Bicategories of fractions for groupoids in monadic categories

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Dedicated to Marco Grandis on the occasion of his 70th birthday

Abstract

The bicategory of fractions of the 2-category of internal groupoids and internal functors in groups with respect to weak equivalences (i.e., functors which are internally full, faithful and essentially surjective) has an easy description: one has just to replace internal functors by monoidal functors. In the present paper, we generalize this result from groups to any monadic category over a regular category $\mathcal{C}$, assuming that the axiom of choice holds in $\mathcal{C}$. For $\mathbb{T}$ a monad on $\mathcal{C}$, the bicategory of fractions of $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to weak equivalences is now obtained replacing internal functors by what we call $\mathbb{T}$-monoidal functors. The notion of $\mathbb{T}$-monoidal functor is related to the notion of pseudo-morphism between strict algebras for a pseudo-monad on a 2-category.

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1 Introduction

Let $F\colon \mathbb{A} \to \mathbb{B}$ be an internal functor between small and strict categorical groups (that is, internal groupoids in the category Gp of groups), and consider the induced functor $UF\colon U\mathbb{A} \to U\mathbb{B}$ between the underlying groupoids in Set. If F is a weak equivalence (that is, it is full, faithful, and essentially surjective), then UF also is a weak equivalence and therefore (admitting the axiom of choice in Set) it is an equivalence of groupoids. Moreover, the quasi-inverses $(UF)^*\colon U\mathbb{B} \to U\mathbb{A}$ of UF are monoidal functors, but in general they fail to be internal functors in Gp (precisely because in Gp the axiom of choice does not hold). This simple fact has been formalized by the second author in [14], where it is shown that the inclusion

$$\mathrm{Grpd}(\mathrm{Gp}) \hookrightarrow \mathrm{MON}$$

where MON is the 2-category of internal groupoids in Gp and monoidal functors, is the bicategory of fractions of Grpd(Gp) with respect to weak equivalences. In [14] it is also shown that a similar result holds when the category of groups is replaced by the category Lie_k of Lie algebras over a field k, and monoidal functors are replaced by homomorphisms of strict Lie-2-algebras.

Both categories Gp and Lie_k are monadic over regular categories where the axiom of choice holds (Set and Vect_k, respectively). The idea of the present paper is, therefore, to generalize the results established in [14] to groupoids internal to $\mathcal{C}^{\mathbb{T}}$, where $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$ and the axiom of choice holds in the regular category $\mathcal{C}$. We look for a simple description of the bicategory

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of fractions of $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to weak equivalences. Of course, the problem is to find a convenient notion of '$\mathbb{T}$-monoidal functor' between internal groupoids in $\mathcal{C}^{\mathbb{T}}$ in order to describe the desired bicategory of fractions as the (not full) embedding

$$\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$$

In some sense, such a notion already appears in the literature: $\mathbb{T}$-MON is the 2-category of strict algebras and pseudo-morphisms for a pseudo-monad $\mathcal{T}$ on the 2-category $\mathrm{Grpd}(\mathcal{C})$. However, in general, such a pseudo-monad $\mathcal{T}$ does not exist!

More precisely, if the functor part $T\colon \mathcal{C} \to \mathcal{C}$ of the monad $\mathbb{T}$ preserves pullbacks, then $\mathbb{T}$ induces a pseudo-monad $\mathcal{T}$ on $\mathrm{Grpd}(\mathcal{C})$ whose 2-category $\mathrm{Alg}(\mathcal{T})$ of strict algebras and strict morphisms is isomorphic to $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$. Moreover, as for any pseudo-monad, we can consider also the 2-category $\text{Ps-Alg}(\mathcal{T})$ of pseudo-algebras and pseudo-morphisms and, as an intermediate situation, the 2-category $\mathcal{T}$-MON of pseudo-morphisms between strict algebras

$$\mathrm{Alg}(\mathcal{T}) \hookrightarrow \mathcal{T}\text{-MON} \hookrightarrow \text{Ps-Alg}(\mathcal{T})$$

Now, when $T\colon \mathcal{C} \to \mathcal{C}$ does not preserve pullbacks, the pseudo-monad $\mathcal{T}$ on $\mathrm{Grpd}(\mathcal{C})$ does not exist because T destroys the internal composition of an internal groupoid $\mathbb{A}$, so that $\mathcal{T}(\mathbb{A})$ is a reflexive graph but not an internal groupoid. Nevertheless, we can still define pseudo-morphisms since, for doing that, only 2-cells of the form

$$\mathcal{T}(\mathbb{A}) \underset{\alpha\Downarrow}{\rightrightarrows} \mathbb{B}$$

are needed, and to express the naturality of α one uses the internal composition in $\mathbb{B}$, not in $\mathcal{T}(\mathbb{A})$. With this idea in mind, we can define the 2-category $\mathbb{T}$-MON for every monad $\mathbb{T}$ on $\mathcal{C}$ and, assuming the axiom of choice in the regular category $\mathcal{C}$, we can prove that the embedding

$$\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$$

is the bicategory of fractions of $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to weak equivalences.

The layout of this paper is as follows. In Section 2 we recall from [6] the notion of weak equivalence between internal groupoids. In Section 3 we introduce the 2-category $\mathbb{T}$-MON constructed from any monad $\mathbb{T}$ on a category $\mathcal{C}$ with pullbacks. Section 4 is devoted to the proof that, under the mentioned conditions, $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\text{-MON}$ is the bicategory of fractions with respect to weak equivalences. The case of groups and the case of Lie-algebras are briefly discussed in Sections 5 and 6. Observe that, since the categories of groups and of Lie algebras are semi-abelian, the bicategories of fractions of $\mathrm{Grpd}(\mathrm{Gp})$ and of $\mathrm{Grpd}(\mathrm{Lie}_k)$ with respect to weak equivalences can also be described using 'butterflies', see [11] and [1]. Finally, in Section 7 we recall the notions of pseudo-monad and pseudo-morphisms between (strict or pseudo-) algebras which form the background to understand the definitions given in Section 3. We also show that, under suitable conditions on a 2-category $\mathcal{B}$ and on a pseudo-monad $\mathcal{T}$ on $\mathcal{B}$, the not full inclusion $\mathrm{Alg}(\mathcal{T}) \hookrightarrow \mathcal{T}\text{-MON}$ is the bicategory of fractions with respect to those arrows in $\mathrm{Alg}(\mathcal{T})$ which are equivalences in $\mathcal{B}$. The needed conditions on $\mathcal{B}$ and $\mathcal{T}$ are satisfied when $\mathcal{B} = \mathrm{Grpd}(\mathcal{C})$ for regular category $\mathcal{C}$ where the axiom of choice holds and $\mathcal{T}$ is induced by a pullback-preserving monad $\mathbb{T}$ on $\mathcal{C}$. Nevertheless, the main result of Section 4 is not a special case of the result in this final section, because in Section 4 we do not assume that $\mathbb{T}$ preserves pullbacks.

Throughout this paper, we use the terminology of Chapter 7 of [4] for 2-categories.

2 Weak equivalences

We recall in this section the definition of weak equivalences in the 2-category of internal groupoids. Let us first fix notations. If $\mathcal{C}$ is a category with pullbacks, an internal groupoid $\mathbb{A}$ in $\mathcal{C}$ is represented by

$$A_1 \times_{c,d} A_1 \xrightarrow{m} A_1 \underset{c}{\overset{d}{\rightrightarrows}} A_0 \quad (e : A_0 \to A_1) \qquad\qquad A_1 \xrightarrow{i} A_1$$

where

$$\begin{array}{ccc} A_1 \times_{c,d} A_1 & \xrightarrow{\pi_2} & A_1 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ A_1 & \xrightarrow[c]{} & A_0 \end{array}$$

is a pullback.

An internal functor $F : \mathbb{A} \to \mathbb{B}$ in $\mathcal{C}$ is represented by

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & B_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array}$$

and an internal natural transformation $\alpha : F \Rightarrow G : \mathbb{A} \to \mathbb{B}$ is represented by $\alpha : A_0 \to B_1$.

