An embedding theorem for regular Mal'tsev categories

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Abstract

In this paper, we obtain a non-abelian analogue of Lubkin's embedding theorem for abelian categories. Our theorem faithfully embeds any small regular Mal'tsev category $\mathbb{C}$ in an n-th power of a particular locally finitely presentable regular Mal'tsev category. The embedding preserves and reflects finite limits, isomorphisms and regular epimorphisms, as in the case of Barr's embedding theorem for regular categories. Furthermore, we show that we can take n to be the (cardinal) number of subobjects of the terminal object in $\mathbb{C}$.

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1 Introduction

One of the most famous results in category theory might be the Yoneda lemma and as a corollary, the Yoneda embedding. As we know, for every small category $\mathbb{C}$, it constructs a fully faithful embedding $\mathbb{C} \hookrightarrow \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ which preserves limits. Therefore, for some type of statements about limits, one only needs to produce the proof in Set to get the result in any category. One can thus say that the category Set of sets 'represents' all categories.

Moreover, Set also represents regular categories (i.e., finitely complete categories with coequalisers of kernel pairs and pullback stable regular epimorphisms [2]). Indeed, Barr's embedding theorem [3] enables us to restrict to Set the proof of some statements about finite limits and regular epimorphisms in regular categories. Note that the key ingredients of regularity are precisely about finite limits and regular epimorphisms. In the same way, a wide range of statements about finite limits and finite colimits in abelian categories can be restricted to Ab, the category of abelian groups [17, 11, 20].

Whereas abelian categories do not cover important algebraic examples such as the categories of groups or rings, the notion of a regular category is in some sense too general because every (quasi)-variety of universal algebras is a regular category. Therefore, one needs some intermediate classes of categories to study the categorical properties of groups. To achieve this, regular Mal'tsev categories have been introduced in [8] as regular categories in which the composition of equivalence relations is commutative. This is equivalent to the property that every reflexive relation is an equivalence [8]. This condition is equivalent in the more general context of finitely complete categories to the condition that each relation is difunctional; and this is the property that defines Mal'tsev categories in this context [9]. Their name comes from the mathematician Mal'tsev who characterised [19] (one-sorted finitary) algebraic categories in which this property holds as the ones whose corresponding theory admits a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x = p(y, y, x)$. These characterisations are recalled in Section 2.

The aim of this paper is to prove an embedding theorem for regular Mal'tsev categories. The 'representing category' will be the category of models of a finitary essentially algebraic theory (i.e.,

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a locally finitely presentable category [12, 1]). As in the algebraic case, objects in such categories are given by an S-sorted set A (i.e., an object in Set^S, for a set S of sorts) endowed with finitary operations $A_{s_1} \times \cdots \times A_{s_n} \to A_s$ satisfying some given equations. The difference is that some of these operations can be only partially defined (and defined exactly for those n-tuples satisfying some given equations involving totally defined operations). We recall this concept in Section 3. We also characterise there those categories of models which are regular and those that are Mal'tsev (via a ternary Mal'tsev term as in the varietal case, see Theorem 3.4). We then construct the 'representing' regular Mal'tsev category $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ using those characterisations.

Section 4 is devoted to the proof of our embedding Theorem 4.4: every small regular Mal'tsev category $\mathbb{C}$ admits a faithful embedding into $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. Here, $\mathrm{Sub}(1)$ denotes the set of subobjects of the terminal object 1 in $\mathbb{C}$. This proof uses three main ingredients: approximate Mal'tsev co-operations (introduced in [7] and recalled in Section 2), the Yoneda embedding $\mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$ (which lifts the property of being regular Mal'tsev from $\mathbb{C}$ to $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$, see Proposition 4.3) and a $\mathbb{C}$-projective covering of $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$ [3, 14]. We can notice that, with this technique, we could also have embedded each small regular Mal'tsev category in a power of the category of approximate Mal'tsev algebras. These are pairs of sets A, B together with two operations $p\colon A^3 \to B$, $a\colon A \to B$ satisfying the axioms $p(x, y, y) = a(x) = p(y, y, x)$. However, this category is not a Mal'tsev category and therefore we have had to refine this argument considering the essentially algebraic category $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$, which is a regular Mal'tsev category.

Due to this embedding theorem, one can reduce the proof of some propositions about finite limits and regular epimorphisms in a regular Mal'tsev category to the particular case of $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$. One is then allowed to use elements and (approximate) Mal'tsev operations to prove some statements in a regular Mal'tsev context. An example of such an application is given in Section 5. Note that our embedding is not full, but it reflects isomorphisms, which is enough for such applications. Indeed, fullness of an embedding $\mathbb{C} \hookrightarrow \mathbb{M}^{\mathbb{P}}$ is not helpful when we look at the components $\mathrm{ev}_P\colon \mathbb{M}^{\mathbb{P}} \to \mathbb{M}$ (which is what we do when we reduce a proof to $\mathbb{M}$).

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2 Regular Mal'tsev categories

A *regular category* is a finitely complete category with coequalisers of kernel pairs and pullback stable regular epimorphisms [2]. They admits a (regular epi, mono)-factorisation system (i.e., a factorisation system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of all regular epimorphisms and $\mathcal{M}$ is the class of all monomorphisms). Moreover, in such categories, two relations $R \rightarrowtail X \times Y$ and $S \rightarrowtail Y \times Z$ can be composed to form a relation $S \circ R \rightarrowtail X \times Z$. This gives rise to the so called 'calculus of relations'.

In this context of regular categories, Mal'tsev categories were introduced in [8]. Two years later, in [9], the authors enlarged this notion of a Mal'tsev category in the context of finitely complete categories.

Definition 2.1. [9] A finitely complete category $\mathbb{C}$ is a Mal'tsev category if every reflexive relation $r\colon R \rightarrowtail X \times X$ is an equivalence relation.

In order to recall some well-known characterisations of Mal'tsev categories, we first need to recall what are difunctional relations.

Definition 2.2. [9] A difunctional relation in a finitely complete category $\mathbb{C}$ is an internal relation $r = (r_1, r_2)\colon R \rightarrowtail X \times Y$ such that, when we consider the following diagram where both squares are pullbacks and tw the twisting isomorphism,

$$\begin{array}{ccccccc}
S & \longrightarrow & & & R & \longleftarrow & T \\
\downarrow & & & & \downarrow{\scriptstyle r} & & \downarrow \\
R \times R & \xrightarrow[r_2 \times r_1]{} & Y \times X & \xrightarrow[\text{tw}]{} & X \times Y & \xleftarrow[r_1 \times r_2]{} & R \times R
\end{array}$$

the canonical monomorphism $S \cap T \rightarrowtail T$ is an isomorphism.

In the category Set of sets (or any algebraic category), a relation $R \subseteq X \times Y$ is difunctional if it satisfies

$$(xRy \wedge xRy' \wedge x'Ry') \Rightarrow x'Ry$$

for all $x, x' \in X$ and $y, y' \in Y$.

Theorem 2.3. [9] Let $\mathbb{C}$ be a finitely complete category. The following statements are equivalent.

1. $\mathbb{C}$ is a Mal'tsev category.
2. Any reflexive relation in $\mathbb{C}$ is symmetric.
3. Any reflexive relation in $\mathbb{C}$ is transitive.
4. Any relation $r\colon R \rightarrowtail X \times Y$ is difunctional.
5. Any relation $r\colon R \rightarrowtail X \times X$ is difunctional.

In a regular context, we have even more characterisations.

Theorem 2.4. [8] Let $\mathbb{C}$ be a regular category. The following statements are equivalent.

1. $\mathbb{C}$ is a Mal'tsev category.
2. The composite of two equivalence relations on the same object is an equivalence relation.
3. If R and S are equivalence relations on the same object, then $R \circ S = S \circ R$.
4. For every reflexive graph,

$$G \underset{c}{\overset{d}{\rightrightarrows}} X, \qquad s\colon X \to G$$

the pullback of (d, c) along (c, d) is a regular epimorphism.

$$\begin{array}{ccc}
P & \longrightarrow & G \\
\downarrow & & \downarrow{\scriptstyle (d,c)} \\
G & \xrightarrow[(c,d)]{} & X \times X
\end{array}$$

The name of Mal'tsev categories comes from the following result of A.I. Mal'tsev.

