in the next section.

6. Fibrations and homotopy pullbacks

To define fibrant diagrams we first recall the notion of a fibration.

Definition 6.1. A continuous map $p\colon E \to B$ is said to have the *homotopy lifing property with respect to a space* Z if for any commutative diagram of continuous maps of the form

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "F_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "f"] & B \end{tikzcd}$$

there exists a continuous map $F\colon Z \times [0,1] \to E$ that extends F_0 and lifts f along p, i.e. $F|Z \times \{0\} = F_0$ and $pF = f$. In other words there is a map F which makes the following diagram commutes

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "F_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "f"] \arrow[ur, "F"] & B \end{tikzcd} .$$

A continuous map $p\colon E \to B$ is called a *(Serre) fibration* (or simply a fibration in the sequel) if it has the homotopy lifing property with respects to all disks, $n \geq 0$,

$$Z = D^n = \{x \in \mathbb{R}^n \colon \|x\| \leq 1\}.$$

Thius is equivalent to say that p has the homotopy lifing property[3]with respects to all spaces CW-complexes Z.

Definition 6.2. A diagram $\mathbb{D} := \ X \xrightarrow{f} B \xleftarrow{g} Y$ is called *fibrant* if both f and g are fibrations.

Exercise 6.3. *Show that if $\alpha\colon \mathbb{D} \to \mathbb{D}'$ is a weak equivalence of diagrams (of the shape of a pullback as in* (1)*) and if both diagrams are fibrant then the induced map between the limits,*

$$\lim\alpha\colon \lim\mathbb{D} \to \lim\mathbb{D}'$$

is a weak equivalence of spaces (Hint: use the fact that in a pullback of a fibrant diagram the fibres of parallel vertical maps are weakly equivalent and use the long exact sequence of homotopy groupsof fibrations and the five lemma to deduce the result.)

Exercise 6.4. *(not easy) Show that if $\mathbb{D}$ is fibrant (in the sense of Hurewicz) then $\eta_{\mathbb{D}}$ is a homotopy equivalence.*

Remark: If the diagram $\mathbb{D}$ is fibrant in the sense of Serre then $\eta_{\mathbb{D}}$ is a weak equivalence, i.e. it induces isomorphisms between homotopy groups.

Exercise 6.5. *Show that for any diagram $\mathbb{D}$ there exists a fibrant diagram $\mathbb{F}$ and a natural objectwise homotopy equivalence $\alpha\colon \mathbb{D} \to \mathbb{F}$. (Hint: show first that any continuous map $f\colon X \to B$ can be factored into $X \overset{h}{\to} X' \overset{p}{\to} B$ where h is a homotopy equivalence and p is a fibration.)*

All of this establish property (D) for homotopy pullbacks.

7. Spaces of natural transformations and (homotopy) pullbacks

We give here a useful alternative description of the (homotopy) pullback in terms of spaces of natural transformations. Remember that the category Top is enriched over itself, which means that for two spaces X_0 and X the set of morphisms

$$\mathrm{Map}(X_0, X) := \{f\colon X_0 \to X : f \text{ is continuous}\}$$

is itself a space (when equipped with the compact open topology[4] More generally given two diagrams

$$\begin{aligned} \mathbb{D}_0 \ &:= \ X_0 \xrightarrow{f_0} B_0 \xleftarrow{g_0} Y_0 \\ \mathbb{D} \ &:= \ X \xrightarrow{f} B \xleftarrow{g} Y \end{aligned}$$

[3]When p has the homotopy lifing property with respects to all spaces, we say that p is a Hurewica fibration, which is a stronger notion that we will not need.

[4]The compact open topolgy on $\mathrm{Map}(X,Y)$ is the topolgy generated by the elementary open sets $\langle, K, U\rangle$ defined, for compact subspace $K \subset X$ and open subsets $U \subset Y$, as

$$\langle, K, U\rangle := \{f\colon X \to Y : F(K) \subset U\}.$$

When X is compact Hausdorff and Y is a mtetric space, this is nothing else than the topolgy induced by the uniform distance.
In other to have a closed cartesian category we would like the functor $\mathrm{Map}(-X,-)\colon \mathrm{Top} \to \mathrm{Top}$ to be the right adjoint of the functor $- \times X\colon \mathrm{Top} \to \mathrm{Top}$. For this to be true we need to work in the subcategory of compactly generated weakly Hausdorff spaces. This is a suitable subcategory of Top that we will not define here but by abuse of notation we assume that Top is that subcategory.

the set of natural transformations

$$\mathrm{Nat}(\mathbb{D}_0,\mathbb{D})\colon\ \{\alpha=(\alpha_X,\alpha_B,\alpha_Y)\colon \mathbb{D}_0\to\mathbb{D}\}$$

is a subset of the product

$$\mathrm{Map}(X_0,X)\times\mathrm{Map}(B_0,B)\times\mathrm{Map}(Y_0,Y)$$

and we can endow it with the subspace topology.

Suppose that $\mathbb{D}_0$ is the diagram in which each space is the one-point space:

$$\mathbb{D}_0 := \ * \longrightarrow * \longleftarrow *.$$

Proposition 7.1. *Given a diagram* $\mathbb{D} := \ X \xrightarrow{f} B \xleftarrow{g} Y$ *, we have a homeomorphism*

$$\lim\mathbb{D}\cong\mathrm{Nat}(\mathbb{D}_0,\mathbb{D})$$

where $\mathbb{D}_0$ *is the diagram of one-point spaces.*

Proof. Obvious. □

The above can be summarized by the formula

$$\lim\left(X \xrightarrow{f} B \xleftarrow{g} Y \right) \quad\cong\quad \mathrm{Nat}\left(\left(* \longrightarrow * \longleftarrow * \right)\ ,\ \left(X \xrightarrow{f} B \xleftarrow{g} Y \right)\right).$$

Consider now the diagram

$$\widetilde{\mathbb{D}_0} := \qquad \{0\} \xhookrightarrow{i_0} [0,1] \xhookleftarrow{i_1} \{1\}$$

where i_0 and i_1 are the inclusions of the two ends of the unit interval $[0,1]$. It is immediate to check from the definition of the homotopy pullback that $\mathrm{holim}\,\mathbb{D}$ is exactly $\mathrm{Nat}(\widetilde{\mathbb{D}_0},\mathbb{D})$, in other words

$$\mathrm{holim}\left(X \xrightarrow{f} B \xleftarrow{g} Y \right) \quad\cong\quad \mathrm{Nat}\left(\left(\{0\} \xrightarrow{i_0} [0,1] \xleftarrow{i_1} \{1\} \right)\ ,\ \left(X \xrightarrow{f} B \xleftarrow{g} Y \right)\right).$$

In view of the above it is interesting to compare diagrams $\mathbb{D}_0$ and $\widetilde{\mathbb{D}_0}$. An important feature of both diagrams is that they consist of contractible spaces. Diagram $\widetilde{\mathbb{D}_0}$ has another property that guarantees that $\mathrm{Nat}(\widetilde{\mathbb{D}_0}\,,-)$ is homotopy invariant, namely it is a free diagram[5] in a sense made precise in [2] and that we will explain in Section 8.