We denote by $\mathrm{Grpd}(\mathcal{C})$ the 2-category of internal groupoids, internal functors and internal natural transformations in $\mathcal{C}$. Now, it is worth remarking that, if $\mathcal{D}$ has pullbacks and if $U : \mathcal{D} \to \mathcal{C}$ preserves them, it induces a 2-functor (also denoted U by abuse of notation)

$$\begin{array}{rcl} U : \ \mathrm{Grpd}(\mathcal{D}) & \longrightarrow & \mathrm{Grpd}(\mathcal{C}) \\ \mathbb{A} & \longmapsto & U\mathbb{A} = (UA_0, UA_1, Ud, Uc, Ue, Um, Ui) \\ F & \longmapsto & UF = (UF_0, UF_1) \\ \alpha & \longmapsto & U\alpha. \end{array}$$

The definition of weak equivalences has been introduced by M. Bunge and R. Paré in [6].

Definition 2.1 (Bunge-Paré)**.** *Let $\mathcal{C}$ be a category with pullbacks and $F : \mathbb{A} \to \mathbb{B}$ be an internal functor between internal groupoids in $\mathcal{C}$.*

- *We say that F is full and faithful if*

$$\begin{array}{ccccccc} & & & A_1 & & & \\ & {\scriptstyle d}\swarrow & & \downarrow{\scriptstyle F_1} & & \searrow{\scriptstyle c} & \\ A_0 & & & B_1 & & & A_0 \\ & {\scriptstyle F_0}\searrow & \swarrow{\scriptstyle d} & & {\scriptstyle c}\searrow & \swarrow{\scriptstyle F_0} & \\ & & B_0 & & B_0 & & \end{array}$$

  is a limit diagram.

Moreover, if $\mathcal{C}$ is regular,

- *F is said to be essentially surjective if*

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

  is a regular epimorphism, where t_2 is given by the pullback

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array}$$

- *F is a weak equivalence if it is full and faithful and essentially surjective.*

We notice that if $\mathcal{C} = \mathrm{Set}$, this corresponds to the usual notion of fully faithful and essentially surjective functor.

We can immediately deduce the following lemma.

Lemma 2.2. *Let $U : \mathcal{D} \to \mathcal{C}$ be a pullback preserving functor between categories with pullbacks and let $F : \mathbb{A} \to \mathbb{B}$ be an internal functor between internal groupoids in $\mathcal{D}$. Then,*

- *If U reflects finite limits, F is full and faithful if and only if UF is.*
- *If $\mathcal{C}$ and $\mathcal{D}$ are regular and if U preserves and reflects regular epimorphisms, then F is essentially surjective if and only if UF is.*

Proof. The 'only if parts' follow from the preserving hypothesis while the 'if parts' follow from the reflecting hypothesis. □

We conclude this section by a well-known lemma. A proof can be found in [14].

Lemma 2.3. *Let $\mathcal{C}$ be a category with pullbacks. An internal functor $F : \mathbb{A} \to \mathbb{B}$ between internal groupoids in $\mathcal{C}$ is an equivalence if and only if it is full and faithful and*

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

is a split epimorphism.

We know from this lemma that in a regular category where the axiom of choice holds (every regular epimorphism splits), weak equivalences coincide with equivalences.

3 $\mathbb{T}$-monoidal functors

Monoidal functors $F : \mathbb{A} \to \mathbb{B}$ between small strict cat-groups (i.e., internal groupoids in the category Gp of groups) can be seen as 'quasi-internal functors'. Indeed, since for all $X, Y \in \mathbb{A}$ we have an isomorphism $F(X) + F(Y) \cong F(X+Y)$, F is not an internal functor in Gp, but it is only an 'internal functor up to isomorphism'. The aim of this section is to generalize this notion of 'quasi-internal functor' replacing Gp by any monadic category.

So, we are given a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$ with pullbacks. We thus have a forgetful functor from the Eilenberg-Moore category

$$\begin{aligned} U : \quad \mathcal{C}^{\mathbb{T}} &\to \mathcal{C} \\ (A, \alpha) &\mapsto A \\ f &\mapsto f \end{aligned}$$

which preserves, reflects and creates pullbacks (see [5], Proposition 4.3.1). It induces a 2-functor $U : \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) \to \mathrm{Grpd}(\mathcal{C})$. If $\mathbb{A}$ is a groupoid in $\mathcal{C}^{\mathbb{T}}$ and if we denote the groupoid $U\mathbb{A}$ in $\mathcal{C}$ by

$$U\mathbb{A} = \left(\begin{array}{c} A_1 \overset{d}{\underset{c}{\rightrightarrows}} A_0 , m, i \\ \text{with } e : A_0 \to A_1 \end{array} \right),$$

then $\mathbb{A}$ is of the form

$$\mathbb{A} = \left(\begin{array}{c} (A_1, a_1) \overset{d}{\underset{c}{\rightrightarrows}} (A_0, a_0) , m, i \\ \text{with } e : (A_0,a_0) \to (A_1,a_1) \end{array} \right)$$

where $a_1 : TA_1 \to A_1$ and $a_0 : TA_0 \to A_0$ are $\mathbb{T}$-algebras and d, c, e, m, i are $\mathbb{T}$-algebra homomorphisms. In particular, this means that

$$\begin{array}{ccc} T(A_1 \times_{c,d} A_1) & \xrightarrow{Tm} & T(A_1) \\ {\scriptstyle (a_1 T\pi_1, a_1 T\pi_2)}\downarrow & & \downarrow{\scriptstyle a_1} \\ A_1 \times_{c,d} A_1 & \xrightarrow[m]{} & A_1 \end{array}$$

commutes since the left downwards arrow is the $\mathbb{T}$-algebra on $A_1 \times_{c,d} A_1$.

We are now able to define $\mathbb{T}$-monoidal functors.

Definition 3.1. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$ with pullbacks. We define the 2-category $\mathbb{T}$-MON as follows:*

- *Objects are internal groupoids in $\mathcal{C}^{\mathbb{T}}$.*
- *1-cells are $\mathbb{T}$-monoidal functors $(F, \varphi) : \mathbb{A} \to \mathbb{B}$. These are the data of a functor $F : U\mathbb{A} \to U\mathbb{B}$ in $\mathcal{C}$ and an arrow $\varphi : TA_0 \to B_1$ such that*

$$\begin{array}{ccc} TA_0 & \xrightarrow{\varphi} & B_1 \\ {\scriptstyle TF_0}\downarrow & & \downarrow{\scriptstyle d} \\ TB_0 & \xrightarrow[b_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} TA_0 & \xrightarrow{\varphi} & B_1 \\ {\scriptstyle a_0}\downarrow & & \downarrow{\scriptstyle c} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} TA_1 & \xrightarrow{(b_1 TF_1, \varphi Tc)} & B_1 \times_{c,d} B_1 \\ {\scriptstyle (\varphi Td, F_1 a_1)}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

$$\begin{array}{ccc} A_0 & \xrightarrow{\eta_{A_0}} & TA_0 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \varphi} \\ A_1 & \xrightarrow[F_1]{} & B_1 \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} T^2A_0 & \xrightarrow{\mu_{A_0}} & TA_0 \\ {\scriptstyle (b_1 T\varphi, \varphi Ta_0)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

  commute.
- $1_{\mathbb{A}} = (1_{U\mathbb{A}}, ea_0)$.
- *The composition of $\mathbb{A} \xrightarrow{(F,\varphi)} \mathbb{B} \xrightarrow{(G,\psi)} \mathbb{C}$ is $(GF, m(\psi TF_0, G_1\varphi))$.*
- *2-cells are $\mathbb{T}$-monoidal natural transformations $\alpha : (F, \varphi) \Rightarrow (F', \varphi') : \mathbb{A} \to \mathbb{B}$. These are natural transformations $\alpha : F \Rightarrow F'$ in $\mathcal{C}$ such that*

$$\begin{array}{ccc} TA_0 & \xrightarrow{(\varphi, \alpha a_0)} & B_1 \times_{c,d} B_1 \\ {\scriptstyle (b_1 T\alpha, \varphi')}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

  commutes.
- *Identities and vertical and horizontal compositions of 2-cells are as in Grpd($\mathcal{C}$).*

Proposition 3.2. *The above definition makes sense, i.e. $\mathbb{T}$-MON is a 2-category. Moreover, every 2-cell in $\mathbb{T}$-MON is invertible.*

Proof. The proof being quite long, we only prove as example that the composition of $\mathbb{T}$-monoidal functors is still $\mathbb{T}$-monoidal.