Theorem 2.5. [19] Let $\mathcal{T}$ be a finitary one-sorted algebraic theory. Then, the category of $\mathcal{T}$-algebras $\mathrm{Alg}_{\mathcal{T}}$ is a Mal'tsev category if and only if $\mathcal{T}$ contains a ternary term $p(x, y, z)$ satisfying the identities

$$p(x, y, y) = x = p(y, y, x).$$

Finally, we will need one more characterisation of Mal'tsev categories in a regular context. In [7], the authors define an *approximate Mal'tsev co-operation on* X (for an object X in a finitely complete category with binary coproducts) as a morphism $p\colon Y \to 3X$ together with an *approximation* $a\colon Y \to X$ such that the square

$$\begin{array}{ccc}
Y & \xrightarrow{p} & 3X \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{pmatrix}} \\
X & \xrightarrow[(\iota_1,\iota_2)]{} & (2X)^2
\end{array}$$

commutes. For each object X, one can build the *universal approximate Mal'tsev co-operation* (p^X, a^X) *on* X by considering the following pullback.

$$\begin{array}{ccc} M(X) & \xrightarrow{p^X} & 3X \\ {\scriptstyle a^X}\downarrow & \lrcorner & \downarrow{\scriptstyle \begin{pmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{pmatrix}} \\ X & \xrightarrow[(\iota_1,\iota_2)]{} & (2X)^2 \end{array}$$

Theorem 2.6 (Theorem 4.2 in [7])**.** Let $\mathbb{C}$ be a regular category with binary coproducts. The following statements are equivalent:

1. $\mathbb{C}$ is a Mal'tsev category.
2. For each $X \in \mathbb{C}$, there is an approximate Mal'tsev co-operation on X for which the approximation a is a regular epimorphism.
3. For each $X \in \mathbb{C}$, the universal approximate Mal'tsev co-operation on X is such that the approximation a^X is a regular epimorphism.

3 Finitary essentially algebraic categories

A *locally finitely presentable category* is a cocomplete category which has a strong set of generators formed by finitely presentable objects. We know from [12] that locally finitely presentable categories are, up to equivalence, exactly the categories of the form $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})$ for a small finitely complete category $\mathbb{C}$ (i.e., the category of finite limit preserving functors from $\mathbb{C}$ to Set). Moreover, they have been further characterised as 'finitary essentially algebraic categories'.

3.1 Essentially algebraic theories and their models

An essentially algebraic category is, roughly speaking, a category of (many-sorted) algebraic structures with partial operations. The domains of definition of these partial operations are themselves defined as the solution sets of some totally defined equations.

More precisely, an *essentially algebraic theory* is a quintuple $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ where S is a set (the set of sorts), Σ is an S-sorted signature of algebras, E is a set of Σ-equations, $\Sigma_t \subseteq \Sigma$ is the subset of 'total operation symbols' and Def is a function assigning to each operation symbol $\sigma\colon \prod_{i\in I} s_i \to s$ in $\Sigma \setminus \Sigma_t$ a set $\mathrm{Def}(\sigma)$ of Σ_t-equations in the variables x_i of sort s_i $(i \in I)$.

A *model* A of an essentially algebraic theory Γ is an S-sorted set $(A_s)_{s\in S} \in \mathrm{Set}^S$ together with, for each operation symbol $\sigma\colon \prod_{i\in I} s_i \to s$ in Σ, a partial function

$$\sigma^A\colon \prod_{i\in I} A_{s_i} \to A_s$$

such that:

- for each $\sigma \in \Sigma_t$, σ^A is defined everywhere,
- for each $\sigma\colon \prod_{i\in I} s_i \to s$ in $\Sigma \setminus \Sigma_t$ and any $(a_i \in A_{s_i})_{i\in I}$, $\sigma^A((a_i)_{i\in I})$ is defined if and only if the elements a_i's satisfy the equations of $\mathrm{Def}(\sigma)$ in A,
- A satisfies the equations of E wherever they are defined.

If A and B are two Γ-models, a *homomorphism* $f\colon A \to B$ *of models* is an S-sorted function $(f_s\colon A_s \to B_s)_{s\in S}$ such that, for each $\sigma\colon \prod_{i\in I} s_i \to s$ in Σ and any $(a_i \in A_{s_i})_{i\in I}$ such that $\sigma^A((a_i)_{i\in I})$ is defined,

$$f_s(\sigma^A((a_i)_{i\in I})) = \sigma^B((f_{s_i}(a_i))_{i\in I}). \tag{1}$$

Notice that if (1) holds for all $\sigma \in \Sigma_t$, then, for each $\sigma' \in \Sigma \setminus \Sigma_t$, $\sigma'^B((f_{s_i}(a_i))_{i\in I})$ is defined if $\sigma'^A((a_i)_{i\in I})$ is, while the converse does not hold in general. The category of Γ-models and their homomorphisms is denoted by Mod(Γ). A category which is equivalent to some model category Mod(Γ) for an essentially algebraic theory Γ is called *essentially algebraic.*

If all arities of Σ are finite, if each equation of E and of all Def(σ)'s uses only a finite number of variables and if all sets Def(σ)'s are also finite, Γ is called a *finitary essentially algebraic theory.* A category which is equivalent to some category Mod(Γ) for a finitary essentially algebraic theory Γ is called a *finitary essentially algebraic category.* As mentioned above, they are exactly the locally finitely presentable categories.

Theorem 3.1. [12, 1] A category is locally finitely presentable if and only if it is a finitary essentially algebraic category.

The basic examples of finitary essentially algebraic categories are finitary (many-sorted) quasi-varieties and so in particular finitary (many-sorted) varieties. The category Cat of small categories is also finitary essentially algebraic.

Let us now focus our attention to the forgetful functor $U\colon \mathrm{Mod}(\Gamma) \to \mathrm{Set}^S$ for a finitary essentially algebraic theory Γ. As expected, we can easily prove that U creates small limits so that they exists in Mod(Γ) and are computed in each sort as in Set. Moreover, U preserves and reflects monomorphisms and isomorphisms. Thus a homomorphism of Γ-models is a monomorphism (resp. an isomorphism) if and only if it is injective (resp. bijective) in each sort. In view of Theorem 3.1, Mod(Γ) is also cocomplete, but colimits are generally harder to describe. In addition, we do not have an easy characterisation of regular epimorphisms and Mod(Γ) is in general not regular.

We are now going to describe a left adjoint for the forgetful functor $U\colon \mathrm{Mod}(\Gamma) \to \mathrm{Set}^S$. In order to do so, we refer the reader to [1] for the definition of terms in Σ in the variables of an S-sorted set X. By abuse of notation, by a (finitary) term $\tau\colon \prod_{i=1}^n s_i \to s$ of Σ, we mean a term of Σ of sort s over the S-sorted set X which contains exactly one formal symbol x_i of sort s_i for each $1 \leqslant i \leqslant n$. Since Mod(Γ) is cocomplete, U has a left adjoint as long as a reflection of those finite S-sorted sets X along U exists. Let us describe it. If $\tau, \tau'\colon \prod_{i=1}^n s_i \to s$ are two terms of Σ, we say that $\tau = \tau'$ is a *theorem* of Γ if $\tau(a_1,\dots,a_n) = \tau'(a_1,\dots,a_n)$ holds in any Γ-model A and for any interpretation of the variables of X in A (i.e., S-sorted function $X \to U(A)$) such that both $\tau(a_1,\dots,a_n)$ and $\tau'(a_1,\dots,a_n)$ are defined. Then, we define the set of *everywhere-defined terms* $\prod_{i=1}^n s_i \to s$ as the smallest subset of the set of terms of Σ in the variables of X such that:

- for each element $x_k \in X_{s_k}$, the k-th projection $\prod_{i=1}^n s_i \to s_k$ is an everywhere-defined term,
- if $(\tau_j\colon \prod_{i=1}^n s_i \to s^j)_{j\in\{1,\dots,m\}}$ are everywhere-defined terms and $\sigma\colon \prod_{j=1}^m s^j \to s$ is an operation symbol of Σ such that, either $\sigma \in \Sigma_t$ or $\sigma \in \Sigma \setminus \Sigma_t$ and, for each equation (μ,μ') of Def(σ), $\mu(\tau_1,\dots,\tau_n) = \mu'(\tau_1,\dots,\tau_n)$ is a theorem of Γ, then $\sigma(\tau_1,\dots,\tau_n)\colon \prod_{i=1}^n s_i \to s$ is everywhere-defined.

Now, $\mathrm{Fr}(X)_s$ is the set of equivalence classes of everywhere-defined terms $\tau\colon \prod_{i=1}^n s_i \to s$ of Σ, where we identify the terms τ and τ' if and only if $\tau = \tau'$ is a theorem of Γ. The operations on Fr(X) and the S-sorted function $X \to U(\mathrm{Fr}(X))$ are defined in the obvious way. The fact that this S-sorted function is the reflection of X along U can be deduced easily from the definitions. We thus have an adjunction $\mathrm{Fr} \dashv U$.