[5]For the readers familiar with the notion of Quillen category, the important property of $\widetilde{\mathbb{D}_0}$ is that it is a cofibrant diagram for a suitable Quillen structure on the category of diagrams. But it is very important to notice than the Quillen structure for which we ask $\widetilde{\mathbb{D}_0}$ to be cofibrant **is not** the same as the Quillen structure for which we asked the diagram $\mathbb{F}$ to be fibrant in property (D) of homotopy pullbacks. Actually there are two natural but different Quillen model structures on the category of diagrams; see [3] for details.

8. Limits and homotopy limits for general diagrams

In this section we generalize what we have done for pullbacks to more general diagrams. First we formalize the notion of a diagram of topological spaces of any shape.

Definition 8.1. Let I be a small category. A *diagram of topological spaces of shape* I or I-*diagram of spaces* is a functor $\mathbb{D}\colon I \to \mathrm{Top}$. A morphism between two diagrams $\mathbb{D}, \mathbb{D}'$ of the same shape is a natural transformation $\alpha\colon \mathbb{D} \to \mathbb{D}'$. There is a category whose objects are diagrams of spaces of shape I and morphisms are natural transformations. This category of I-diagrams of spaces is denoted by Top^I.

Example 8.2. Let I_{pb} be the category with 3 objects, u, v, and w and with only two non-identity morphisms $u \to v$ and $w \to v$. Then diagrams of shape I_{pb} are exactly diagrams studied in Section 1. Indeed an I_{pb}-diagram $\mathbb{D}\colon I_{pb} \to \mathrm{Top}$ is characterized by

$$\mathbb{D}(u) \xrightarrow{\mathbb{D}(u\to v)} \mathbb{D}(v) \xleftarrow{\mathbb{D}(w\to v)} \mathbb{D}(w)\ .$$

Many other examples of shape of diagrams are given in Section 11.

Fix a small category I. The limit of a diagram $\mathbb{D}\colon I \to \mathrm{Top}$ can be defined by the universal property, generalizing the pullback. In more detail, $\lim \mathbb{D}$ is a space P equipped with continuous maps $p_i\colon P \to \mathbb{D}(i)$ for each object i of I such that for any morphism $\phi\colon i \to j$ in I the following diagram

$$\begin{tikzcd} P \arrow[r, "p_i"] \arrow[dr, "p_j"'] & \mathbb{D}(i) \arrow[d, "\mathbb{D}(\phi)"] \\ & \mathbb{D}(j) \end{tikzcd}$$

is commutative. To be called the limit this couple $(P, \{p_i\}_{i\in ob(I)})$ needs to be universal with respect to the above property, as was the pullback.

Exercise 8.3. *State precisely this universal property.*

Actually the limit is given by the explicit formula

$$\lim \mathbb{D} = \left\{ (x_i)_{i\in ob(I)} \in \prod_{i\in ob(I)} \mathbb{D}(i) \text{ such that } \forall \phi\colon i \to j, \mathbb{D}(\phi)(x_i) = x_j \right\}. \tag{5}$$

topologized as a subspace of the product $\prod_{i\in ob(I)} \mathbb{D}(i)$.

Exercise 8.4. *Compare this formula when $I = I_{pb}$ with the definition of the pullback.*

Exercise 8.5. *Show that the right hand side of Equation* (5) *has the universal property.*

Moreover a morphism of diagrams $\alpha\colon \mathbb{D} \to \mathbb{D}'$ induces an obvious map $\lim \alpha\colon \lim \mathbb{D} \to \lim \mathbb{D}'$. In other words we have a functor

$$\lim : \mathrm{Top}^I \to \mathrm{Top}.$$

We can also see this functor lim as a right adjoint. Indeed to any space X one can associate the "constant" diagram δ_X of shape I such that for each object i of I, $\delta_X(i) = X$, and for each map $\phi\colon i \to j$, the map $\delta_X(\phi)$ is the identity of X. For $f\colon X \to Y$ we have an obvious induced map of diagrams $\delta_f\colon \delta_X \to \delta_Y$. This defines a functor

$$\delta:\ \mathrm{Top} \to \mathrm{Top}^I\,,\ X \mapsto \delta_X.$$

Exercise 8.6. *Show that the universal property of the limit is equivalent to the fact that* lim *is a right adjoint to the functor δ.*

Also the limit can be rephrased in terms of some space of natural transformations. Let $\mathbb{D}_0\colon I \to \mathrm{Top}$ be the diagram of shape I such that each space $\mathbb{D}_0(i)$ is the one point space $*$. Using Equation (5) clearly we have a homeomorphism

$$\lim \mathbb{D} \quad \cong \quad \mathrm{Nat}(\mathbb{D}_0, \mathbb{D}).$$

Of course the limit is in general not homotopy invariant since it is was not for pullback diagrams. But one can prove that there exists a notion of homotopy limits of I-diagrams with the following properties:

(A) holim: $\mathrm{Top}^I \to \mathrm{Top}$ is a well defined functor;
(B) the homotopy limit is homotopy invariant: if $\alpha\colon \mathbb{D} \to \mathbb{D}'$ is a morphism of diagram such that for each object i of I, $\alpha(i)$ is a homotopy equivalence, then $\mathrm{holim}\,\alpha\colon\ \mathrm{holim}\,\mathbb{D} \to \mathrm{holim}\,\mathbb{D}'$ is also a homotopy equivalence;
(C) there is a natural transformation $\eta\colon\ \lim \to \mathrm{holim}$, that is for each diagram $\mathbb{D}$ we have a continuous map $\eta_{\mathbb{D}}\colon\ \lim \mathbb{D} \to \mathrm{holim}\,\mathbb{D}$ making commute the obvious diagrams;
(D) there exists a class of diagrams $\mathbb{F}$ (called *fibrant diagrams*) for which the map $\eta_{\mathbb{F}}$ is a homotopy equivalence and this class of diagrams is large in the sense that for any diagram $\mathbb{D}$ there exists a fibrant diagram $\mathbb{F}$ and a homotopy equivalence $\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{F}$ (we say that $\mathbb{F}$ is a *fibrant replacement* of the diagram $\mathbb{D}$.)

If we want to emphasize the shape I of the diagram we write $\lim_I \mathbb{D}$ or $\mathrm{holim}_I\,\mathbb{D}$.

The question is of course how to define the functor $\mathrm{holim}_I\colon \mathrm{Top}^I \to \mathrm{Top}$ for a general shape I. There are many approches but we will focus on the original one, due to Bousfield-Kan [1] and wich used the classifying space pf the over-category, and then we will develop another approach due to Dror-Farjoun [2] which is sometimes more computable but involve some choices (there are altenative ways due to Dwyer-Kan, Dwyer-Spalinski [3], etc... all of which being interesting).