- $$dm(\psi TF_0, G_1\varphi) = d\psi TF_0 = c_0 T(G_0 F_0).$$

- $$cm(\psi TF_0, G_1\varphi) = cG_1\varphi = G_0c\varphi = G_0F_0a_0.$$

- $$\begin{aligned} m(c_1TG_1TF_1, m(\psi TF_0, G_1\varphi)Tc) &= m(m(c_1TG_1TF_1, \psi TF_0Tc), G_1\varphi Tc) \\ &= m(m(c_1TG_1, \psi Tc)TF_1, G_1\varphi Tc) \\ &= m(m(\psi Td, G_1b_1)TF_1, G_1\varphi Tc) \\ &= m(m(\psi TF_0Td, G_1b_1TF_1), G_1\varphi Tc) \\ &= m(\psi TF_0Td, G_1m(b_1TF_1, \varphi Tc)) \\ &= m(\psi TF_0Td, G_1m(\varphi Td, F_1a_1)) \\ &= m(m(\psi TF_0, G_1\varphi)Td, G_1F_1a_1). \end{aligned}$$

- $$\begin{aligned} m(\psi TF_0, G_1\varphi)\eta_{A_0} &= m(\psi TF_0\eta_{A_0}, G_1\varphi\eta_{A_0}) \\ &= m(\psi\eta_{B_0}F_0, G_1F_1e) = m(G_1eF_0, G_1F_1e) = G_1F_1e. \end{aligned}$$

- $$\begin{aligned} m(\psi TF_0, G_1\varphi)\mu_{A_0} &= m(\psi TF_0\mu_{A_0}, G_1\varphi\mu_{A_0}) \\ &= m(\psi\mu_{B_0}TTF_0, G_1m(b_1T\varphi, \varphi Ta_0)) \\ &= m(m(c_1T\psi, \psi Tb_0)TTF_0, m(G_1b_1T\varphi, G_1\varphi Ta_0)) \\ &= m(m(c_1T\psi TTF_0, \psi TdT\varphi), m(G_1b_1T\varphi, G_1\varphi Ta_0)) \\ &= m(c_1T\psi TTF_0, m(m(\psi Td, G_1b_1)T\varphi, G_1\varphi Ta_0)) \\ &= m(c_1T\psi TTF_0, m(m(c_1TG_1, \psi Tc)T\varphi, G_1\varphi Ta_0)) \\ &= m(c_1T\psi TTF_0, m(m(c_1TG_1T\varphi, \psi TF_0Ta_0), G_1\varphi Ta_0)) \\ &= m(m(c_1T\psi TTF_0, c_1TG_1T\varphi), m(\psi TF_0Ta_0, G_1\varphi Ta_0)) \\ &= m(m(c_1T\pi_1, c_1T\pi_2)T(\psi TF_0, G_1\varphi), m(\psi TF_0Ta_0, G_1\varphi Ta_0)) \\ &= m(c_1TmT(\psi TF_0, G_1\varphi), m(\psi TF_0, G_1\varphi)Ta_0). \end{aligned}$$

For the second part of the proof, we notice that if $\alpha : (F, \varphi) \Rightarrow (F', \varphi')$ is $\mathbb{T}$-monoidal, then so is α^{-1} (we also omit the computations).

□

Diagrams involved in Definition 3.1 might be thought as unintuitive at a first glance. The example of the free group monad on Set is treated in Section 5 while an explanation where these axioms come from can be found in Section 7 in the context of strict algebras for a pseudo-monad.

Remark that we have two 2-functors

$$\begin{aligned} I : \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) &\hookrightarrow \mathbb{T}\text{-MON} \\ \mathbb{A} &\longmapsto \mathbb{A} \\ F &\longmapsto (UF, eb_0TF_0) \\ \alpha &\longmapsto U\alpha \end{aligned} \qquad \text{and} \qquad \begin{aligned} J : \mathbb{T}\text{-MON} &\longrightarrow \mathrm{Grpd}(\mathcal{C}) \\ \mathbb{A} &\longmapsto U\mathbb{A} \\ (F, \varphi) &\longmapsto F \\ \alpha &\longmapsto \alpha. \end{aligned}$$

Thus, by abuse of notation, we say that a $\mathbb{T}$-monoidal functor $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ is internal in $\mathcal{C}^{\mathbb{T}}$ when $\varphi = eb_0TF_0$. We will often identify an internal functor F in $\mathcal{C}^{\mathbb{T}}$ with (UF, eb_0TF_0).

It is a well-know fact that if a monoidal functor between monoidal categories has a pseudo-inverse, then, this pseudo-inverse can be equipped with a monoidal structure. The next lemma asserts that the same occurs for $\mathbb{T}$-monoidal functors.

Lemma 3.3. *A $\mathbb{T}$-monoidal functor $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ is a $\mathbb{T}$-monoidal equivalence (i.e., an equivalence in $\mathbb{T}$-MON) if and only if $F : U\mathbb{A} \to U\mathbb{B}$ is an equivalence in Grpd($\mathcal{C}$).*

Proof. The 'only if part' is clear. Let us prove the 'if part'. Suppose we have a functor $G : \mathbb{B} \to \mathbb{A}$ and natural isomorphisms $\alpha : GF \Rightarrow 1_{\mathbb{A}}$ and $\beta : FG \Rightarrow 1_{\mathbb{B}}$ in $\mathcal{C}$. Without loss of generality, we can assume that $\beta \star 1_F = 1_F \star \alpha$ and $1_G \star \beta = \alpha \star 1_G$. Since F is full and faithful, there exists a unique $\psi : TB_0 \to A_1$ such that $d\psi = a_0 TG_0$, $c\psi = G_0 b_0$ and $F_1\psi = m(i\varphi TG_0, m(b_1 T\beta, i\beta b_0))$. This makes (G, ψ) and β $\mathbb{T}$-monoidal. Moreover, since α and β satisfy the triangular identities, α is also $\mathbb{T}$-monoidal. Therefore, (F, φ) is a $\mathbb{T}$-monoidal equivalence. □

Remark 3.4. We can notice here that, even if $\varphi = eb_0TF_0$, this does not imply that $\psi = ea_0TG_0$. So, an internal functor in $\mathcal{C}^{\mathbb{T}}$ can be an equivalence in $\mathbb{T}$-MON, without being an equivalence in Grpd($\mathcal{C}^{\mathbb{T}}$).

4 $\mathbb{T}$-MON as bicategory of fractions

In [12], D. Pronk defined bicategories of fractions as the 2-dimensional analogous to the categories of fractions introduced by P. Gabriel and M. Zisman in [7]. In this section, we first recall this notion. Afterwards, we prove that the bicategory of fractions of Grpd($\mathcal{C}^{\mathbb{T}}$) with respect to weak equivalences is $\mathbb{T}$-MON.

We use the term 'pseudo-functor' for 'homomorphism' of bicategories. An introduction to bicategories can be found in [3].

Definition 4.1 (Pronk)**.** *Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The bicategory of fractions of $\mathcal{B}$ with respect to Σ is a pseudo-functor*

$$P_\Sigma : \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

which sends elements of Σ to equivalences and which is universal with this property:

$$- \circ P_\Sigma : PsFunct(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \to PsFunct_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence for every bicategory $\mathcal{A}$, where $PsFunct(\mathcal{B}[\Sigma^{-1}], \mathcal{A})$ is the bicategory of pseudo-functors $\mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$ and $PsFunct_\Sigma(\mathcal{B}, \mathcal{A})$ is the bicategory of pseudo-functors $\mathcal{B} \to \mathcal{A}$ which send elements of Σ to equivalences.

Similarly, admitting a right calculus of fractions for a class of 1-cells in a bicategory is the 2-dimensional version of the 1-dimensional case. We refer the reader to [12] for definitions. The results we will use in this section are the following two.

Proposition 4.2 (Pronk)**.** *Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$ which has a right calculus of fractions. Consider a pseudo-functor $F : \mathcal{B} \to \mathcal{A}$ which sends elements of Σ to equivalences and let $\hat{F} : \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$ its extension. Suppose the following conditions hold.*

EF1. F is essentially surjective on objects.

EF2. F is full and faithful on 2-cells.