3.2 Regular Mal'tsev finitary essentially algebraic categories

For a small finitely complete category $\mathbb{C}$, Lex($\mathbb{C}$, Set) is a regular category if and only if $\mathbb{C}$ is weakly coregular [10]. Moreover, in [13], the authors describe the categories $\mathbb{C}$ for which Lex($\mathbb{C}$, Set) is a regular Mal'tsev category. On the other hand, the categories of the form Mod(Γ) for a finitary Γ can equivalently be written as Lex($\mathbb{C}$, Set) for some small finitely complete category $\mathbb{C}$. This category $\mathbb{C}$ can be chosen (up to equivalence) as the dual of the full subcategory of finitely presentable objects in Mod(Γ) [12]. However, those objects are hard to describe in general and it is not easy to derive a direct characterisation of those Γ's for which Mod(Γ) is regular or regular Mal'tsev from the previous ones. In this subsection we give a direct characterisation of those finitary essentially algebraic theories whose categories of models are regular, and separately, those whose categories of models are Mal'tsev categories. The more general case of a (non necessarily finitary) essentially algebraic theory is similar and appears in [15].

We start by describing a (strong epi, mono)-factorisation system in $\mathrm{Mod}(\Gamma)$, for an arbitrary finitary essentially algebraic theory Γ (by a (strong epi, mono)-factorisation system we mean a factorisation system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of all strong epimorphisms and $\mathcal{M}$ is the class of all monomorphisms). Let $f\colon A \to B$ be a homomorphism of Γ-models. By abuse of notation, we will often write $f(a)$ instead of $f_s(a)$ for $s \in S$ and $a \in A_s$. We consider the submodel I of B such that, for all $s \in S$,

$$I_s = \{\tau(f(a_1), \dots, f(a_n)) \,|a_i \in A_{s_i} \text{ and } \tau\colon \prod_{i=1}^{n} s_i \to s \text{ is a finitary term of } \Sigma \text{ which}$$
$$\text{is defined in } B \text{ on } (f(a_1), \dots, f(a_n))\}.$$

We can thus view I as the smallest submodel of B for which $f_s(a) \in I_s$ for all $s \in S$ and $a \in A_s$. This means that the corestriction $p\colon A \to I$ of f to I is a strong epimorphism and f factors as $f = ip$ with i the inclusion $I \hookrightarrow B$. As usual, we will refer to $I = \mathrm{Im}(f)$ as the *image of* f.

Before being able to describe those Γ's for which $\mathrm{Mod}(\Gamma)$ is regular, we need the following lemma.

Lemma 3.2. Let Γ be a finitary essentially algebraic theory and $\theta\colon \prod_{i=1}^{n} s_i \to s$ a finitary term of Γ. If $(a_i \in A_{s_i})_{i\in\{1,\dots,n\}}$ are elements of a Γ-model A, we can find a strong epimorphism $q\colon A \twoheadrightarrow B$ in $\mathrm{Mod}(\Gamma)$ such that $\theta(q(a_1), \dots, q(a_n))$ is defined and if $f\colon A \to C$ is a homomorphism such that $\theta(f(a_1), \dots, f(a_n))$ is defined, then f factors uniquely through q.

Proof. We are going to prove this lemma by induction on the number of steps used in the construction of the term θ. If θ is a projection (or any everywhere-defined term), 1_A is the homomorphism we are looking for. Now, suppose θ uses the operation symbols or projections $\sigma_1, \dots, \sigma_m \in \Sigma \cup \{p_k\colon \prod_{i=1}^{n} s_i \to s_k \,|\, 1 \leqslant k \leqslant n\}$ as first step of its construction. Thus, θ can be written as

$$\theta(x_1, \dots, x_n) = \theta'(\sigma_1(x_1, \dots, x_n), \dots, \sigma_m(x_1, \dots, x_n))$$

where θ' uses less steps than θ to be constructed. Let R be the smallest submodel of $A \times A$ which contains $(\chi(a_1, \dots, a_n), \chi'(a_1, \dots, a_n))$ for all equations $(\chi, \chi') \in \mathrm{Def}(\sigma_j)$ and all j such that $\sigma_j \in \Sigma \setminus \Sigma_t$. Let q_1 be the coequaliser of r_1 and r_2

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \overset{q_1}{\twoheadrightarrow} B_1$$

where $r_i = p_i r$ with r the inclusion $R \hookrightarrow A \times A$ and p_1 and p_2 the projections. Thus, in B_1, all $\sigma_j(q_1(a_1), \dots, q_1(a_n))$ are defined. Now, we use the induction hypothesis on θ' to build a universal strong epimorphism $q_2\colon B_1 \twoheadrightarrow B$ such that

$$\begin{aligned}
&\theta'(q_2(\sigma_1(q_1(a_1), \dots, q_1(a_n))), \dots, q_2(\sigma_m(q_1(a_1), \dots, q_1(a_n)))) \\
&= \theta'(\sigma_1(q_2q_1(a_1), \dots, q_2q_1(a_n)), \dots, \sigma_m(q_2q_1(a_1), \dots, q_2q_1(a_n))) \\
&= \theta(q_2q_1(a_1), \dots, q_2q_1(a_n))
\end{aligned}$$

is defined. Let us prove that q_2q_1 is the strong epimorphism we are looking for. Let $f\colon A \to C$ be a homomorphism such that $\theta(f(a_1), \dots, f(a_n))$ is defined. Since the kernel pair $R[f]$ of f contains $(\chi(a_1, \dots, a_n), \chi'(a_1, \dots, a_n))$ for all equations $(\chi, \chi') \in \mathrm{Def}(\sigma_j)$ and all j such that $\sigma_j \in \Sigma \setminus \Sigma_t$, we have $R \subseteq R[f]$ and $fr_1 = fr_2$. Therefore, f factors through q_1 as $f = gq_1$. Finally, g factors through q_2 since

$$\begin{aligned}
&\theta(f(a_1), \dots, f(a_n)) \\
&= \theta(gq_1(a_1), \dots, gq_1(a_n)) \\
&= \theta'(\sigma_1(gq_1(a_1), \dots, gq_1(a_n)), \dots, \sigma_m(gq_1(a_1), \dots, gq_1(a_n))) \\
&= \theta'(g(\sigma_1(q_1(a_1), \dots, q_1(a_n))), \dots, g(\sigma_m(q_1(a_1), \dots, q_1(a_n))))
\end{aligned}$$

is defined. □

We are now able to describe regular finitary essentially algebraic categories.

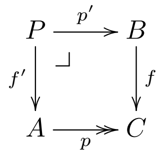
Proposition 3.3. Let Γ be a finitary essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a regular category if and only if, for each finitary term $\theta\colon \prod_{i=1}^n s_i \to s$ of Γ, there exists in Γ:

- a finitary term $\pi\colon \prod_{j=1}^m s'_j \to s$,
- for each $1 \leqslant j \leqslant m$, an everywhere-defined term $\alpha_j\colon s \to s'_j$ and
- for each $1 \leqslant j \leqslant m$, an everywhere-defined term $\mu_j\colon \prod_{i=1}^n s_i \to s'_j$

such that

- $\pi(\alpha_1(x), \dots, \alpha_m(x))$ is everywhere-defined,
- $\pi(\alpha_1(x), \dots, \alpha_m(x)) = x$ is a theorem of Γ,
- $\alpha_j(\theta(x_1, \dots, x_n)) = \mu_j(x_1, \dots, x_n)$ is a theorem of Γ for each $1 \leqslant j \leqslant m$.

Proof. Since $\mathrm{Mod}(\Gamma)$ is complete and has a (strong epi, mono)-factorisation system, it is regular if and only if strong epimorphisms are pullback stable (see e.g. Proposition 2.2.2 in the second volume of [4]). So, let us suppose that the condition in the statement holds in Γ and consider a pullback square



in $\mathrm{Mod}(\Gamma)$ with p a strong epimorphism. We have to prove that $\mathrm{Im}(p') = B$. So, let $b \in B_s$ for some $s \in S$. Since p is a strong epimorphism, there exists a finitary term $\theta\colon \prod_{i=1}^n s_i \to s$ of Σ and elements $a_i \in A_{s_i}$ for each $1 \leqslant i \leqslant n$ such that $\theta(p(a_1), \dots, p(a_n))$ is defined and is equal to $f(b)$. Let the terms π, α_j's and μ_j's be given by the assumption for this θ. For each $j \in \{1, \dots, m\}$,

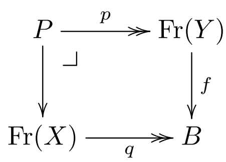
$$\begin{aligned} f(\alpha_j(b)) = \alpha_j(f(b)) &= \alpha_j(\theta(p(a_1), \dots, p(a_n))) \\ &= \mu_j(p(a_1), \dots, p(a_n)) = p(\mu_j(a_1, \dots, a_n)) \end{aligned}$$

since α_j and μ_j are everywhere-defined. But small limits in $\mathrm{Mod}(\Gamma)$ are computed in each sort as in Set. Hence, $d_j = (\mu_j(a_1, \dots, a_n), \alpha_j(b)) \in P_{s'_j}$ with

$$b = \pi(\alpha_1(b), \dots, \alpha_m(b)) = \pi(p'(d_1), \dots, p'(d_m)).$$

Therefore, $b \in \mathrm{Im}(p')_s$ and p' is a strong epimorphism.