Both Bousfield and Kan and Dror-Farjoun start by fixing a particular diagram

$$\widetilde{\mathbb{D}_0}\colon I \to \mathrm{Top}$$

with some good properties and they set

$$\mathrm{holim}_I\,\mathbb{D} := \mathrm{Nat}(\widetilde{\mathbb{D}_0}, \mathbb{D}).$$

Bousfield-Kan defines $\widetilde{\mathbb{D}_0}$ to be the diagram obtained from the classifying space of the over category,

$$\widetilde{\mathbb{D}_0}(i) = B(I \downarrow i).$$

Here, $I \downarrow i$ is the over-category whose objects are morphisms $f_1\colon i_1 \to i$ of I with codomain i, and whose morphisms in $I \downarrow i$ from $f_1\colon i_1 \to i$ to $f_2\colon i_2 \to i$ are the morphisms $g\colon i_1 \to i_2$ in I such that $f_1 = f_2 g$. Then $B(I \downarrow i)$ is the classifying space of the category $I \downarrow i$. Remember that for any small category, $\mathcal{C}$, its nerve, $N_\bullet(\mathcal{C})$, is the simplicial set whose n-simplices are sequences

$$i_0 \xrightarrow{f_1} i_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} i_n$$

of n composable morphisms. In particular the 0-simplices are the objects of the category and the 1-simplices are the morphisms. The boundaries are obtained by composing two consecutive morphisms and the degeneracies correspond to inserting an identity map. The classifying space of $\mathcal{C}$ is the geometric realization of its nerve, $B\mathcal{C} := |N_\bullet(\mathcal{C})|$. An excellent reference for the notion of classifying space of a category is the eponym paper of G. Segal at Publ. IHES.

Exercise 8.7. *Consider the shape $I_{pb} = \{u \to v \leftarrow w\}$ of Example 8.2 corresponding to pullbacks. Determine the over-categories $I_{pb} \downarrow i$ for the three objects $i = u,\ v,$ and w. Determine their classifying spaces $B(I_{pb} \downarrow i)$ and show that the corresponding diagram $B(I_{pb} \downarrow -)$ is naturally homeomorphic to the diagram*

$$\{0\} \xhookrightarrow{i_0} [0,1] \xhookleftarrow{i_1} \{1\}\ .$$

Exercise 8.8. *Show that for any small category I all the spaces $B(I \downarrow i)$ are contractible. (Hint: check that $I \downarrow i$ has a final object.)*

Definition 8.9. Let I be a small category and let $\mathbb{D}\colon I \to \mathrm{Top}$ be an I-diagram of spaces. The homotopy limit of that diagram is

$$\mathrm{holim}_I\, \mathbb{D} := \mathrm{Nat}(B(I \downarrow -), \mathbb{D}).$$

A natural map $\alpha\colon \mathbb{D} \to \mathbb{D}'$ induces a map $\mathrm{holim}_I\, \alpha := Nat(B(I \downarrow -), \alpha)$. Hence we get a functor $\mathrm{holim}_I\colon \mathrm{Top}^I \to \mathrm{Top}$.

The previous definition is not always practical because it requires the determination of the classifying spaces of the over-categories. We explain an alternative approach from [2]. The main idea is that we can replace $B(I \downarrow -)$ by any diagram $\widetilde{\mathbb{D}_0}$ as soon as it has the two following properties:

(1) for each object i in I, $\widetilde{\mathbb{D}_0}(i)$ is contractible, and
(2) the diagram $\widetilde{\mathbb{D}_0}$ is *free* as an I-diagram of sets in a sense that we explain below.

Let explain what Dror-Farjoun means by a free I-diagram of sets. For an object i_0 of I we consider for each object i of I the set $\hom_I(i_0, i)$ of morphisms in I from i_0 to i. This defines an I-diagram of sets

$$\hom(i_0, -)\colon I \to \mathrm{Set}\,,\ i \mapsto \hom_I(i_0, i)$$

that we call the *free I-diagram of sets generated by the object i_0*. Given a family $F_\gamma\colon I \to \mathrm{Set}$ of I-diagram of sets we define their disjoint union as the I-diagram $\coprod_\gamma F_\gamma\colon I \to \mathrm{Set}$ defined by $\left(\coprod_\gamma F_\gamma\right)(i) = \coprod_\gamma (F_\gamma(i))$. An I-diagram of sets $\mathbb{D}$ is said to be *free* if it is a disjoint union of diagrams each free generated by some object i_γ of I, i.e. $\mathbb{D} \cong \coprod_\gamma \hom_I(i_\gamma, -)$ as I-diagrams of sets. The condition (2) above means that $U(\widetilde{\mathbb{D}_0})$ is a free diagram of sets where $U\colon \mathrm{Top} \to \mathrm{Set}$ is the forgetful functor.

Exercise 8.10. *Compute the free I_{pb}-diagrams of sets generated by u, v, and w respectively. Show that the diagram*

$$\{0\} \xhookrightarrow{i_0} [0,1] \xhookleftarrow{i_1} \{1\}$$

is an I_{pb}-free diagram of sets. Show that on the contrary the constant diagram $\mathbb{D}_0 = \delta_* = (* \to * \leftarrow *)$ is not free.

It turns out that the diagram $B(I \downarrow -)$ of Bousfield-Kan is always free in the sense of Dror-Farjoun (exercise) and objectwise contractible. Therefore the first approach is a special case of the second. Notice that different choices of a free diagram $\widetilde{\mathbb{D}_0}$ objectwise contractible will lead to different spaces $\mathrm{Nat}(\widetilde{\mathbb{D}_0}, -)$, even if they are all homotopy equivalent. The advantage of the Bousfield-Kan definition is that there is no undeterminacy. However the Dror-Farjoun approach simplify sometimes the determination of the homotopy type of the holim.

The diagram $\mathbb{D}_0$ consisting of one-point spaces is terminal in the category of diagrams, hence there exists a unique morphism $\alpha_0 \colon \widetilde{\mathbb{D}_0} \to \mathbb{D}_0$ which induces a natural map

$$\eta\colon \ \mathrm{Nat}(\mathbb{D}_0, -) \to \mathrm{Nat}(\widetilde{\mathbb{D}_0}, -).$$

It turns out that $\operatorname{holim} \mathbb{D} := \mathrm{Nat}(\widetilde{\mathbb{D}_0}, -)$ is homotopy invariant thanks to the freeness of $\widetilde{\mathbb{D}_0}$. This is the content of Proposition 2.9 of [2].

Finally in order to check property (D) we would need to define a class of fibrant diagrams. This notion is certainly more complicated for general shape than it was for pullback diagrams, and the naive generalization of Definition 6.2 is certainly not the right notion. In other words it is **not enough** in general that each map in a diagram $\mathbb{D}$ is a fibration in order to guarantee that $\mathbb{D}$ is a fibrant diagram (but it is a necessary condition). We will not explain the right notion of a fibrant diagram here but a first taste of this notion (in the case of a "very small" category I) can be found in [3] in the setting of Quillen model categories.