EF3. For each 1-cell $f : F(A) \to F(B)$ in $\mathcal{A}$, there exist 1-cells $g : C \to B$ and $w : C \to A$ in $\mathcal{B}$ such that $w \in \Sigma$ and $F(g) \cong f \circ F(w)$.

Then, $\hat{F}$ is a biequivalence and $F : \mathcal{B} \to \mathcal{A}$ is the bicategory of fractions of $\mathcal{B}$ with respect to Σ.

A proof of this proposition can be found in [12], while the next one is proved in [14].

Proposition 4.3. *Let $\mathcal{C}$ be a regular category and Σ be the class of weak equivalences in Grpd($\mathcal{C}$). Then Σ has a right calculus of fractions.*

In order to use this proposition for weak equivalences in $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$, we need the following lemma, stated without proof in [13].

Lemma 4.4. *Let $\mathcal{C}$ be a regular category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Then,*

1. *T preserves regular epimorphisms if and only if $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ does;*
2. *if T preserves regular epimorphisms, U reflects them. In this case, $\mathcal{C}^{\mathbb{T}}$ is regular.*

Proof. 1. Firstly, suppose U preserves regular epimorphisms. Since $T = UF$ where $F : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ is the left adjoint to U, T preserves regular epimorphisms since U and F do. Conversely, suppose T preserves regular epimorphisms. Let $f : (A, \alpha) \to (B, \beta)$ be a regular epimorphism in $\mathcal{C}^{\mathbb{T}}$. Then, f factors as

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle m} & \\ & I & \end{array}$$

in $\mathcal{C}$, with p a regular epimorphism and m a monomorphism. So, Tp is a regular epimorphism and there exists a unique i making the following diagram commute.

$$\begin{array}{ccc} TA & \overset{Tp}{\twoheadrightarrow} & TI \\ {\scriptstyle p\alpha}\downarrow & \overset{i}{\swarrow} & \downarrow{\scriptstyle \beta Tm} \\ I & \underset{m}{\rightarrowtail} & B \end{array}$$

Since m is a monomorphism and (B, β) a $\mathbb{T}$-algebra, (I, i) is also a $\mathbb{T}$-algebra and p and m are $\mathbb{T}$-algebra homomorphisms. Therefore, since f is the coequalizer of two parallel arrows in $\mathcal{C}^{\mathbb{T}}$, p is also their coequalizer in $\mathcal{C}^{\mathbb{T}}$. Thus, m is an isomorphism and f a regular epimorphism in $\mathcal{C}$.

2. Since U creates finite limits, $\mathcal{C}^{\mathbb{T}}$ has them. Now, let $f : (A, \alpha) \to (B, \beta)$ be a $\mathbb{T}$-algebra homomorphism such that $f : A \twoheadrightarrow B$ is a regular epimorphism in $\mathcal{C}$. Denote by

$$R[f] \underset{f_2}{\overset{f_1}{\rightrightarrows}} (A, \alpha)$$

the kernel pair of f in $\mathcal{C}^{\mathbb{T}}$. Since U preserves limits, it is also its kernel pair in $\mathcal{C}$. So, f coequalizes the pair (f_1, f_2) in $\mathcal{C}$. Using this and the epimorphism Tf in $\mathcal{C}$, one proves that f coequalizes them also in $\mathcal{C}^{\mathbb{T}}$. Therefore, U reflects regular epimorphisms. Moreover, since U preserves and reflects regular epimorphisms and since they are stable under pullbacks in $\mathcal{C}$, they are also in $\mathcal{C}^{\mathbb{T}}$. Finally, the construction done in point 1 shows that $\mathcal{C}^{\mathbb{T}}$ inherits the factorisation system (regular epi - mono) from $\mathcal{C}$, since U reflects monomorphisms and regular epimorphisms. □

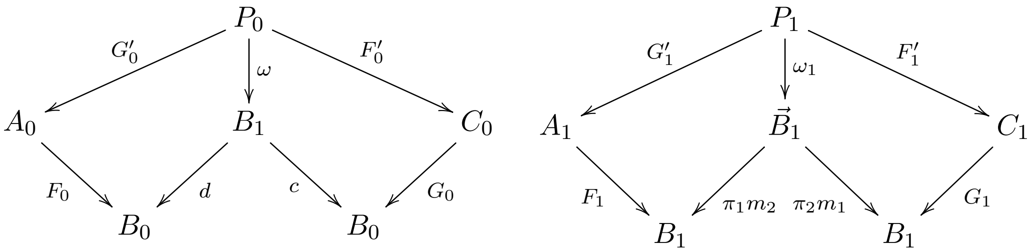
As in [14], the key lemma (4.5) in proving that $\mathbb{T}$-MON is the bicategory of fractions of $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ with respect to weak equivalences is the fact that the bipullback of two $\mathbb{T}$-monoidal functors exists and the legs can be chosen to be in $\mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$. Firstly, from [14], we know that if $\mathcal{C}$ has pullbacks, then $\mathrm{Grpd}(\mathcal{C})$ has bipullbacks, constructed as follows: Given

$$\begin{array}{ccc} & & \mathbb{C} \\ & & \downarrow{\scriptstyle G} \\ \mathbb{A} & \underset{F}{\longrightarrow} & \mathbb{B} \end{array}$$

in $\mathrm{Grpd}(\mathcal{C})$, we construct the pullback in $\mathcal{C}$

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\quad m_2 \quad} & B_1 \times_{c,d} B_1 \\ {\scriptstyle m_1}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[\quad m \quad]{} & B_1 \end{array}$$

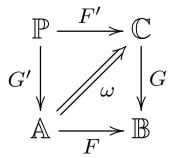
and define P_0 and P_1 to be the limits



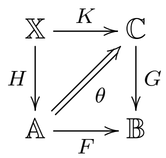
It turns out that $\mathbb{P}$ has a groupoid structure in $\mathcal{C}$ and that

$$\omega_1 = ((\omega d, G_1 F'_1), (F_1 G'_1, \omega c)).$$

Finally, the bipullback of F and G is given by



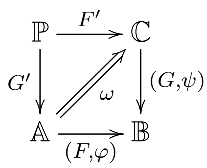
As far as its universal property is concerned, if



is another square, the triple $(L : \mathbb{X} \to \mathbb{P}, \alpha : G'L \Rightarrow H, \beta : F'L \Rightarrow K)$ defined by $G'L = H$, $F'L = K$, $\omega L_0 = \theta$, $\omega_1 L_1 = ((\theta d, G_1 K_1), (F_1 H_1, \theta c))$, $\alpha = 1_H$ and $\beta = 1_K$ is a fill-in. Thus, if $(L' : \mathbb{X} \to \mathbb{P}, \alpha' : G'L' \Rightarrow H, \beta' : F'L' \Rightarrow K)$ is another fill-in, the unique $\lambda : L' \Rightarrow L$ such that $1_{G'} \star \lambda = \alpha'$ and $1_{F'} \star \lambda = \beta'$ occurs to be the unique $\lambda : X_0 \to P_1$ such that $G'_1 \lambda = \alpha'$, $\omega_1 \lambda = ((\omega L'_0, G_1 \beta'), (F_1 \alpha', \theta))$ and $F'_1 \lambda = \beta'$.

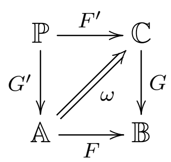
Now, we can prove that $\mathbb{T}$-MON has bipullbacks.

Lemma 4.5. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$ with pullbacks. Then, the 2-category $\mathbb{T}$-MON has bipullbacks. Moreover, given $\mathbb{T}$-monoidal functors $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ and $(G, \psi) : \mathbb{C} \to \mathbb{B}$, it is possible to choose a bipullback of (F, φ) and (G, ψ)*



in such a way that F' and G' are internal functors in $\mathcal{C}^{\mathbb{T}}$.