Conversely, let us suppose that $\mathrm{Mod}(\Gamma)$ is regular and let $\theta\colon \prod_{i=1}^n s_i \to s$ be a finitary term of Σ. Let X be the S-sorted set which contains exactly, for each $i \in \{1, \dots, n\}$, an element x_i of sort s_i and Y the S-sorted set which contains exactly one element y of sort s. We consider also the strong epimorphism $q\colon \mathrm{Fr}(X) \twoheadrightarrow B$ given by Lemma 3.2, for the term θ and the elements $x_i \in \mathrm{Fr}(X)_{s_i}$. Thus $\theta(q(x_1), \dots, q(x_n))$ is defined. Let $f\colon \mathrm{Fr}(Y) \to B$ be the unique map such that $f(y) = \theta(q(x_1), \dots, q(x_n))$ and consider the pullback of q along f.



Since $\mathrm{Mod}(\Gamma)$ is regular, p is also a strong epimorphism. So, $y \in \mathrm{Im}(p)_s$ which means, using the descriptions of P, $\mathrm{Fr}(X)$ and $\mathrm{Fr}(Y)$, that there exist finitary terms $\pi\colon \prod_{j=1}^m s'_j \to s$, $\alpha_j\colon s \to s'_j$ and $\mu_j\colon \prod_{i=1}^n s_i \to s'_j$ for each $1 \leqslant j \leqslant m$ such that the α_j's and μ_j's are everywhere-defined, the equalities

$$\begin{aligned} y &= \pi(p(\mu_1(x_1, \dots, x_n), \alpha_1(y)), \dots, p(\mu_m(x_1, \dots, x_n), \alpha_m(y))) \\ &= \pi(\alpha_1(y), \dots, \alpha_m(y)) \end{aligned} \qquad (2)$$

hold and, for each $j \in \{1, \dots, m\}$,

$$\begin{aligned} \mu_j(q(x_1), \dots, q(x_n)) &= q(\mu_j(x_1, \dots, x_n)) \\ &= f(\alpha_j(y)) \\ &= \alpha_j(f(y)) \\ &= \alpha_j(\theta(q(x_1), \dots, q(x_n))). \end{aligned} \tag{3}$$

Equalities (2) mean that $\pi(\alpha_1(x), \dots, \alpha_m(x))$ is everywhere-defined and

$$\pi(\alpha_1(x), \dots, \alpha_m(x)) = x$$

is a theorem of Γ. With the universal properties of $\mathrm{Fr}(X)$ and q, equalities (3) mean that $\alpha_j(\theta(a_1, \dots, a_n)) = \mu_j(a_1, \dots, a_n)$ holds in any Γ-model as soon as $\theta(a_1, \dots, a_n)$ is defined. □

We characterise now those Γ's for which $\mathrm{Mod}(\Gamma)$ is a Mal'tsev category. This theorem can be seen as a generalisation of Mal'tsev's Theorem 2.5.

Theorem 3.4. Let Γ be a finitary essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a Mal'tsev category if and only if, for each sort $s \in S$, there exists in Γ a term $p^s \colon s^3 \to s$ such that

- $p^s(x, x, y)$ and $p^s(x, y, y)$ are everywhere-defined and
- $p^s(x, x, y) = y$ and $p^s(x, y, y) = x$ are theorems of Γ.

Proof. Since finite limits in $\mathrm{Mod}(\Gamma)$ are computed in each sort as in Set, a relation $R \rightarrowtail A \times B$ in $\mathrm{Mod}(\Gamma)$ can be seen as a submodel of $A \times B$ and it is difunctional exactly when, for each sort $s \in S$, $R_s \subseteq A_s \times B_s$ is difunctional as a relation in Set.

Suppose first that such terms are given. Let $R \subseteq A \times B$ be a binary relation in $\mathrm{Mod}(\Gamma)$, $s \in S$, $a, a' \in A_s$ and $b, b' \in B_s$ such that $(a, b), (a, b')$ and (a', b') are in R_s. Since $p^s(a, a, a') \in A_s$ and $p^s(b, b', b') \in B_s$ are defined, so is $p^s((a, b), (a, b'), (a', b'))$ in the product $A \times B$. Thus,

$$(a', b) = (p^s(a, a, a'), p^s(b, b', b')) = p^s((a, b), (a, b'), (a', b')) \in R_s$$

and R is difunctional.

Conversely, let us suppose that $\mathrm{Mod}(\Gamma)$ is a Mal'tsev category. Let $s \in S$ be a sort and X the S-sorted set such that $X_s = \{x, y\}$ and $X_{s'} = \varnothing$ for $s' \neq s$. We denote by R the smallest homomorphic binary relation on $\mathrm{Fr}(X)$ which contains $(x, x), (x, y)$ and (y, y). It is easy to prove that this submodel of $\mathrm{Fr}(X) \times \mathrm{Fr}(X)$ is actually given by

$$\begin{aligned} R_{s'} = \{(\tau(x, x, y), \tau(x, y, y)) \,|\, \tau \colon s^3 \to s' &\text{ is a term and} \\ &\tau(x, x, y) \text{ and } \tau(x, y, y) \text{ are everywhere-defined terms}\} \end{aligned}$$

for all $s' \in S$. Since $\mathrm{Mod}(\Gamma)$ is supposed to be a Mal'tsev category, R is difunctional and $(y, x) \in R_s$. This gives the expected term p^s. □

3.3 A finitary essentially algebraic regular Mal'tsev category

In this subsection we are going to construct a finitary essentially algebraic theory Γ_{Mal} for which $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ is a regular Mal'tsev category. This category of models is the one we need for our embedding theorem.

Firstly, if Γ and Γ' are two finitary essentially algebraic theories, we will write $\Gamma \subseteq \Gamma'$ to mean $S \subseteq S'$, $\Sigma \subseteq \Sigma'$, $E \subseteq E'$, $\Sigma_t \subseteq \Sigma'_t$, $\Sigma \setminus \Sigma_t \subseteq \Sigma' \setminus \Sigma'_t$ and $\mathrm{Def}(\sigma) = \mathrm{Def}'(\sigma)$ for all $\sigma \in \Sigma \setminus \Sigma_t$. In this case, we have a forgetful functor $U \colon \mathrm{Mod}(\Gamma') \to \mathrm{Mod}(\Gamma)$.

We are going to construct recursively a series of finitary essentially algebraic theories

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Gamma^n \subseteq \Delta^{n+1} \subseteq \cdots$$

We define Γ^0 as $S^0 = \{\star\}$ and $\Sigma^0 = \Sigma^0_t = E^0 = \varnothing$. Thus $\mathrm{Mod}(\Gamma^0) \cong \mathrm{Set}$. Now, suppose we have defined

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^n \subseteq \Gamma^n$$

with $\Gamma^n = (S^n, \Sigma^n, E^n, \Sigma^n_t, \mathrm{Def}^n)$. We are going to construct

$$\Delta^{n+1} = (S'^{n+1}, \Sigma'^{n+1}, E'^{n+1}, \Sigma'^{n+1}_t, \mathrm{Def}'^{n+1})$$

first (below, $S^{-1} = \varnothing$):

$$S'^{n+1} = S^n \cup \{(s,0),(s,1) \mid s \in S^n \setminus S^{n-1}\} \cong S^n \sqcup (S^n \setminus S^{n-1}) \sqcup (S^n \setminus S^{n-1}),$$

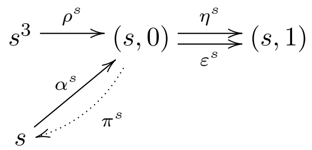
$$\begin{aligned}\Sigma'^{n+1}_t = \Sigma^n_t &\cup \{\alpha^s \colon s \to (s,0) \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\rho^s \colon s^3 \to (s,0) \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\eta^s, \varepsilon^s \colon (s,0) \to (s,1) \mid s \in S^n \setminus S^{n-1}\},\end{aligned}$$

$$\Sigma'^{n+1} = \Sigma^n \cup \Sigma'^{n+1}_t \cup \{\pi^s \colon (s,0) \to s \mid s \in S^n \setminus S^{n-1}\},$$

$$\begin{aligned}E'^{n+1} = E^n &\cup \{\rho^s(x,y,y) = \alpha^s(x) \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\rho^s(x,x,y) = \alpha^s(y) \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\eta^s(\alpha^s(x)) = \varepsilon^s(\alpha^s(x)) \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\pi^s(\alpha^s(x)) = x \mid s \in S^n \setminus S^{n-1}\}\\ &\cup \{\alpha^s(\pi^s(x)) = x \mid s \in S^n \setminus S^{n-1}\},\end{aligned}$$

$$\begin{cases}\mathrm{Def}'^{n+1}(\sigma) = \mathrm{Def}^n(\sigma) \text{ if } \sigma \in \Sigma^n \setminus \Sigma^n_t\\ \mathrm{Def}'^{n+1}(\pi^s) = \{\eta^s(x) = \varepsilon^s(x)\} \text{ for } s \in S^n \setminus S^{n-1}.\end{cases}$$

This gives $\Gamma^n \subseteq \Delta^{n+1}$.