9. Colimits and homotopy colimits

There is a notion of a (homotopy) colimit dual to that of a (homotopy) limit. Most of the definitions and constructions for colimits can be obtained by reverting the arrows in those for limits. In particular the dual of Example 8.6 is that $\operatorname{colim}\colon \mathrm{Top}^I \to \mathrm{Top}$ is a *left* adjoint to the functor $\delta\colon \mathrm{Top} \to \mathrm{Top}^I$ of constant diagrams.

However the formulation in terms of space of natural transformations has to be replaced by the following. Consider a diagram $\mathbb{D}\colon I \to \mathrm{Top}$. Let I^{op} be the opposite category of I and let $\mathbb{D}^0\colon I^{\mathrm{op}} \to \mathrm{Top}$ be an I^{op}-diagram of spaces. In other words $\mathbb{D}^0$ can be thought as a *contravariant* functor from I to Top. We define the tensor product of $\mathbb{D}$ and $\mathbb{D}^0$ over I as the space

$$\mathbb{D}^0 \otimes_I \mathbb{D} := \left(\coprod_{i \in ob(I)} \mathbb{D}^0(i) \times \mathbb{D}(i) \right) / \sim$$

where the equivalence relation $\sim$ is generated by

$$(x, \mathbb{D}(\phi)(y)) \sim (\mathbb{D}^0(\phi)(x), y)$$

for each morphism $\phi\colon i \to j$ in I, $x \in \mathbb{D}^0(j)$ and $y \in \mathbb{D}(i)$.

Exercise 9.1. *Show that if* $\mathbb{D}^0\colon I^{\mathrm{op}} \to \mathrm{Top}$ *is the* I^{op}*-diagram consisting of one-point spaces and* $\mathbb{D}\colon I \to \mathrm{Top}$ *is any* I*-diagram of spaces then*

$$\operatorname{colim}_I \mathbb{D} \cong \mathbb{D}^0 \otimes_I \mathbb{D}.$$

(Hint: show that the right hand side has the universal property).

Definition 9.2. Let I be a small category. Fix a diagram $\widetilde{\mathbb{D}^0}\colon I^{\mathrm{op}} \to \mathrm{Top}$ that is a free I^{op}-diagram consisting of contractible spaces. The *homotopy colimit* of a diagram $\mathbb{D}\colon I \to \mathrm{Top}$ is defined as

$$\mathrm{hocolim}_I\, \mathbb{D} := \widetilde{\mathbb{D}^0} \otimes_I \mathbb{D}.$$

Remark: The previous definition involves the choice of a diagram $\widetilde{\mathbb{D}^0}$. To get rid off this indeterminacy one can take $\widetilde{\mathbb{D}^0} := B(I^{\mathrm{op}} \downarrow -)$.

Example 9.3. Consider the category $I_{po} = \{u^* \leftarrow v^* \rightarrow w^*\}$ consisting of three objects u^*, v^*, and w^* and two non-identity maps $v^* \to u^*$ and $v^* \to w^*$. Thus $I_{po} = (I_{pb})^{\mathrm{op}}$ (of course here "po" stands for push-out and "op" for opposite). We already know that the I_{pb}-diagram

$$\widetilde{\mathbb{D}^0}\colon (I_{po})^{\mathrm{op}} = I_{pb} \to \mathrm{Top} \quad = \quad \{0\} \xrightarrow{i_0} [0,1] \xleftarrow{i_1} \{1\}$$

is a free diagram of contractible spaces. Given an I_{po}-diagram of spaces $\mathbb{D} := (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)$ we have

$$\mathrm{hocolim}\, \mathbb{D} = \widetilde{\mathbb{D}^0} \otimes_{I_{po}} \mathbb{D}.$$

This defines the homotopy pushout of $\mathbb{D}$.

Exercise 9.4. *Check that the homotopy push-outof $\mathbb{D}$ is the double mapping cylinder of f and g*

We also have for homotopy colimits properties (A)-(D) dual to the corresponding properties of homotopy limits.

10. Homotopy (co)limits for pointed and unpointed spaces

We can repeat the above discussions for diagrams of *pointed* spaces, $\mathbb{D}\colon I \to \mathrm{Top}_*$. It is important to notice that the (homotopy) colimits are in general different in the category of pointed spaces than in the category of unpointed spaces. Indeed the coproduct in Top is the disjoint union but the coproduct in Top_* is the wedge. Since the coproduct is used in the definition of $- \otimes_I -$ which in turn is used to construct the (homotopy) colimits this difference also appears for the (homotopy) colimits of more general diagrams of pointed/unpointed spaces.

On the other hand the categorical product is the same in Top and Top_* (the smash product is the adjoint of the pointed mapping space but is not the categorical product in Top_*.) More generally, for any shape I the (homotopy) limits give the same result for a diagram of pointed spaces as if we forget the base point. Notice however that the homotopy limit of a diagram of pointed spaces is never empty although a diagram of unpointed spaces can be empty.

In summary, if $U\colon \mathrm{Top}_* \to \mathrm{Top}$ is the forgetful functor and if $\mathbb{D}\colon I \to \mathrm{Top}_*$ is a diagram of pointed spaces then

$$U(\mathrm{holim}_I\, \mathbb{D}) = \mathrm{holim}_I(U\mathbb{D}) \neq \emptyset$$

but in general

$$U(\mathrm{hocolim}_I\, \mathbb{D}) \not\simeq \mathrm{hocolim}_I(U\mathbb{D}).$$

11. A zoo of homotopy limits and colimits

In this section we give a list of classical examples of (homotopy) (co)limits for different shapes of diagrams.

Let fix the notation we use in this section. We consider a given small category I. We will consider a diagram $\mathbb{D}\colon I \to \mathrm{Top}$. We will denote by $\mathbb{D}_0$ the I-diagram of one-point spaces and by $\widetilde{\mathbb{D}_0}\colon I \to \mathrm{Top}$ some free I-diagrams of contractible spaces, as needed in the determination of the homotopy limit in the Dror-Farjoun approach. The diagram $\widetilde{\mathbb{D}_0}$ that we will take will be the first that come in mind, and will not always be $B(I \downarrow -)$, as it should for the strict definition of the homotopy limit. Similarly $\mathbb{D}^0\colon I^{\mathrm{op}} \to \mathrm{Top}$ is the I^{op}-diagram of one-point spaces and $\widetilde{\mathbb{D}^0}\colon I^{\mathrm{op}} \to \mathrm{Top}$ is a free I^{op}-diagram of contractible spaces.

As an exercise the reader should try to check all these assertions below.

(a) The empty category, $I = \{\quad\}$.