Proof. Let



be the bipullback in Grpd($\mathcal{C}$) described above. We now turn $\mathbb{P}$ into an internal groupoid in $\mathcal{C}^{\mathbb{T}}$ in the following way. P_0 being a limit, there exists a unique $p_0 : TP_0 \to P_0$ such that $G'_0 p_0 = a_0 T G'_0$,

$\omega p_0 = m(i\varphi TG'_0, m(b_1 T\omega, \psi TF'_0))$ and $F'_0 p_0 = c_0 TF'_0$. Moreover, P_1 being also a limit, there exists a unique $p_1 : TP_1 \to P_1$ such that $G'_1 p_1 = a_1 TG'_1$, $\pi_1 m_1 \omega_1 p_1 = \omega p_0 Td$, $\pi_2 m_1 \omega_1 p_1 = G_1 c_1 TF'_1$, $\pi_1 m_2 \omega_1 p_1 = F_1 a_1 TG'_1$, $\pi_2 m_2 \omega_1 p_1 = \omega p_0 Tc$ and $F'_1 p_1 = c_1 TF'_1$. This makes $\mathbb{P}$ an internal groupoid in $\mathcal{C}^{\mathbb{T}}$, F' and G' internal functors in $\mathcal{C}^{\mathbb{T}}$ and ω a $\mathbb{T}$-monoidal natural transformation.

If

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{(K,k)} & \mathbb{C} \\ {\scriptstyle (H,h)}\big\downarrow & \overset{\theta}{\Nearrow} & \big\downarrow{\scriptstyle (G,\psi)} \\ \mathbb{A} & \xrightarrow[(F,\varphi)]{} & \mathbb{B} \end{array}$$

is another square, the triple $(L : \mathbb{X} \to \mathbb{P}, 1_H, 1_K)$ defined above for the square

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{K} & \mathbb{C} \\ {\scriptstyle H}\big\downarrow & \overset{\theta}{\Nearrow} & \big\downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in Grpd($\mathcal{C}$) can be turned in a fill-in in $\mathbb{T}$-MON. Indeed, it suffices to set $l : TX_0 \to P_1$ as the unique arrow such that $G'_1 l = h$, $\pi_1 m_1 \omega_1 l = \omega p_0 TL_0$, $\pi_2 m_1 \omega_1 l = G_1 k$, $\pi_1 m_2 \omega_1 l = F_1 h$, $\pi_2 m_2 \omega_1 l = \omega L_0 x_0$ and $F'_1 l = k$.

Finally, if

$$\big((L', l') : \mathbb{X} \to \mathbb{P}, \alpha' : G' \circ (L', l') \Rightarrow (H, h), \beta' : F' \circ (L', l') \Rightarrow (K, k)\big)$$

is another fill-in, the unique $\lambda : L' \Rightarrow L$ such that $1_{G'} \star \lambda = \alpha'$ and $1_{F'} \star \lambda = \beta'$ is a $\mathbb{T}$-monoidal natural transformation $(L', l') \Rightarrow (L, l)$. □

We are now able to prove our main result.

Proposition 4.6. *Let $\mathcal{C}$ be a regular category where the axiom of choice holds and let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. Then, the inclusion 2-functor*

$$I\colon\ \textit{Grpd}(\mathcal{C}^{\mathbb{T}}) \hookrightarrow \mathbb{T}\textit{-MON}$$

is the bicategory of fractions of Grpd($\mathcal{C}^{\mathbb{T}}$) with respect to the class of weak equivalences.

Proof. Since regular epimorphisms are split, T preserves them. By Lemma 4.4, the forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ preserves and reflects regular epimorphisms and $\mathcal{C}^{\mathbb{T}}$ is regular. Let Σ be the class of weak equivalences in Grpd($\mathcal{C}^{\mathbb{T}}$). Then, we know from Proposition 4.3 that Σ has a right calculus of fractions.

Now, let $F \in \Sigma$. By Lemma 2.2, we know that $UF \in$ Grpd($\mathcal{C}$) is a weak equivalence. Since $\mathcal{C}$ has the axiom of choice, UF is actually an equivalence. Thus, by Lemma 3.3, $I(F)$ is an equivalence and I sends elements of Σ to equivalences.

Therefore, it remains to prove that I satisfies conditions EF1, EF2 and EF3 of Proposition 4.2. EF1 is obvious and EF2 is the fact that, between internal functors in $\mathcal{C}^{\mathbb{T}}$, $\mathbb{T}$-monoidal natural transformations are exactly internal natural transformations in $\mathcal{C}^{\mathbb{T}}$. Let us prove EF3. Given $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ in $\mathbb{T}$-MON, consider the bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{G} & \mathbb{B} \\ {\scriptstyle W}\big\downarrow & \overset{\omega}{\Nearrow} & \big\downarrow{\scriptstyle I} \\ \mathbb{A} & \xrightarrow[(F,\varphi)]{} & \mathbb{B} \end{array}$$

given by Lemma 4.5, in such a way that G and W are internal functors in $\mathcal{C}^{\mathbb{T}}$. Thus, $G \cong (F, \varphi) \circ W$. Since bipullbacks preserve equivalences, W is an equivalence in $\mathbb{T}$-MON and thus in Grpd($\mathcal{C}$). By Lemma 2.2 again, this implies that $W \in \Sigma$. □

Corollary 4.7. *Let $\mathcal{C}$ be a regular category where the axiom of choice holds and $G : \mathcal{D} \to \mathcal{C}$ be a monadic functor. Denote by $\mathbb{T} = (T, \eta, \mu)$ the monad induced by the adjunction $F \dashv G$ on $\mathcal{C}$ and $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ the comparison functor. Then, the composite*

$$\mathit{Grpd}(\mathcal{D}) \xrightarrow{K} \mathit{Grpd}(\mathcal{C}^{\mathbb{T}}) \overset{I}{\hookrightarrow} \mathbb{T}\text{-}\mathit{MON}$$

is the bicategory of fractions of $\mathit{Grpd}(\mathcal{D})$ with respect to weak equivalences.

Proof. Since G is monadic, $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an equivalence and $K : \mathrm{Grpd}(\mathcal{D}) \to \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ is a biequivalence of 2-categories. In addition, by Lemma 2.2, $W \in \mathrm{Grpd}(\mathcal{D})$ is a weak equivalence if and only if $K(W) \in \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}})$ is. Thus, IK satisfies conditions EF1, EF2 and EF3 of Proposition 4.2 since I does. □

5 The case of groups

We study in this section the particular case of the monadic forgetful functor $U : \mathrm{Gp} \to \mathrm{Set}$. Set is a regular category with the axiom of choice. Therefore, $\mathbb{T}$-MON is the bicategory of fractions of Grpd(Gp) with respect to Σ, the class of weak equivalences. In order to explain the axioms of Definition 3.1, we make them explicit in this context. Let $\mathbb{A}$ and $\mathbb{B}$ be two groupoids in $\mathrm{Set}^{\mathbb{T}}$ (i.e. in Gp by the biequivalence K). A $\mathbb{T}$-monoidal functor $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ is a functor $F : \mathbb{A} \to \mathbb{B}$ between the underlying categories, together with a function $\varphi : \mathrm{Fr}(A_0) \to B_1$ (where $\mathrm{Fr}(A_0)$ is the free group on the set of objects of $\mathbb{A}$) satisfying the following axioms:

- 1 and 2: For all $a_1, \dots, a_n \in A_0$ and $i_k \in \{-1, 1\}$, $\varphi(a_1^{i_1} \dots a_n^{i_n})$ is an arrow

$$F(a_1)^{i_1} \otimes \cdots \otimes F(a_n)^{i_n} \xrightarrow{\varphi(a_1^{i_1} \dots a_n^{i_n})} F(a_1^{i_1} \otimes \cdots \otimes a_n^{i_n}).$$

- 3: For all $f_1, \dots, f_n \in A_1$ and $i_k \in \{-1, 1\}$,

$$\begin{array}{ccc} F(a_1)^{i_1} \otimes \cdots \otimes F(a_n)^{i_n} & \xrightarrow{F(f_1)^{i_1} \otimes \cdots \otimes F(f_n)^{i_n}} & F(a'_1)^{i_1} \otimes \cdots \otimes F(a'_n)^{i_n} \\ \Big\downarrow {\scriptstyle \varphi(a_1^{i_1} \dots a_n^{i_n})} & & \Big\downarrow {\scriptstyle \varphi(a'^{i_1}_1 \dots a'^{i_n}_n)} \\ F(a_1^{i_1} \otimes \cdots \otimes a_n^{i_n}) & \xrightarrow[F(f_1^{i_1} \otimes \cdots \otimes f_n^{i_n})]{} & F(a'^{i_1}_1 \otimes \cdots \otimes a'^{i_n}_n) \end{array}$$

  commutes.