Now let T^{n+1} be the set of finitary terms $\theta\colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} which are not terms of Σ'^n (where we consider $\Sigma'^0 = \varnothing$). We then define Γ^{n+1} as:

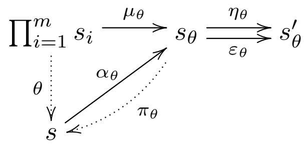
$$S^{n+1} = S'^{n+1} \cup \{s_\theta, s'_\theta \mid \theta \in T^{n+1}\} \cong S'^{n+1} \sqcup T^{n+1} \sqcup T^{n+1},$$

$$\begin{aligned}\Sigma^{n+1}_t = \Sigma'^{n+1}_t &\cup \{\alpha_\theta \colon s \to s_\theta \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}\\ &\cup \{\mu_\theta \colon \prod_{i=1}^m s_i \to s_\theta \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}\\ &\cup \{\eta_\theta, \varepsilon_\theta \colon s_\theta \to s'_\theta \mid \theta \in T^{n+1}\},\end{aligned}$$

$$\Sigma^{n+1} = \Sigma'^{n+1} \cup \Sigma^{n+1}_t \cup \{\pi_\theta \colon s_\theta \to s \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\},$$

$$
\begin{aligned}
E^{n+1} = E'^{n+1} &\cup \{\eta_\theta(\alpha_\theta(x)) = \varepsilon_\theta(\alpha_\theta(x)) \mid \theta \in T^{n+1}\} \\
&\cup \{\pi_\theta(\alpha_\theta(x)) = x \mid \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\pi_\theta(x)) = x \mid \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\theta(x_1, \ldots, x_m)) = \mu_\theta(x_1, \ldots, x_m) \mid \theta \colon \prod_{i=1}^{m} s_i \to s \in T^{n+1}\},
\end{aligned}
$$

$$
\begin{cases}
\mathrm{Def}^{n+1}(\sigma) = \mathrm{Def}'^{n+1}(\sigma) \text{ if } \sigma \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1} \\
\mathrm{Def}^{n+1}(\pi_\theta) = \{\eta_\theta(x) = \varepsilon_\theta(x)\} \text{ for } \theta \in T^{n+1}.
\end{cases}
$$



We have constructed $\Delta^{n+1} \subseteq \Gamma^{n+1}$ and this completes the recursive definition of

$$\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots$$

Let Γ_{Mal} be the union of these finitary essentially algebraic theories. By that we obviously mean $S_{\mathrm{Mal}} = \bigcup_{n\geqslant 0} S^n$, $\Sigma_{\mathrm{Mal}} = \bigcup_{n\geqslant 0} \Sigma^n$, $E_{\mathrm{Mal}} = \bigcup_{n\geqslant 0} E^n$, $\Sigma_{t,\mathrm{Mal}} = \bigcup_{n\geqslant 0} \Sigma_t^n$ and $\mathrm{Def}_{\mathrm{Mal}}(\sigma) = \mathrm{Def}^n(\sigma)$ for all $n \geqslant 0$ and all $\sigma \in \Sigma^n \setminus \Sigma_t^n$. Remark that, for each $\pi \colon s' \to s \in \Sigma_{\mathrm{Mal}} \setminus \Sigma_{t,\mathrm{Mal}}$, there are three corresponding operation symbols in $\Sigma_{t,\mathrm{Mal}}$, these are $\alpha \colon s \to s'$ and $\eta, \varepsilon \colon s' \to s''$.

Proposition 3.5. $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ is a regular Mal'tsev category.

Proof. The 'Δ ingredient' of the construction of Γ_{Mal} ensures that $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ is a Mal'tsev category. Indeed, the terms $\pi^s \circ \rho^s \colon s^3 \to s$ satisfy the conditions of Theorem 3.4.

On the other hand, the 'Γ part' of Γ_{Mal} makes $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ a regular category since each finitary term θ of Σ_{Mal} is in T^{n+1} for some $n \geqslant 0$, which makes the conditions of Proposition 3.3 hold. □

4 The embedding theorem

The aim of this section is to prove that, for each small regular Mal'tsev category $\mathbb{C}$, there exists a faithful embedding $\phi \colon \mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. In order to do this, we still need to recall/prove some other propositions about the embedding $\mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$.

4.1 The embedding $\mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$

Let us now turn our attention to the Yoneda embedding $i \colon \mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}} = \widetilde{\mathbb{C}}$ for a small category $\mathbb{C}$ with finite limits. Due to this embedding, we will treat $\mathbb{C}$ as a full subcategory of $\widetilde{\mathbb{C}}$. Firstly, let us recall the following theorems.

Theorem 4.1. [12] Let $\mathbb{C}$ be a small finitely complete category. The following statements hold.

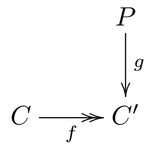
1. $\widetilde{\mathbb{C}}$ is complete and cocomplete.
2. In $\widetilde{\mathbb{C}}$, cofiltered limits commute with limits and finite colimits.
3. The embedding $i \colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ preserves all colimits and finite limits.
4. For all $A \in \widetilde{\mathbb{C}}$, $(A, (c)_{(C,c)\in(A\downarrow i)})$ is the cofiltered limit of the functor

$$
\begin{aligned}
(A \downarrow i) &\longrightarrow \widetilde{\mathbb{C}} \\
c \colon A \to i(C) &\longmapsto i(C).
\end{aligned}
$$

5. $i\colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ is the free cofiltered limit completion of $\mathbb{C}$.

For precise definitions of cofiltered limits and their commutativity with limits and finite colimits, we refer the reader to Sections 2.12 and 2.13 of the first volume of [4].

We recall that $P \in \widetilde{\mathbb{C}}$ is *regular* $\mathbb{C}$*-projective* (abbreviated here by $\mathbb{C}$*-projective*) if, for any diagram



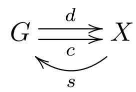
where $C, C' \in \mathbb{C}$ and f is a regular epimorphism, there exists a morphism $h\colon P \to C$ such that $fh = g$. By the Yoneda lemma, if we consider P as a finite limit preserving functor $\mathbb{C} \to \mathrm{Set}$, morphisms $P \to C'$ in $\widetilde{\mathbb{C}}$ are in 1-1 correspondence with elements of $P(C')$. Thus, $P \in \widetilde{\mathbb{C}}$ is $\mathbb{C}$-projective if and only if $P\colon \mathbb{C} \to \mathrm{Set}$ preserves regular epimorphisms.

Theorem 4.2. (Theorems 2.2 and 2.7 in [3]) Let $\mathbb{C}$ be a small regular category. Then $\widetilde{\mathbb{C}}$ is regular and each object $X \in \widetilde{\mathbb{C}}$ admits a $\mathbb{C}$-projective cover, i.e., a regular epimorphism $e_X : \widehat{X} \twoheadrightarrow X$ where $\widehat{X}$ is $\mathbb{C}$-projective.

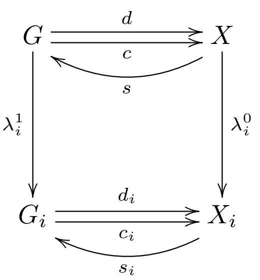
We now prove that the regular Mal'tsev property is also 'preserved' by the embedding $i\colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$.

Proposition 4.3. Let $\mathbb{C}$ be a small regular Mal'tsev category. Then $\widetilde{\mathbb{C}}$ is also a regular Mal'tsev category.

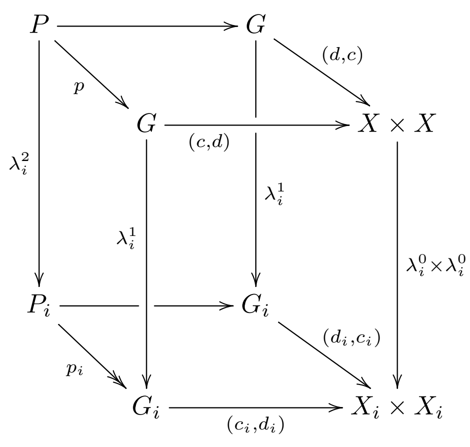
Proof. By Theorem 4.2, we already know that $\widetilde{\mathbb{C}}$ is regular. We are going to prove that $\widetilde{\mathbb{C}}$ is a Mal'tsev category using Theorem 2.4.4. So, let



be a reflexive graph in $\widetilde{\mathbb{C}}$. By Lemma 5.1 in [18], it is a cofiltered limit of reflexive graphs in $\mathbb{C}$.