Let $I = \emptyset$ be the empty category with no object. Then
- $\lim_\emptyset \mathbb{D} = \mathrm{holim}_\emptyset \mathbb{D} = *$ is the one-point space which is the final object in Top;
- $\mathrm{colim}_\emptyset \mathbb{D} = \mathrm{hocolim}_\emptyset \mathbb{D} = \emptyset$ is the empty space which is the initial object in Top.

Notice that in the category of pointed set the (ho)colim of the empty diagram is the one-point space which is the initial object in Top_*.

(b) The discrete category, $I = \{\bullet \quad \bullet \quad \cdots \quad \bullet\}$

Let I be a small discrete category, i.e. a category whose only morphisms are identity maps. Then we can take $\widetilde{\mathbb{D}_0} = \mathbb{D}_0$ and $\widetilde{\mathbb{D}^0} = \mathbb{D}^0$. We get
- $\lim \mathbb{D} = \mathrm{holim}\, \mathbb{D} = \prod_{i \in ob(I)} \mathbb{D}(i)$;
- $\mathrm{colim}\, \mathbb{D} = \mathrm{hocolim}\, \mathbb{D} = \coprod_{i \in ob(I)} \mathbb{D}(i)$.

The empty category is of course a particular case of a discrete category. In the case of pointed spaces the (homotopy) colimit is the wedge instead of the disjoint union.

(c) The pullback category, $I = I_{pb} = \{\bullet \to \bullet \leftarrow \bullet\}$

Let $I = I_{pb} = (u \to v \leftarrow w)$ be the category of Example 8.2 shaping pullbacks diagrams. Then we can take $\widetilde{\mathbb{D}_0} := \left(\{0\} \xrightarrow{i_0} [0,1] \xleftarrow{i_1} \{1\} \right)$ and $\widetilde{\mathbb{D}^0} = \mathbb{D}^0 = (* \leftarrow * \to *)$. Notice that $\widetilde{\mathbb{D}^0}$ is not the same as $B(I^{\mathrm{op}} \downarrow -)$ but it is free and contractible, which is good enough to determine the *homotopy type* of the hocolim. For $\mathbb{D} = X \xrightarrow{f} B \xleftarrow{g} Y$ we have
- $\lim \mathbb{D} = X \times_B Y = \{(x,y) \in X \times Y | f(x) = g(y)\}$ is the usual pullback;
- $\mathrm{holim}\, \mathbb{D} = \{(x,\beta,y) \in X \times B^{[0,1]} \times Y | \beta(0) = f(x), \beta(1) = g(y)\}$ is the homotopy pullback;
- $\mathrm{colim}\, \mathbb{D} = B = \mathbb{D}(v)$ because v is a terminal object in I;
- $\mathrm{hocolim}\, \mathbb{D} = B$.

To compute the homotopy colim in the strict sense we should use $\widetilde{\mathbb{D}^0} = B(I^{\mathrm{op}} \downarrow -)$ which is naturally homeomorphic to the $(I_{pb})^{\mathrm{op}}$-diagram $[0,1] \leftarrow \{0\} \to [0,1]$. Thus $\mathrm{hocolim}\, \mathbb{D} = (X \times [0,1] \coprod B \times \{0\} \coprod Y \times [0,1])/\sim$ with $\sim$ generated by $(f(x),0) \sim (x,0)$ and $(g(y),0) \sim (y,0)$ for $x \in X$ and $y \in Y$. That space has the same homotopy type as B.

(d) The pushout category, $I = I_{po} = \{\bullet \leftarrow \bullet \to \bullet\}$

For the pushout category $I = I_{po} = I_{pb}^{\mathrm{op}}$ we have results dual to the case of the pullback diagram. For $\mathbb{D} = X \xleftarrow[f]{} A \xrightarrow[g]{} Y$ we have
- $\mathrm{colim}\, \mathbb{D} = (X \coprod Y)/\sim$ where $\sim$ is generated by $f(a) \sim g(a)$, for $a \in A$;

- $\operatorname{hocolim}\mathbb{D} = (X \coprod A \times [0,1] \coprod Y)/\sim$ where $\sim$ is generated by $f(a) \sim (a,0)$ and $g(a) \sim (a,1)$, for $a \in A$;
- $\lim \mathbb{D} = \operatorname{holim}\mathbb{D} = A$.

(e) The category associated to a group, $I = \{\circlearrowleft \bullet \circlearrowright\}$
Let G be a discrete group. We define a category $I = G[e]$ with a single object e and whose set of morphisms is $\hom_I(e,e) = G$ with the composition corresponding to the multiplication in G.

Notice first that a diagram $\mathbb{D}\colon G[e] \to \mathrm{Top}$ is the same as a left G-space, that is a topological space X equipped with a continuous left action of G. Indeed for a given such diagram $\mathbb{D}$ we can take $X = \mathbb{D}(e)$ and the action $x \mapsto g\cdot x$ is induced by the map $\mathbb{D}(e \xrightarrow{g} e)$. Conversely to a space X equipped with a continuous left action of G one can associates a diagram $\mathbb{D}$ with $\mathbb{D}(e) = X$ and $\mathbb{D}(e \xrightarrow{g} e)$ being the map $x \mapsto g\cdot x$. Similarly a $(G[e])^{\mathrm{op}}$-diagram is the same as a space equipped with a right action of G (which can be understood as a left action by the inverse elements.)

The diagram $\mathbb{D}_0$ corresponds to the one-point space with the trivial G-action. For any discrete group G there exists a contractible space EG on which G acts freely. This has been constructed by Milnor as the infinite join $EG := *_{i=1}^{\infty} G$ with the diagonal action but other models for EG are available. For example, for $G = \mathbb{Z}$ one can take $EG = \mathbb{R}$ with the action by translation. For $G = \mathbb{Z}/2$ one can take the infinite-dimensional sphere $EG = S^\infty = \cup_{n=1}^{\infty} S^n$ with the antipodal action. We take for $\widetilde{\mathbb{D}_0}$ the $G[e]$-diagram corresponding to this contractible free left G-space EG. For $\widetilde{\mathbb{D}^0}$ we take the diagram associated to the contractible free right G-space.

Let X be the G-space associated to some $G[e]$-diagram $\mathbb{D}$. Then
- $\operatorname{colim} D = X/G$ is the orbit space;
- $\operatorname{hocolim} D = EG \times_G X$ is the homotopy orbit space (or Borel construction) also denoted by X_{hG};
- $\lim D = X^G$ is the space of fixed points;
- $\operatorname{holim} D = \operatorname{Map}_G(EG, X)$ is the space of G-equivariant maps from EG to X also called homotopy fixed-point space and denoted by X^{hG}.

Exercise 11.1. *Let $G = \mathbb{Z}$ and let $*$ be the one-point space with the trivial $\mathbb{Z}$-action. Let $\mathbb{D}\colon \mathbb{Z}[e] \to \mathrm{Top}$ be the associated $\mathbb{Z}[e]$-diagram. Show that $\lim \mathbb{D} = *$ and $\operatorname{holim}\mathbb{D} = S^1$.*

(f) The telescope category, $I = \{\bullet \to \bullet \to \bullet \to \cdots\}$
Let I be the category whose objects are natural integers and where $\hom_I(m,n)$ is a singleton if $m \le n$ and is empty otherwise[6].