- 4: For all $a \in A_0$, $\varphi(a) = 1_{F(a)} : F(a) \to F(a)$.

- 5: For all $a_{11}, \dots, a_{n_1 1}, \dots, a_{1k}, \dots, a_{n_k k} \in A_0$,

$$\begin{array}{ccc} F(a_{11}) \otimes \cdots \otimes F(a_{n_k k}) & \xrightarrow{\varphi(a_{11} \dots a_{n_k k})} & F(a_{11} \otimes \cdots \otimes a_{n_k k}) \\ \Big\downarrow {\scriptstyle \varphi(a_{11} \dots a_{n_1 1}) \otimes \cdots \otimes \varphi(a_{1k} \dots a_{n_k k})} & \nearrow {\scriptstyle \varphi((a_{11} \otimes \cdots \otimes a_{n_1 1}) \dots (a_{1k} \otimes \cdots \otimes a_{n_k k}))} & \\ F(a_{11} \otimes \cdots \otimes a_{n_1 1}) \otimes \cdots \otimes F(a_{1k} \otimes \cdots \otimes a_{n_k k}) & & \end{array}$$

  commutes (for the sake of simplicity, we only express axiom 5 with exponents 1).

In [14], it is shown that the bicategory of fractions $\mathrm{Grpd}(\mathrm{Gp})[\Sigma^{-1}]$ is MON, the 2-category of groupoids in Gp (i.e. small strict cat-groups), monoidal functors and monoidal natural transformations. Thus, MON and $\mathbb{T}$-MON are biequivalent. This biequivalence $\widetilde{K} : \mathrm{MON} \to \mathbb{T}\text{-MON}$

makes the diagram

$$
\begin{array}{ccccccc}
\mathrm{Grpd(Gp)} & & \hookrightarrow & & \mathrm{MON} & \longrightarrow & \mathrm{Grpd(Set)} \\
\| & & & & \downarrow{\scriptstyle \widetilde{K}}\,\simeq & & \| \\
\mathrm{Grpd(Gp)} & \xrightarrow[K]{\simeq} & \mathrm{Grpd(Set^{\mathbb{T}})} & \xhookrightarrow[I]{} & \mathbb{T}\text{-}\mathrm{MON} & \xrightarrow[J]{} & \mathrm{Grpd(Set)}
\end{array}
$$

commutative. Moreover, it can be described by

$$
\begin{array}{rccc}
\widetilde{K}: & \mathrm{MON} & \longrightarrow & \mathbb{T}\text{-}\mathrm{MON} \\
& \mathbb{A} & \longmapsto & K\mathbb{A} \\
& \mathbb{A} \xrightarrow{(F,F_2)} \mathbb{B} & \longmapsto & K\mathbb{A} \xrightarrow{(F,\varphi)} K\mathbb{B} \\
& \alpha & \longmapsto & \alpha
\end{array}
$$

where F_2 is the monoidal structure of F and $\varphi : \mathrm{Fr}(A_0) \to B_1$ is defined on the word $a_1^{i_1} \cdots a_n^{i_n}$ ($a_k \in A_0$ and $i_k \in \{-1, 1\}$) to be the arrow part of the sum

$$\big(F(a_1), 1_{F(a_1)}, a_1\big)^{i_1} + \cdots + \big(F(a_n), 1_{F(a_n)}, a_n\big)^{i_1} .$$

This sum is calculated in the group of triples $(b \in B_0, f : b \to F(a), a \in A_0)$ and defined by

$$(b, f, a) + (b', f', a') = \left(b + b',\ b + b' \xrightarrow{f+f'} F(a) + F(a') \xrightarrow{F_2^{a,a'}} F(a + a')\ , a + a' \right) .$$

6 The case of Lie algebras

The aim of this section is to make the link between a result in [14] and our Corollary 4.7 for the particular monadic adjunction $U : \mathrm{Lie}_k \to \mathrm{Vect}_k$, where Lie_k and Vect_k are the categories of Lie algebras and vector spaces respectively (for a fixed field k). Vect_k is a regular category with the axiom of choice since every vector space is free (admits a basis). Thus, we can apply our Corollary 4.7 to deduce that $\mathbb{T}$-MON is the bicategory of fractions of $\mathrm{Grpd}(\mathrm{Lie}_k)$ with respect to weak equivalences.

Besides, it is shown in [14], that this bicategory of fractions is LIE_k, the 2-category of internal groupoids in Lie_k, homomorphisms and 2-homomorphisms (see [2]). Therefore, $\mathbb{T}$-MON and Lie_k are biequivalent. As for groups, this biequivalence $\widetilde{K} : \mathrm{LIE}_k \to \mathbb{T}\text{-MON}$ makes the diagram

$$
\begin{array}{ccccccc}
\mathrm{Grpd}(\mathrm{Lie}_k) & & \hookrightarrow & & \mathrm{LIE}_k & \longrightarrow & \mathrm{Grpd}(\mathrm{Vect}_k) \\
\| & & & & \downarrow{\scriptstyle \widetilde{K}}\,\simeq & & \| \\
\mathrm{Grpd}(\mathrm{Lie}_k) & \xrightarrow[K]{\simeq} & \mathrm{Grpd}(\mathrm{Vect}_k^{\mathbb{T}}) & \xhookrightarrow[I]{} & \mathbb{T}\text{-}\mathrm{MON} & \xrightarrow[J]{} & \mathrm{Grpd}(\mathrm{Vect}_k)
\end{array}
$$

commute. Moreover, it can be described by

$$
\begin{array}{rccc}
\widetilde{K}: & \mathrm{LIE}_k & \longrightarrow & \mathbb{T}\text{-}\mathrm{MON} \\
& \mathbb{A} & \longmapsto & K\mathbb{A} \\
& \mathbb{A} \xrightarrow{(F,F_2)} \mathbb{B} & \longmapsto & K\mathbb{A} \xrightarrow{(F,\varphi)} K\mathbb{B} \\
& \alpha & \longmapsto & \alpha
\end{array}
$$

where $\varphi : TA_0 \to B_1$ is defined as follows. TA_0 is the free Lie algebra of the underlying vector space of A_0. It is actually the Lie subalgebra generated by A_0 of the tensor algebra

$$\mathsf{T}(A_0) = K \oplus A_0 \oplus (A_0 \otimes A_0) \oplus (A_0 \otimes A_0 \otimes A_0) \oplus \cdots .$$

To each element $v \in TA_0$, we associated a triple $\hat{\varphi}(v) = (b \in B_0, f : b \to F(a), a \in A_0)$ by induction:

- if $a \in A_0$, $\hat{\varphi}(a) = (F(a), 1_{F(a)}, a)$;
- if $k \in K$ and $v \in TA_0$, $\hat{\varphi}(kv) = (kb, kf, ka)$ where $(b, f, a) = \hat{\varphi}(v)$;
- if $v_1, v_2 \in K$, $\hat{\varphi}(v_1 + v_2) = (b_1 + b_2, f_1 + f_2, a_1 + a_2)$ where $(b_i, f_i, a_i) = \hat{\varphi}(v_i)$ for $i \in \{1, 2\}$;
- if $v_1, v_2 \in K$, $\hat{\varphi}([v_1, v_2]) = ([b_1, b_2], F_2^{a_1,a_2}[f_1, f_2], [a_1, a_2])$ where $(b_i, f_i, a_i) = \hat{\varphi}(v_i)$ for $i \in \{1, 2\}$.

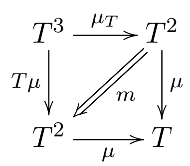
Let us recall that $F_2^{a_1,a_2} : [F(a_1), F(a_2)] \to F([a_1, a_2])$ is the isomorphism making (F, F_2) an homomorphism of groupoids in Lie_k (also called small strict Lie 2-algebras in [2]). Then, $\varphi(v) = f$ is the arrow part of this triple $\hat{\varphi}(v) = (b, f, a)$.

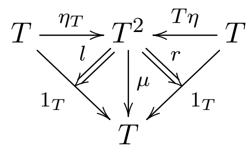
7 Pseudo-algebras

The aim of this section is to give an intuition where the axioms of Definition 3.1 of $\mathbb{T}$-monoidal functors come from. We will see that these axioms are actually particular cases of coherence axioms defining pseudo-morphisms between strict algebras for a pseudo-monad. We adopt the following definitions from [10] and [9].