Since limits commute with limits, the pullback of (d, c) along (c, d) is the limit of the corresponding pullbacks arising from the reflexive graphs in $\mathbb{C}$.



Similarly, the kernel pair of p is the cofiltered limit of the kernel pairs of the p_i's.

$$\begin{array}{ccccc} R & \overset{r}{\underset{s}{\rightrightarrows}} & P & \xrightarrow{p} & G \\ \downarrow{\scriptstyle \lambda_i^3} & & \downarrow{\scriptstyle \lambda_i^2} & & \downarrow{\scriptstyle \lambda_i^1} \\ R_i & \overset{r_i}{\underset{s_i}{\rightrightarrows}} & P_i & \underset{p_i}{\twoheadrightarrow} & G_i \end{array}$$

Since $\mathbb{C}$ is a Mal'tsev category, the p_i's are regular epimorphisms, and so coequalisers of r_i and s_i. By Theorem 4.1, cofiltered limits commute with coequalisers in $\widetilde{\mathbb{C}}$. Thus, p, which is the limit of the coequalisers of the r_i's and s_i's, is the coequaliser of their limits r and s. Therefore p is a regular epimorphism and $\widetilde{\mathbb{C}}$ is a regular Mal'tsev category. □

Note that this preservation property of the embedding $\mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ can be generalised to a wide range of properties, see [15, 16].

4.2 Proof of the embedding theorem

We are now able to prove our main theorem.

Theorem 4.4. Let $\mathbb{C}$ be a small regular Mal'tsev category and Sub(1) the set of subobjects of its terminal object 1. Then, there exists a faithful embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. Moreover, for each morphism $f\colon C \to C'$ in $\mathbb{C}$, each $I \in \mathrm{Sub}(1)$ and each $s \in S_{\mathrm{Mal}}$,

$$(\mathrm{Im}\,\phi(f)_I)_s = \{(\phi(f)_I)_s(x) \mid x \in (\phi(C)_I)_s\}.$$

Proof. We know that $\widetilde{\mathbb{C}}$ is a regular Mal'tsev category (Proposition 4.3). In what follows, we will denote by $\widehat{X}$ the $\mathbb{C}$-projective covering of $X \in \widetilde{\mathbb{C}}$ given by Theorem 4.2. If $C \in \mathbb{C}$ and $P \in \mathrm{Sub}(1)$, we are going to construct $\phi(C)_P \in \mathrm{Mod}(\Gamma_{\mathrm{Mal}})$. More precisely, we are going to construct a Γ_{Mal}-model $\phi(C)_P$ satisfying the following conditions:

1. For each $s \in S_{\mathrm{Mal}}$, $(\phi(C)_P)_s = \widetilde{\mathbb{C}}(P_s, C)$ for some $\mathbb{C}$-projective object $P_s \in \widetilde{\mathbb{C}}$.
2. For each $\pi\colon s' \to s \in \Sigma_{\mathrm{Mal}} \setminus \Sigma_{t,\mathrm{Mal}}$ and its corresponding $\alpha\colon s \to s'$, there is a given regular epimorphism $l_\alpha\colon P_{s'} \twoheadrightarrow P_s$ in $\widetilde{\mathbb{C}}$ such that

$$\begin{aligned} \alpha\colon \widetilde{\mathbb{C}}(P_s, C) &\to \widetilde{\mathbb{C}}(P_{s'}, C) \\ f &\mapsto f l_\alpha \end{aligned}$$

and

$$\begin{aligned} \pi\colon \widetilde{\mathbb{C}}(P_{s'}, C) &\to \widetilde{\mathbb{C}}(P_s, C) \\ g &\mapsto \text{the unique } f \text{ such that } f l_\alpha = g \end{aligned}$$

where π is defined if and only if such an f exists. For the corresponding $\eta, \varepsilon\colon s' \to s''$, we consider the kernel pair (v, w) of l_α.

$$\widehat{R} \overset{e_R}{\twoheadrightarrow} R \overset{v}{\underset{w}{\rightrightarrows}} P_{s'} \overset{l_\alpha}{\twoheadrightarrow} P_s$$

We require then $P_{s''} = \widehat{R}$,

$$\begin{aligned} \eta\colon \widetilde{\mathbb{C}}(P_{s'}, C) &\to \widetilde{\mathbb{C}}(P_{s''}, C) \\ g &\mapsto g v e_R \end{aligned}$$

and

$$\begin{aligned} \varepsilon\colon \widetilde{\mathbb{C}}(P_{s'}, C) &\to \widetilde{\mathbb{C}}(P_{s''}, C) \\ g &\mapsto g w e_R. \end{aligned}$$

3. For each sort $s \in S_{\text{Mal}}$, we consider the universal approximate Mal'tsev co-operation (p^{P_s}, a^{P_s}) on P_s

$$\begin{array}{ccccc} \widehat{M(P_s)} & \xrightarrow{e_{M(P_s)}} & M(P_s) & \xrightarrow{p^{P_s}} & 3P_s \\ & & {\scriptstyle a^{P_s}}\downarrow \quad \lrcorner & & \downarrow {\scriptstyle \left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)} \\ & & P_s & \xrightarrow[(\iota_1,\iota_2)]{} & (2P_s)^2 \end{array}$$

where a^{P_s} is a regular epimorphism by Theorem 2.6. We require then $P_{(s,0)} = \widehat{M(P_s)}$ and

$$\rho^s \colon \widetilde{\mathbb{C}}(P_s, C)^3 \to \widetilde{\mathbb{C}}(P_{(s,0)}, C)$$
$$(f, g, h) \mapsto \begin{pmatrix} f \\ g \\ h \end{pmatrix} p^{P_s} e_{M(P_s)}.$$

4. For each finitary term $\theta \colon \prod_{i=1}^m s_i \to s$ of Σ_{Mal}, there is a given morphism $l_{\mu_\theta} \colon P_{s_\theta} \to P_{s_1} + \cdots + P_{s_m}$ such that

$$\mu_\theta \colon \widetilde{\mathbb{C}}(P_{s_1}, C) \times \cdots \times \widetilde{\mathbb{C}}(P_{s_m}, C) \to \widetilde{\mathbb{C}}(P_{s_\theta}, C)$$
$$(f_1, \dots, f_m) \mapsto \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} l_{\mu_\theta}.$$

Since Γ_{Mal} is the union of the series

$$\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots$$

of essentially algebraic theories, to construct a Γ_{Mal}-model $\phi(C)_P$, it is enough to construct recursively a Γ^n-model for each $n \geqslant 0$ such that they agree on the common sorts and operations. Firstly, to define a Γ^0-model, we set $P_\star$ to be the coproduct of the $\widehat{C'}$'s for all $C' \in \mathbb{C}$ such that the image of the unique morphism $C' \to 1$ is $P \in \text{Sub}(1)$. This object $P_\star$ is $\mathbb{C}$-projective since it is the coproduct of $\mathbb{C}$-projective objects.

Now, we suppose we have defined a Γ^n-model satisfying the above conditions. We are going to extend it to a Γ^{n+1}-model with the same properties. Firstly, we extend it to a Δ^{n+1}-model. Let $s \in S^n \setminus S^{n-1}$. Condition 3 above imposes the constructions of $P_{(s,0)}$ and ρ^s. Moreover, condition 2 with $l_{\alpha^s} = a^{P_s} e_{M(P_s)}$ from condition 3 defines α^s, π^s, $P_{(s,1)}$, η^s and ε^s. It follows then from the definitions that this indeed gives a Δ^{n+1}-model which satisfies conditions 1–4.

It remains to extend it to a Γ^{n+1}-model. In order to simplify the proof, we are going to construct P_{s_θ}, l_{μ_θ} and l_{α_θ} for each finitary term $\theta \colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} such that it matches the previous construction if θ is actually a term of Σ'^n. Then, condition 2 will define α_θ, π_θ, $P_{s'_\theta}$, η_θ and ε_θ, while condition 4 imposes the definition of μ_θ. We are going to do it recursively in such a way that the equality

$$\alpha_\theta(\theta(f_1, \dots, f_m)) = \mu_\theta(f_1, \dots, f_m)$$

holds for any cospan $(f_i \colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$ such that $\theta(f_1, \dots, f_m)$ is defined.