The set of all morphisms in $\mathbb{D}$ is generated by the successor morphisms $n \to (n+1)$, for $n \ge 0$, that is every morphism in $\mathbb{D}$ is a composite of these successors morphisms. Therefore a diagram $\mathbb{D}\colon I \to \mathrm{Top}$ is characterized by the sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ with $X_n = \mathbb{D}(n)$ and by the map $f_n = \mathbb{D}(n \to n+1)\colon X_n \to X_{n+1}$. Moreover there are no relations between the successors morphisms in $\mathbb{D}$, that is every morphism can be written in a unique way as a composite of the successors. This implies that conversely any sequence of space $\{X_n\}_{n\ge 0}$ and of maps $f_n\colon X_n \to X_{n+1}$ corresponds to a telescope diagram $\mathbb{D}\colon I \to \mathrm{Top}$.

[6]A small category whose sets of morphisms are either empty or a singleton is called a poset, i.e. partially orderly set, because the category is completely characterized by the order on the set of objects given by $x \le y$ if and only if $\hom(x,y) \ne \emptyset$.

We define a diagram $\widetilde{\mathbb{D}_0}\colon I^{\rm op}\to \mathrm{Top}$ by $\widetilde{\mathbb{D}_0}(n)=[n,+\infty[$ and for a morphism $\phi\colon n\to m$ in $I^{\rm op}$ (hence $n\geq m$) we set $\widetilde{\mathbb{D}_0}(n)$ to be the inclusion map. This is a free $I^{\rm op}$-diagram. (Check that the one-point spaces diagram $\mathbb{D}^0$ is not free!)

Then

$$\operatorname{hocolim}\mathbb{D}=\left(\coprod_{n=0}^{\infty}X_n\times[n,+\infty[\right)/\sim$$

with $\sim$ generated by $(x,t)\sim(f_{n,m}(x),t)$ for $x\in X_n$, $t\geq m\geq n\geq 0$ and $f_{n,m}=\mathbb{D}(n\to m)$. It can be simplified by noting that there is a homeomorphism

$$\operatorname{hocolim}\mathbb{D}\cong\left(\coprod_{n=0}^{\infty}X_n\times[n,n+1]\right)/\sim$$

with $\sim$ generated by $(x,n+1)\sim(f_n(x),n+1)$ for $x\in X_n$ and $n\geq 0$. This construction is called the telescope.

(g) Inverse limit of a tower, $I=\{\cdots\to\bullet\to\bullet\to\bullet\}$

We consider the opposite category of the category used for the telescope. Namely I is the poset of natural integers with the reversed order. In other words $\hom_I(m,n)$ is a singleton when $m\geq n$ and empty otherwise. An I-diagram $\mathbb{D}\colon I\to\mathrm{Top}$ is characterized by the sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ with $X_n=\mathbb{D}(n)$ and by the map $f_n=\mathbb{D}((n+1)\to n)\colon X_{n+1}\to X_n$. The homotopy limit of that diagram is called the homotopy inverse limit of the tower $\{X_n\}$. We have

- $\lim\mathbb{D}=\{(x_n)_{n\geq 0}\in\prod_{n=0}^{\infty}X_n : f_n(x_{n+1})=x_n, n\geq 0\}$
- $\operatorname{holim}\mathbb{D}\cong\{(\beta_n\colon[0,1]\to X_n)_{n\geq 0}\in\prod_{n=0}^{\infty}X_n : f_n(\beta_{n+1}(0))=\beta_n(1), n\geq 0\}$

When each of the map f_n is a fibration, the homotopy limit is homotopy equivalent to the genuine limit.

(h) Cosimplicial spaces

The simplicial category $I=\Delta$ is defined as follows. The objects of Δ are the finite ordered sets $[n]:=\{0,1,\ldots,n\}$, $n\geq 0$. The morphisms of Δ are the non-decreasing functions between two such finite ordered sets. A *cosimplicial space* is a diagram $\mathbb{D}\colon\Delta\to\mathrm{Top}$.

For $0\leq i\leq n$ and $1\leq j\leq n$ let $d^i\colon[n]\to[n+1]$ be the only non decreasing injective function such that $i\notin\mathrm{im}(d^i)$ and let $s^j\colon[n]\to[n-1]$ be the only non decreasing surjective function such that $s^j(j-1)=s^j(j)$. The maps d^i are called *cofaces* and the s^j are called *degeneracies*. It is straightforwrd to check that any morphism in Δ is a composite of cofaces and codegeneracies. Moreover the relations between these maps can be explicitely listed and are called the cosimplicial identities (to be found everywhere in the litterature, eg in [5]).

Therefore a cosimplicial space $\mathbb{D}\colon\Delta\to\mathrm{Top}$ is characterized by the spaces $X^n=\mathbb{D}(n)$, $n\geq 0$, cofaces maps $d^i:=\mathbb{D}(d^i)\colon X^n\to X^{n+1}$ and the codegeneracies maps $s^{\cdot}j=\mathbb{D}(s^j)\colon X^n\to X^{n-1}$ for $n\geq 0$, $0\leq i\leq n$ and $1\leq j\leq n$.

Conversely the data of a sequence of spaces X^n together with maps d^i and s^j satisfying the cosimplicial relations determines a diagram $\mathbb{D}\colon\Delta\to\mathrm{Top}$. Such a cosimplicial space $\{X^n,d^i,s^j\}_{n\geq 0}$ is denoted by $X^\bullet$.

An important example of a cosimplical space is given by the geometric simplices. Let $\underline{\Delta}^n=\{(x_0,\cdots,x_n)\in\mathbb{R}^{n+1}|\sum x_i=1 \text{ and } x_i\geq 0\}$ be the standard geometric n-simplex. The inclusion of faces are maps $d^i\colon\underline{\Delta}^n\to\underline{\Delta}^{n+1}$ (obtained by inserting a 0 at the i-th cordinate) and summing of the $(j-1)$-th and j-th coordinates defines a codegeneracy map $s^j\colon\underline{\Delta}^n\to\underline{\Delta}^{n-1}$. The corresponding cosimplicial space is denoted by $\underline{\Delta}^\bullet$.

The *totalisation* of a cosimplicial space $X^\bullet$ is by definition the space

$$\mathrm{Tot}(X^\bullet) := \mathrm{Nat}_\Delta(\underline{\Delta}^\bullet, X^\bullet).$$

Each space of $\underline{\Delta}^n$ is contractible but it is not true that $\underline{\Delta}^\bullet$ is a free Δ-diagram, therefore there is no reason for $\mathrm{Tot}(X^\bullet)$ to be equal to $\mathrm{holim}_\Delta X^\bullet$.