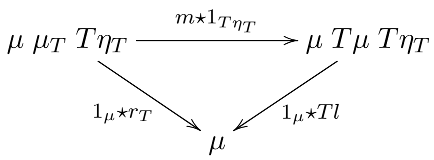
Definition 7.1. *Let $\mathcal{B}$ be a 2-category. A pseudo-monad $\mathcal{T}$ on $\mathcal{B}$ is the data of*

- *a 2-functor $T : \mathcal{B} \to \mathcal{B}$,*
- *two pseudo-natural transformations $\eta : 1_{\mathcal{B}} \to T$ and $\mu : T^2 \to T$,*
- *three isomodifications (modifications which are isomorphisms) m, l and r*

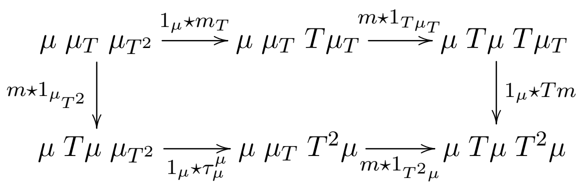




such that the two diagrams



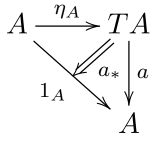
and

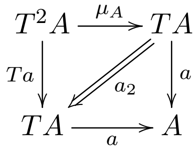


commute, where we denoted by τ^μ_f the isomorphism 2-cell $Tf\ \mu_A \to \mu_B\ T^2 f$ given by the pseudo-naturality of μ for all arrows $f : A \to B$ (and similarly for the pseudo-naturality of η).

Definition 7.2. *Let $\mathcal{B}$ be a 2-category and $\mathcal{T}$ a pseudo-monad on it. We define the 2-category Ps-Alg$(\mathcal{T})$ as follows:*

- *The objects are the pseudo-algebras of $\mathcal{T}$, i.e. quadruples (A, a, a_*, a_2) where A is an object of $\mathcal{B}$, $a : TA \to A$ is an arrow and a_* and a_2 are two invertible 2-cells*





such that

$$
\begin{array}{ccc}
a\,\mu_A\,T\eta_A & \xrightarrow{a_2\star 1_{T\eta_A}} & a\,Ta\,T\eta_A \\
& {\scriptstyle 1_a\star r_A}\searrow \quad \swarrow{\scriptstyle 1_a\star Ta_*} & \\
& a &
\end{array}
$$

and

$$
\begin{array}{ccccc}
a\,\mu_A\,\mu_{TA} & \xrightarrow{1_a\star m_A} & a\,\mu_A\,T\mu_A & \xrightarrow{a_2\star 1_{T\mu_A}} & a\,Ta\,T\mu_A \\
{\scriptstyle a_2\star 1_{\mu_{TA}}}\big\downarrow & & & & \big\downarrow{\scriptstyle 1_a\star Ta_2} \\
a\,Ta\,\mu_{TA} & \xrightarrow[1_a\star\tau^\mu_a]{} & a\,\mu_A\,T^2a & \xrightarrow[a_2\star 1_{T^2a}]{} & a\,Ta\,T^2a
\end{array}
$$

commute.

- *Arrows* $(A,a,a_*,a_2)\to(B,b,b_*,b_2)$ *are pseudo-morphisms of pseudo-algebras, i.e. pairs* (f,φ) *where* $f:A\to B$ *is an arrow and* φ *a 2-cell isomorphism*

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
{\scriptstyle a}\big\downarrow & \overset{\varphi}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle b} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

such that

$$
\begin{array}{ccc}
b\,Tf\,\eta_A & \xrightarrow{1_b\star\tau^\eta_f} & b\,\eta_B\,f \\
{\scriptstyle \varphi\star 1_{\eta_A}}\big\downarrow & & \big\downarrow{\scriptstyle b_*\star 1_f} \\
f\,a\,\eta_A & \xrightarrow[1_f\star a_*]{} & f
\end{array}
$$

and

$$
\begin{array}{ccccc}
b\,Tf\,\mu_A & \xrightarrow{\varphi\star 1_{\mu_A}} & f\,a\,\mu_A & \xrightarrow{1_f\star a_2} & f\,a\,Ta \\
{\scriptstyle 1_b\star\tau^\mu_f}\big\downarrow & & & & \big\uparrow{\scriptstyle \varphi\star 1_{Ta}} \\
b\,\mu_B\,T^2f & \xrightarrow[b_2\star 1_{T^2f}]{} & b\,Tb\,T^2f & \xrightarrow[1_b\star T\varphi]{} & b\,Tf\,Ta
\end{array}
$$

commute.

- *2-cells* $\alpha:(f,\varphi)\Rightarrow(g,\psi)$ *are 2-cells* $\alpha:f\Rightarrow g$ *such that*

$$
\begin{array}{ccc}
b\,Tf & \xrightarrow{1_b\star T\alpha} & b\,Tg \\
{\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
fa & \xrightarrow[\alpha\star 1_a]{} & ga
\end{array}
$$

commutes.

Similarly to Lemma 3.3, a pseudo-morphism (f,φ) is an equivalence in Ps-Alg$(\mathcal{T})$ if and only if f is an equivalence in $\mathcal{B}$.

We are now going to see how pseudo-morphisms are linked with $\mathbb{T}$-monoidal functors. Firstly, if $\mathcal{T}$ is a pseudo-monad, we define $\mathcal{T}$-MON to be the full sub-2-category of Ps-Alg$(\mathcal{T})$ whose objects are the strict algebras, i.e. the pseudo-algebras (A,a,a_*,a_2) with a_* and a_2 being identities. Moreover, we denote by Alg$(\mathcal{T})$ the sub-2-category of $\mathcal{T}$-MON in which we only consider strict morphisms of strict algebras, i.e. the pseudo-morphisms (f,φ) where φ is the identity.

$$
\mathrm{Alg}(\mathcal{T}) \hookrightarrow \mathcal{T}\text{-MON} \hookrightarrow \text{Ps-Alg}(\mathcal{T})
$$

Now, suppose $\mathbb{T}$ is a (1-dimensional) monad on a category $\mathcal{C}$ with pullbacks. If $T\colon \mathcal{C} \to \mathcal{C}$ preserves them, then $\mathbb{T}$ induces a pseudo-monad $\mathcal{T}$ on the 2-category $\mathrm{Grpd}(\mathcal{C})$. Moreover, $\mathcal{T}$ is such that η and μ are 2-natural transformations, given by $\eta_{\mathbb{B}} = (\eta_{B_0}, \eta_{B_1})$ and $\mu_{\mathbb{B}} = (\mu_{B_0}, \mu_{B_1})$ for all $\mathbb{B} \in \mathrm{Grpd}(\mathcal{C})$. For this pseudo-monad, we know that the modifications m, l and r are identities and that the two coherence axioms become trivial. Moreover, we have an isomorphism of 2-categories

$$\begin{aligned} \mathrm{Grpd}(\mathcal{C}^{\mathbb{T}}) &\longrightarrow \mathrm{Alg}(\mathcal{T}) \\ \mathbb{A} &\longmapsto (U\mathbb{A}, a = (a_0, a_1)) \\ F &\longmapsto UF \\ \alpha &\longmapsto U\alpha \end{aligned}$$

where $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is the forgetful functor and $a_i : TA_i \to A_i$ are the $\mathbb{T}$-algebra structures for $i \in \{0, 1\}$. Note that $a = (a_0, a_1)$ is an internal functor since d, c, e, m and i are $\mathbb{T}$-algebra homomorphisms. Notice also that the fact that $U\alpha$ satisfies the coherence axiom for the definition of 2-cells in $\mathrm{Ps\text{-}Alg}(\mathcal{T})$ corresponds to the fact that α is a $\mathbb{T}$-algebra homomorphism.