Firstly, let $\theta = p_j \colon \prod_{i=1}^m s_i \to s_j$ be a projection ($1 \leqslant j \leqslant m$). In this case, we define $P_{s_\theta} = P_{s_j}$, $l_{\mu_\theta} = \iota_j \colon P_{s_j} \to P_{s_1} + \cdots + P_{s_m}$ the coproduct injection and $l_{\alpha_\theta} = 1_{P_{s_j}}$. Obviously, one has

$$\alpha_\theta(\theta(f_1, \dots, f_m)) = f_j = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \iota_j = \mu_\theta(f_1, \dots, f_m)$$

for any cospan $(f_i \colon P_{s_i} \to C)_{i \in \{1,\dots,m\}}$.

Secondly, let $\theta \colon \prod_{i=1}^m s_i \to s'$ be a finitary term of Σ'^{n+1} for which l_{μ_θ} and l_{α_θ} have been constructed. If $\pi \colon s' \to s \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1}$ has corresponding $\alpha \colon s \to s'$, we define $P_{s_{\pi(\theta)}} = P_{s_\theta}$, $l_{\alpha_{\pi(\theta)}} = l_\alpha l_{\alpha_\theta}$ and $l_{\mu_{\pi(\theta)}} = l_{\mu_\theta}$.

$$\begin{array}{ccc} P_{s_{\pi(\theta)}} = P_{s_\theta} & \xrightarrow{l_{\mu_{\pi(\theta)}} = l_{\mu_\theta}} & P_{s_1} + \cdots + P_{s_m} \\ {\scriptstyle l_{\alpha_\theta}}\downarrow & \searrow^{l_{\alpha_{\pi(\theta)}}} & \\ P_{s'} & \xrightarrow[l_\alpha]{} & P_s \end{array}$$

If the cospan $(f_i\colon P_{s_i} \to C)_{i\in\{1,\dots,m\}}$ is such that $\theta(f_1,\dots,f_m)\colon P_{s'} \to C$ is defined, we know from the previous step in the recursion that

$$\theta(f_1,\dots,f_m)l_{\alpha_\theta} = \alpha_\theta(\theta(f_1,\dots,f_m)) = \mu_\theta(f_1,\dots,f_m).$$

If moreover $\pi(\theta(f_1,\dots,f_m))\colon P_s \to C$ is defined, we have

$$\pi(\theta(f_1,\dots,f_m))l_\alpha = \theta(f_1,\dots,f_m).$$

In this case,

$$\begin{aligned}\alpha_{\pi(\theta)}(\pi(\theta(f_1,\dots,f_m))) &= \pi(\theta(f_1,\dots,f_m))l_{\alpha_{\pi(\theta)}}\\ &= \pi(\theta(f_1,\dots,f_m))l_\alpha l_{\alpha_\theta}\\ &= \theta(f_1,\dots,f_m)l_{\alpha_\theta}\\ &= \mu_\theta(f_1,\dots,f_m)\\ &= \mu_{\pi(\theta)}(f_1,\dots,f_m).\end{aligned}$$

Eventually, let us suppose that $\sigma\colon \prod_{i=1}^r s'_i \to s \in \Sigma'^{n+1}_t$ is an operation symbol and, for each $1 \leqslant j \leqslant r$, $\theta_j\colon \prod_{i=1}^m s_i \to s'_j$ is a finitary term of Σ'^{n+1} for which $l_{\mu_{\theta_j}}$ and $l_{\alpha_{\theta_j}}$ have been defined. We already have a corresponding morphism $l_\sigma\colon P_s \to P_{s'_1} + \dots + P_{s'_r}$ such that

$$\begin{aligned}\sigma\colon \widetilde{\mathbb{C}}(P_{s'_1},C) \times \dots \times \widetilde{\mathbb{C}}(P_{s'_r},C) &\to \widetilde{\mathbb{C}}(P_s,C)\\ (f_1,\dots,f_r) &\mapsto \begin{pmatrix} f_1\\ \vdots\\ f_r\end{pmatrix} l_\sigma.\end{aligned}$$

Let us consider the following diagram where the square is a pullback.

$$\begin{array}{ccccccc} P_{s_\theta} = \widehat{U} & \xrightarrow{e_U} & U & \xrightarrow{u_1} & P_{s_{\theta_1}} + \dots + P_{s_{\theta_r}} & \xrightarrow{\begin{pmatrix} l_{\mu_{\theta_1}}\\ \vdots\\ l_{\mu_{\theta_r}}\end{pmatrix}} & P_{s_1} + \dots + P_{s_m}\\ & & {\scriptstyle u_2}\downarrow & \lrcorner & \downarrow{\scriptstyle l_{\alpha_{\theta_1}}+\dots+l_{\alpha_{\theta_r}}} & & \\ & & P_s & \xrightarrow[l_\sigma]{} & P_{s'_1} + \dots + P_{s'_r} & & \end{array}$$

Denoting the term $\sigma(\theta_1,\dots,\theta_r)\colon \prod_{i=1}^m s_i \to s$ by θ, we define $P_{s_\theta} = \widehat{U}$, $l_{\alpha_\theta} = u_2 e_U$ and

$$l_{\mu_\theta} = \begin{pmatrix} l_{\mu_{\theta_1}}\\ \vdots\\ l_{\mu_{\theta_r}}\end{pmatrix} u_1 e_U.$$

Then, if the cospan $(f_i\colon P_{s_i} \to C)_{i\in\{1,\dots,m\}}$ is such that $\theta_j(f_1,\dots,f_m)\colon P_{s'_j} \to C$ is defined for all $1 \leqslant j \leqslant r$,

$$\begin{aligned}\alpha_\theta(\theta(f_1,\dots,f_m)) &= \sigma(\theta_1(f_1,\dots,f_m),\dots,\theta_r(f_1,\dots,f_m))l_{\alpha_\theta}\\ &= \begin{pmatrix} \theta_1(f_1,\dots,f_m)\\ \vdots\\ \theta_r(f_1,\dots,f_m)\end{pmatrix} l_\sigma u_2 e_U\\ &= \begin{pmatrix} \theta_1(f_1,\dots,f_m)\\ \vdots\\ \theta_r(f_1,\dots,f_m)\end{pmatrix} (l_{\alpha_{\theta_1}} + \dots + l_{\alpha_{\theta_r}})u_1 e_U\\ &= \begin{pmatrix} \alpha_{\theta_1}(\theta_1(f_1,\dots,f_m))\\ \vdots\\ \alpha_{\theta_r}(\theta_r(f_1,\dots,f_m))\end{pmatrix} u_1 e_U\\ &= \begin{pmatrix} \mu_{\theta_1}(f_1,\dots,f_m)\\ \vdots\\ \mu_{\theta_r}(f_1,\dots,f_m)\end{pmatrix} u_1 e_U\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \begin{pmatrix} l_{\mu_{\theta_1}} \\ \vdots \\ l_{\mu_{\theta_r}} \end{pmatrix} u_1 e_U \\
&= \mu_\theta(f_1, \dots, f_m)
\end{aligned}
$$

using the previous steps in the recursion.

We have thus defined a Γ^{n+1}-model which satisfies conditions 1–4. This concludes the recursive construction of our Γ^n-model for each $n \geqslant 0$. Considering them all together, we get a Γ_{Mal}-model $\phi(C)_P$.

Now, if $f\colon C \to C' \in \mathbb{C}$ and $P \in \mathrm{Sub}(1)$, we define a morphism $\phi(f)_P \colon \phi(C)_P \to \phi(C')_P$ by

$$
\begin{aligned}
(\phi(f)_P)_s \colon \widetilde{\mathbb{C}}(P_s, C) &\to \widetilde{\mathbb{C}}(P_s, C') \\
g &\mapsto fg
\end{aligned}
$$

for all $s \in S_{\mathrm{Mal}}$. By conditions 2–4, $\phi(f)_P$ is a Γ_{Mal}-homomorphism. This defines the expected functor $\phi\colon \mathbb{C} \to \mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$. Let us now check that it satisfies all the required properties.

Since finite limits in $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$ are computed componentwise, to prove that ϕ preserves them, we only need to prove that $\phi(-)_P \colon \mathbb{C} \to \mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ preserves finite limits for each $P \in \mathrm{Sub}(1)$. Furthermore, since they are computed in each sort as in Set, we only need to check that $(\phi(-)_P)_s \colon \mathbb{C} \to \mathrm{Set}$ preserves finite limits for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. But, by the Yoneda lemma, $(\phi(-)_P)_s$ is isomorphic to $P_s \colon \mathbb{C} \to \mathrm{Set}$ which preserves finite limits by definition. Therefore, ϕ preserves them as well.