However one can consider the *restricted cosimplicial category* Δ^{restr} which is the subcategory of Δ having the same objects $[0], [1], ...$ and whose morphisms are the *injective* non-decrasing functions. The morphisms Δ^{restr} are generated by the cofaces d with the only relations $d^j d^i = d^i d^{j-1}$ for $j > i$. Because of the inclusion $j\colon \Delta^{restr} \hookrightarrow \Delta$ any Δ-diagram $X^\bullet$ induces a Δ^{restr}-diagram $j^*(X^\bullet) = X^\bullet \circ j$. By a result of Dwyer and Dror-Farjoun it turns out that

$$\mathrm{holim}_{\Delta^{restr}} j^*(X^\bullet) \simeq \mathrm{holim}_\Delta X^\bullet.$$

Moreover it is elementary to check that $j^*(\underline{\Delta}^\bullet)$ is a free Δ^{restr}-diagram. Therefore $\mathrm{holim}_{\Delta^{restr}} j^*(X^\bullet) = \mathrm{Nat}_{\Delta^{restr}}(j^*(\underline{\Delta}^\bullet), j^*(X^\bullet))$.

Moreover, when the cosimplicial space is *fibrant* in a sense made precise in [1], then it can be shown that

$$\mathrm{Tot}(X^\bullet) \cong \mathrm{holim}_\Delta X^\bullet.$$

(i) Simplicial spaces

Let Δ^{op} be the opposite category of the cosimplicial category. It is called the *simplicial category.* Its objects are the finite ordered sets $[n]$ for $n \geq 0$ and the morphisms are generated by the faces $\partial_i\colon [n] \to [n-1]$, $0 \leq i \leq n$, and the degeneraties $\sigma_j\colon [n] \to [n+1]$, $1 \leq j \leq n$, dual to the cofaces and codegeneracies. This maps satisfy the simplicial identities.

A diagram $\mathbb{D}\colon \Delta^{\mathrm{op}} \to \mathrm{Top}$ is called a simplicial space and is characterized by the sequence of spaces $X_n = \mathbb{D}(n)$, the faces $\partial_i\colon X_n \to X_{n-1}$ and the degeneracies $\sigma_j\colon X_n \to X_{n+1}$. We denote this simplicial space by $X_\bullet$. The *realization* of the simplicial space $X_\bullet$ is defined as

$$|X_\bullet| = \left(\coprod_{n\geq 0} \underline{\Delta}^n \times X_n\right) / \big((t, \partial_i x) \sim (d^i t, x)\,,\, (t, \sigma_j(x)) \sim (s^j(t), x)\big)\,.$$

In other words

$$|X_\bullet| = \underline{\Delta}^\bullet \otimes_{\Delta^{\mathrm{op}}} X_\bullet.$$

Since $\underline{\Delta}^b ullet$ is not a free diagram, $|X_\bullet|$ is not necesseraly homotopy equivalent to $\mathrm{hocolim}_\Delta X_\bullet$. But it turns out to that for many simplicial spaces (called *good* by Segal) we have a homotopy equivalence

$$|X_\bullet| \simeq \mathrm{hocolim}_\Delta X_\bullet.$$

(j) Cubical and subcubical diagram of spaces

We introduce here cubical and subcubical diagrams which are fundamental in calculus of functors (see [4].

Fix a natural integer k and let $\underline{k} := \{1, \cdots, k\}$ be the set on k elements. Let $\mathcal{P}(\underline{k})$ be the set of subsets of $\underline{k}$. This set is ordered by $S \leq T$ if $S \subset T$ for $S, T \subset \underline{k}$. We consider the category associated to this poset, whose objects are subsets of $\underline{k}$ and morphisms are inclusions. We also denote by $\mathcal{P}(\underline{k})$ this category. An $\mathcal{P}(\underline{k})$-diagram is called a k-*cubical* diagram for obvious reasons.

Each subset $S \subset \underline{k}$ can be seen as a discrete space and we set

$$\widetilde{\mathbb{D}_0}(S) := [0,1]^S = \mathrm{Map}(S, [0,1]) \cong [0,1]^{\#S}$$

which defines a $\mathcal{P}(\underline{k})$-diagram which is actually $B(\mathcal{P}(\underline{k}) \downarrow -)$.

Since $\mathcal{P}(\underline{k})$ has the initial object $\emptyset$, we have that $\mathbb{D}(\emptyset) \simeq \mathrm{holim}_{\mathcal{P}(\underline{k})} \mathbb{D}$. We will use the notation

$$\mathrm{holim}_{S \subset \underline{k}} \mathbb{D}(S) := \mathrm{holim}_{\mathcal{P}(\underline{k})} \mathbb{D}.$$

Consider now the subposet $\mathcal{P}_0(\underline{k}) := \mathcal{P}(\underline{k}) \setminus \{\emptyset\}$ obtained by removing the initial object and denote by the same symbol the corresponding category. An $\mathcal{P}_0(\underline{k})$-diagram is called a k-*subcubical* diagram. We have a functor $j\colon \mathcal{P}_0(\underline{k}) \to \mathcal{P}(\underline{k})$, hence any k-cubical diagram $\mathbb{D}$ induces a k-subcubical diagram $j^*(\mathbb{D})$. The $\mathcal{P}_0(\underline{k})$-diagram $\widetilde{\mathbb{D}'_0} := B(\mathcal{P}_0(\underline{k}) \downarrow -)$ can be described as follows. Set $l := k-1$ and let $\underline{\Delta}^l$ be the standard geometric l-simplex with vertices $v_1, \cdots, v_k$ which are linearly independent points in $\mathbb{R}^k$. For $\emptyset \neq S \subset \underline{k}$ set

$$\widetilde{\mathbb{D}'_0}(S) := \langle v_s : s \in S \rangle = \{\sum_{s \in S} t_s v_s : t_s \geq 0, \sum_{s \in S} t_s = 1\} \cong \underline{\Delta}^{\#S-1}.$$

For $S \subset T$, $\widetilde{\mathbb{D}'_0}(S \subset T)$ is the obvious inclusion.

Using this we can describe

$$\mathrm{holim}_{\emptyset \neq S \subset \underline{k}} \mathbb{D}(S) := \mathrm{holim}_{\mathcal{P}_0(\underline{k})} \mathbb{D} = \mathrm{Nat}(\widetilde{\mathbb{D}'_0}, \mathbb{D})$$

where we commit the abuse of notation of writing $\mathbb{D}$ for $j^*(\mathbb{D})$ when it is clear that the underlying category is $\mathcal{P}_0(\underline{k})$. It is instructive to illustrative this homotopy limits for small values $k = 0, 1, 2$.