With this particular pseudo-monad on $\mathrm{Grpd}(\mathcal{C})$, we remark that, if we extend this isomorphism, $\mathcal{T}$-MON becomes the following 2-category:

- Objects are internal groupoids in $\mathcal{C}^{\mathbb{T}}$.
- A 1-cell $(F, \varphi) : \mathbb{A} \to \mathbb{B}$ is the data of a functor $F : U\mathbb{A} \to U\mathbb{B}$ in $\mathcal{C}$ together with an internal natural isomorphism

$$\begin{array}{ccc} TU\mathbb{A} & \xrightarrow{TF} & TU\mathbb{B} \\ {\scriptstyle a}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \varphi} & \downarrow{\scriptstyle b} \\ U\mathbb{A} & \xrightarrow[F]{} & U\mathbb{B} \end{array}$$

  in $\mathcal{C}$ such that $\varphi \star 1_{\eta_{U\mathbb{A}}} = 1_F$ and $\varphi \star 1_{\mu_{U\mathbb{A}}} = (\varphi \star 1_{Ta})(1_b \star T\varphi)$.

$$\begin{array}{ccc} b\,TF\,\eta_{U\mathbb{A}} & = & b\,\eta_{U\mathbb{B}}\,F \\ {\scriptstyle \varphi\star 1_{\eta_{U\mathbb{A}}}}\downarrow & & \| \\ F\,a\,\eta_{U\mathbb{A}} & = & F \end{array} \qquad \begin{array}{ccccc} b\,TF\,\mu_{U\mathbb{A}} & \xrightarrow{\varphi\star 1_{\mu_{U\mathbb{A}}}} & F\,a\,\mu_{U\mathbb{A}} & = & F\,a\,Ta \\ \| & & & & \uparrow{\scriptstyle \varphi\star 1_{Ta}} \\ b\,\mu_{U\mathbb{B}}\,T^2F & = & b\,Tb\,T^2F & \xrightarrow[1_b\star T\varphi]{} & b\,TF\,Ta \end{array}$$

- A 2-cell $\alpha\colon (F, \varphi) \Rightarrow (G, \psi)\colon \mathbb{A} \to \mathbb{B}$ is an internal natural transformation $\alpha\colon F \Rightarrow G$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} b\,TF & \xrightarrow{1_b\star T\alpha} & b\,TG \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ F\,a & \xrightarrow[\alpha\star 1_a]{} & G\,a \end{array}$$

  commutes.

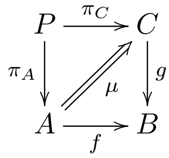
Now, we notice that these are exactly the axioms of the Definition 3.1 of $\mathbb{T}$-MON. Indeed, the first three axioms defining a $\mathbb{T}$-monoidal functor are the fact that the 2-cell $\varphi\colon bTF \Rightarrow F a$ is an internal natural transformation in $\mathcal{C}$ while the last two are the above ones. In other words, if $\mathcal{T}$ is the pseudo-monad on $\mathrm{Grpd}(\mathcal{C})$ induced by a pullback preserving monad $\mathbb{T}$ on $\mathcal{C}$, the 2-categories $\mathcal{T}$-MON and $\mathbb{T}$-MON coincide. What makes it possible to define $\mathbb{T}$-MON even if the monad $\mathbb{T}$ does not preserve pullbacks is the fact that, to express the naturality of $\varphi : bTF \Rightarrow Fa : TU\mathbb{A} \to U\mathbb{B}$, one needs only the composition in the codomain category $U\mathbb{B}$ and not in the domain $TU\mathbb{A}$.

Analogously to Proposition 4.6, we are going to prove that, under some hypothesis,

$$\mathrm{Alg}(\mathcal{T}) \hookrightarrow \mathcal{T}\text{-MON}$$

is the bicategory of fractions of $\mathrm{Alg}(\mathcal{T})$ with respect to a certain class of 1-cells Σ.

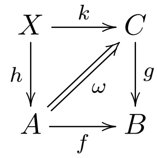
Definition 7.3. *Let $\mathcal{B}$ be a 2-category where every 2-cell is invertible. We say that the diagram*



(1)

is a strong homotopy-pullback of f and g if

1. *for all diagrams*

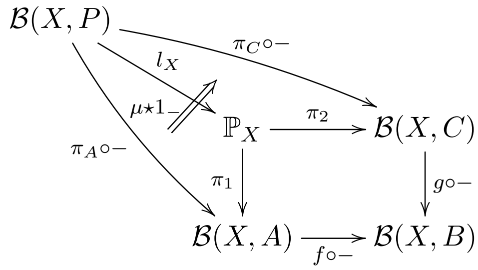


   there exists a unique 1-cell $l : X \to P$ such that $\pi_A l = h$, $\pi_C l = k$ and $\mu \star 1_l = \omega$;

2. *for all 1-cells $l, l' : X \to P$ and for all 2-cells $\alpha : \pi_A l \Rightarrow \pi_A l'$ and $\beta : \pi_C l \Rightarrow \pi_C l'$ such that $(1_g \star \beta)(\mu \star 1_l) = (\mu \star 1_{l'})(1_f \star \alpha)$, there exists a unique 2-cell $\gamma : l \Rightarrow l'$ such that $1_{\pi_A} \star \gamma = \alpha$ and $1_{\pi_C} \star \gamma = \beta$.*

(When only condition 1 is satisfied, P is usually called homotopy-pullback, compare for example with [8].)

Remark 7.4. Here is another way to understand Definition 7.3. If we have a diagram as (1) in such a 2-category $\mathcal{B}$, we can construct, for all objects $X \in \mathcal{B}$, the following diagram in Grpd(Set)

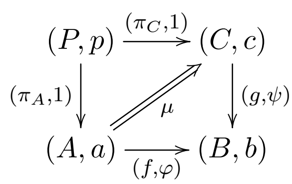


where $\mathbb{P}_X$ is the bipullback of groupoids and l_X the factorisation as constructed in Remark 3.3 in [14]. Then, we have a characterisation of bipullbacks and strong homotopy-pullbacks:

1. The diagram (1) is a bipullback if and only if $l_X : \mathcal{B}(X, P) \to \mathbb{P}_X$ is an equivalence of categories for all objects $X \in \mathcal{B}$.
2. The diagram (1) is a strong homotopy-pullback if and only if $l_X : \mathcal{B}(X, P) \to \mathbb{P}_X$ is an isomorphism of categories for all objects $X \in \mathcal{B}$.

The following lemma is analogous to Lemma 4.5.

Lemma 7.5. *Let $\mathcal{B}$ be a 2-category where every 2-cell is invertible and $\mathcal{T}$ a pseudo-monad on $\mathcal{B}$ such that η and μ are 2-natural transformations. If $\mathcal{B}$ has strong homotopy-pullbacks, so has $\mathcal{T}$-MON. Moreover, given pseudo-morphisms of strict algebras $(f, \varphi) : (A, a) \to (B, b)$ and $(g, \psi) : (C, c) \to (B, b)$, it is possible to choose a strong homotopy-pullback of (f, φ) and (g, ψ)*



in such a way that $(\pi_A, 1)$ and $(\pi_C, 1)$ are strict morphisms.

Proof. Consider the strong homotopy-pullback

$$\begin{array}{ccc} P & \xrightarrow{\pi_C} & C \\ {\scriptstyle \pi_A}\downarrow & \overset{\mu}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

in $\mathcal{B}$. There exists a unique $p : TP \to P$ such that $\pi_A p = aT\pi_A$, $\pi_C p = cT\pi_C$ and $(\mu \star 1_p)(\varphi \star 1_{T\pi_A}) = (\psi \star 1_{T\pi_C})(1_b \star T\mu)$. Is is routine to check that this makes (P, p) a strict algebra and that we have constructed the announced strong homotopy-pullback. □

As for Proposition 4.6, this lemma is the key point to prove next proposition.

Proposition 7.6. *Let $\mathcal{B}$ be a 2-category where every 2-cell is invertible and which has strong homotopy-pullbacks. Let also $\mathcal{T}$ be a pseudo-monad on $\mathcal{B}$ such that η and μ are 2-natural transformations. If Σ is the class of 1-cells $(f, 1)$ of Alg($\mathcal{T}$) such that f is an equivalence in $\mathcal{B}$, then,*

$$\mathit{Alg}(\mathcal{T}) \hookrightarrow \mathcal{T}\text{-}\mathit{MON}$$

is the bicategory of fractions of Alg($\mathcal{T}$) with respect to Σ.

Proof. We know that Σ has a right calculus of fractions since it is a bipullback congruence (see Definition 5.1 and Proposition 5.2 in [14]). The rest of the proof is similar to the one of Proposition 4.6 using Lemma 7.5. □

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