Now, suppose that $f\colon C \to C' \in \mathbb{C}$ is such that $(\phi(f)_P)_s$ is surjective for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. Let

$$
C' \overset{p}{\twoheadrightarrow} I \rightarrowtail 1
$$

be the image factorisation of the unique morphism $C' \to 1$. We recall that $I_\star = \coprod \widehat{C''}$ where the coproduct runs through all C'' such that the image of $C'' \to 1$ is I. For each such C'', there exists a morphism $g_{C''}$ making the diagram

$$
\begin{array}{ccccc}
\widehat{C''} & \overset{e_{C''}}{\twoheadrightarrow} & C'' & & \\
{\scriptstyle g_{C''}}\downarrow & & \downarrow & & \\
C' & \underset{p}{\twoheadrightarrow} & I & \rightarrowtail & 1
\end{array}
$$

commutative since $\widehat{C''}$ is $\mathbb{C}$-projective. We choose $g_{C'} = e_{C'}$ and consider the induced morphism $g\colon I_\star \twoheadrightarrow C'$ which is a regular epimorphism since $g\iota_{\widehat{C'}} = g_{C'}$ is. But we have supposed that $f \circ - \colon \mathbb{C}(I_\star, C) \to \mathbb{C}(I_\star, C')$ is surjective. So, there is a morphism $h\colon I_\star \to C$ such that $fh = g$, which implies that f is also a regular epimorphism. Moreover, since each P_s is $\mathbb{C}$-projective, this means that f is a regular epimorphism if and only if $(\phi(f)_P)_s$ is surjective for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. In particular, ϕ preserves regular epimorphisms.

Now, let $f, f' \colon C \to C'$ be two morphisms of $\mathbb{C}$ such that $(\phi(f)_P)_s = (\phi(f')_P)_s$ for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. Let $e\colon E \rightarrowtail C$ be their equaliser. Since ϕ preserves equalisers, $(\phi(e)_P)_s$ is a bijection for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. Hence, e is a regular epimorphism and $f = f'$. This shows that ϕ is faithful.

Let $f\colon C \to D \in \mathbb{C}$ be such that $(\phi(f)_P)_s$ is injective for all $P \in \mathrm{Sub}(1)$ and all $s \in S_{\mathrm{Mal}}$. We want to prove that f is a monomorphism. So, suppose $h, k\colon C' \to C \in \mathbb{C}$ are such that $fh = fk$. Let $g\colon I_\star \twoheadrightarrow C'$ be the regular epimorphism defined as above. Thus, $fhg = fkg$. Since $f \circ - \colon \widetilde{\mathbb{C}}(I_\star, C) \to \widetilde{\mathbb{C}}(I_\star, D)$ is injective, we know that $hg = kg$. Hence $h = k$ and f is a monomorphism since g is a regular epimorphism. Therefore, ϕ reflects isomorphisms, finite limits and regular epimorphisms.

It remains to check that, for $f\colon C \to C' \in \mathbb{C}$, $P \in \mathrm{Sub}(1)$ and $s \in S_{\mathrm{Mal}}$,

$$
(\mathrm{Im}\, \phi(f)_P)_s = \{(\phi(f)_P)_s(x) \mid x \in (\phi(C)_P)_s\}.
$$

Consider $\pi\colon s' \to s \in \Sigma_{\mathrm{Mal}} \setminus \Sigma_{t,\mathrm{Mal}}$ and $x \in \widetilde{\mathbb{C}}(P_{s'}, C)$ such that $\pi((\phi(f)_P)_{s'}(x))$ is defined. So, there exists $g\colon P_s \to C'$ making the square

$$\begin{array}{ccc} P_{s'} & \xrightarrow{x} & C \\ {\scriptstyle l_\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ P_s & \xrightarrow[g]{} & C' \end{array}$$

commute (with $\alpha\colon s \to s'$ corresponding to π). Let $f = iq$ be the image factorisation of f. Since l_α is a regular epimorphism, there exists $g'\colon P_s \to \mathrm{Im}(f)$ such that $ig' = g$. Since P_s is $\mathbb{C}$-projective, there exists a morphism $y\colon P_s \to C$ such that $qy = g'$. Thus, $fy = g$ and $(\phi(f)_P)_s(y) = g = \pi((\phi(f)_P)_{s'}(x))$. Therefore, in view of the description of the images in categories of Γ-models on page 6 for any finitary essentially algebraic theory Γ, this concludes the proof. □

5 Applications

Analogously to the metatheorems of [5], our embedding theorem gives a way to prove some statements in regular Mal'tsev categories in an 'essentially algebraic way' as follows.

Consider a statement P of the form $\psi \Rightarrow \omega$ where ψ and ω are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. some morphism is a monomorphism,
4. some morphism is a regular epimorphism,
5. some morphism is an isomorphism,
6. some morphism factors through a given monomorphism.

Then, this statement P is valid in all regular Mal'tsev $\mathcal{V}$-categories (for all universes $\mathcal{V}$) if and only if it is valid in $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ (for all universes). Indeed, in view of Proposition 3.5, the 'only if part' is obvious. Conversely, if $\mathbb{C}$ is a regular Mal'tsev category, we can consider it is small up to a change of universe. Then, by Theorem 4.4, it suffices to prove P in $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})^{\mathrm{Sub}(1)}$. Since every part of the statement P is 'componentwise', it is enough to prove it in $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$.

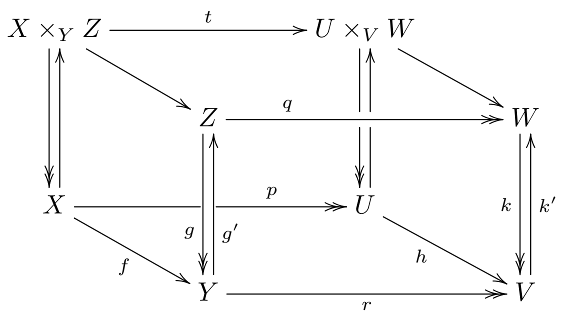
At a first glance, one could think this technique will be hard to use in practice, in view of the difficult definition of $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$. However, due to the additional property in our Theorem 4.4, we can suppose that the homomorphisms $f\colon A \to B$ considered in the statement P have an easy description of their images, i.e.,

$$(\mathrm{Im}\, f)_s = \{f_s(a) \,|\, a \in A_s\}$$

for all $s \in S_{\mathrm{Mal}}$. In particular, if f is a regular epimorphism, f_s will be a surjective function for all $s \in S_{\mathrm{Mal}}$. Therefore, in practice, it seems we will never have to use the operations α_θ, μ_θ, η_θ, ε_θ and π_θ. They were built only to make $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ a regular category.

We illustrate now how to use the embedding theorem to prove a result in a regular Mal'tsev category using an (essentially) algebraic argument. We refer the reader to [5] for a definition of the category $\mathrm{Pt}(\mathbb{C})$ of points of $\mathbb{C}$.

Lemma 5.1. (Proposition 4.1 in [6]) Let $\mathbb{C}$ be a regular Mal'tsev category. Consider a commutative square in the category $\mathrm{Pt}(\mathbb{C})$ of points of $\mathbb{C}$. This yields a cube where the vertical morphisms are split epimorphisms with a given section. Suppose that the left and right faces of this cube are pullbacks and p and q are regular epimorphisms.



Then, the comparison map t is also a regular epimorphism.

Proof. It is enough to prove this lemma in $\mathrm{Mod}(\Gamma_{\mathrm{Mal}})$ supposing that p and q are surjective in each sort. So, let $s \in S_{\mathrm{Mal}}$, $u \in U_s$ and $w \in W_s$ be such that $h(u) = k(w)$ and let us prove $(u, w) \in \mathrm{Im}(t)_s$. Since p and q are surjective, we can find $x \in X_s$ and $z \in Z_s$ such that $p(x) = u$ and $q(z) = w$. Let $z' = \rho^s(g'f(x), g'g(z), z) \in Z_{(s,0)}$. Since $g(z') = \rho^s(f(x), g(z), g(z)) = \alpha^s(f(x)) = f(\alpha^s(x))$, we can consider $(\alpha^s(x), z') \in (X \times_Y Z)_{(s,0)}$. Moreover, since

$$\begin{aligned} q(z') &= \rho^s(qg'f(x), qg'g(z), q(z)) \\ &= \rho^s(k'hp(x), k'kq(z), q(z)) \\ &= \rho^s(k'h(u), k'k(w), w) \\ &= \rho^s(k'k(w), k'k(w), w) \\ &= \alpha^s(w), \end{aligned}$$

we know that $t(\alpha^s(x), z') = (p(\alpha^s(x)), q(z')) = (\alpha^s(u), \alpha^s(w)) = \alpha^s(u, w)$. Therefore, we have $(u, w) = \pi^s(\alpha^s(u, w)) = \pi^s(t(\alpha^s(x), z')) \in \mathrm{Im}(t)_s$. □

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