It is fairly easy to define a natural inclusion (actually objectwise cofibration)

$$\widetilde{\mathbb{D}'_0} \hookrightarrow j^*(\widetilde{\mathbb{D}_0})$$

which induces a fibration

$$\mathrm{holim}_{S \subset \underline{k}} \mathbb{D}(S) \to \mathrm{holim}_{\emptyset \neq S \subset \underline{k}} \mathbb{D}(S).$$

We also have the homotopy equivalence

$$\mathbb{D}(\underline{k}) = \lim_{S \subset \underline{k}} \mathbb{D}(S) \xrightarrow{\simeq} \mathrm{holim}_{S \subset \underline{k}} \mathbb{D}(S).$$

The composition of this two maps gives a map

$$a(\mathbb{D})\colon \mathbb{D}(\underline{k}) \to \mathrm{holim}_{\emptyset \neq S \subset \underline{k}} \mathbb{D}(S).$$

Definition 11.2. A k-cubical diagram $\mathbb{D}$ is called p-cartesian if the map $a(\mathbb{D})$ is p-connected. When the cubical diagram is pointed, the homotopy fibre of the map $a(\mathbb{D})$ over a is called the *total fibre* of the cubical diagram.

Exercise 11.3. *In the case of a cubical diagram of based spaces establish a relation between its "cartesianity" and the connectivity of its total fibre.*

Exercise 11.4. *Show that* $\mathrm{holim}_{\emptyset\neq S\subset \underline{k}}\,\mathbb{D}(S)$ *can be computed by* $k-1$ *iterated homotopy pullbacks.*

(k) Open sets of a manifold

Let M be a smooth manifold and let $\mathcal{O}_M$ be the poset of its open subsets. This define a category, also denoted $\mathcal{O}_M$, whose objects are open sets $U \subset M$ and morphisms are inclusions. For $k \in \mathbb{N} \cup \{\infty\}$ we define the full subcategory $\mathcal{O}_M(k)$ consisting of open subsets diffeomorphic to a finite union of at most k open disks.

For any full subcategory $\mathcal{U} \subset \mathcal{O}_M$ we can consider the "identity functor"

$$\iota\colon \mathcal{U} \to \mathrm{Top}, U \mapsto U.$$

Then we get that $\mathrm{colim}_{U\in\mathcal{U}}\, U := \mathrm{colim}_{\mathcal{U}}\, \iota \cong \cup_{U\in\mathcal{U}} U$. In particular if $\mathcal{U}$ is a covering of M we get that $\mathrm{colim}_{U\in\mathcal{U}}\, U = M$. Moreover because the objects of $\mathcal{U}$ are open sets one can show that the diagram ι is "cofibrant", therefore $\mathrm{hocolim}_{U\in\mathcal{U}}\, U \simeq \mathrm{colim}_{U\in\mathcal{U}}\, U$.

We can consider various contravariant functors $F\colon \mathcal{U} \to \mathrm{Top}$ like

$$\mathrm{Emb}(-,W)\colon \mathcal{O}_M \to \mathrm{Top}, U \mapsto \mathrm{Emb}(U,W)$$

or

$$\mathrm{Imm}(-,W)\colon \mathcal{O}_M \to \mathrm{Top}, U \mapsto \mathrm{Imm}(U,W)$$

where W is a fixed manifold. Since $\mathcal{O}_M$ has M as a final object we have that $\mathrm{holim}_{U\in\mathcal{O}_M} F(U) \simeq \lim_{U\in\mathcal{O}_M} F(U) = F(M)$. An important question is wether this is still true when we restrict the functor F to a subcategory $\mathcal{U} \subset \mathcal{O}_M$ that is still a covering of M, for example when $\mathcal{U} = \mathcal{O}_M(k)$ for $k \geq 1$. The fundamental result of immersion theory is that it is still true for $\mathrm{Imm}(-,W)$ if $\dim M < \dim W$, and embedding calculus asserts that it is true when $\dim M < \dim W - 2$ and $k = \infty$ (and true "in a range" for finite k).

12. More exercises

Exercise 12.1. *Let* I *be a small category and let* $\mathbb{D}_0\colon I \to \mathrm{Top}$ *be the* I*-diagram of one-point spaces. Show that* $\mathrm{holim}_I * := \mathrm{holim}\,\mathbb{D}_0 = *$ *and that* $\mathrm{hocolim}_I * := \mathrm{hocolim}\,\mathbb{D}_0 = BI$.

Exercise 12.2. *Let* I *be the category with four objects* x_1, x_2, y_1, y_2 *and four non-identity morphisms* $x_i \to y_j$, *for* $1 \leq i, j \leq 2$. *Picture this category. Compute* $\mathrm{hocolim}_I *$.

Exercise 12.3. *Let* X *be a simplicial complex. Show that there exists a category* I *such that* $\mathrm{hocolim}_I \simeq X$.

Exercise 12.4. *By definition the* homotopy fibre *of a map* $f\colon X \to Y$ *over a point* $y_0 \in Y$ *is*

$$\mathrm{hofibre}_{y_0}(f) := \mathrm{holim}\left(\{y_0\} \hookrightarrow Y \xleftarrow{f} X\right)$$

. Show that if y_0 *and* y_1 *belongs to the same path component then* $\mathrm{hofibre}_{y_0}(f) \simeq \mathrm{hofibre}_{y_1}(f)$.
The fibre *of the map* f *over the point* y_0 *is the subspace* $f^{-1}(y_0)$. *Show that if* f *is a fibration then the fibre is homotopy equivalent to the homotopy fibre.*
Show that the fibre is absolutely not homotopy invariant: given a map $f\colon X \to Y$ *and any space* Z *one can find a map* $f'\colon X' \to Y'$ *and a homotopy equivalence of maps* $\alpha\colon f \xrightarrow{\simeq} f'$ *such that the fibre of* f' *over some point is* Z.

Exercise 12.5. *Let X be a space and $x_0 \in X$. Show that*
$$\mathrm{hocolim}\,(\{x_0\} \to X \leftarrow \{x_0\}) = \Sigma X.$$

Exercise 12.6. *Show that*
$$\mathrm{Map}(\mathrm{hocolim}_I\,\mathbb{D}, Y) \simeq \mathrm{holim}_I(\mathrm{Map}(\mathbb{D}, Y))$$
and that
$$\mathrm{Map}(X, \mathrm{holim}_I\,\mathbb{D}) \simeq \mathrm{holim}_I(\mathrm{Map}(X, \mathbb{D})).$$

Exercise 12.7. *Let $p\colon E \to B$ be a locally trivial bundle of fibre F with base B a simplicial complex. Constructs a natural category I and a diagram $\mathbb{D}\colon I \to \mathrm{Top}$ such that $\mathrm{hocolim}\,\mathbb{D} = E$ and $\mathrm{holim}\,\mathbb{D} = \Gamma(p)$ which is the space of sections of p.*

Exercise 12.8. *Let $j\colon J \to I$ be a functor between small categories. Construct a natural transformation $j_\#\colon (J \downarrow -) \to j^*(I \downarrow -)$. Deduce a natural map*
$$\mathrm{holim}_J\, j^*(\mathbb{D}) \to \mathrm{holim}_I\,\mathbb{D}.$$

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