ON FIBRATIONS BETWEEN INTERNAL GROUPOIDS AND THEIR NORMALIZATIONS

P.-A. JACQMIN, S. MANTOVANI, G. METERE, E.M. VITALE

ABSTRACT. We characterize fibrations and $*$-fibrations in the 2-category of internal groupoids in terms of the comparison functor from certain pullbacks to the corresponding strong homotopy pullbacks. As an application, we deduce the internal version of the Brown exact sequence for $*$-fibrations from the internal version of the Gabriel-Zisman exact sequence. We also analyse fibrations and $*$-fibrations in the category of arrows and study when the normalization functor preserves and reflects them. This analysis allows us to give a characterization of protomodular categories using strong homotopy kernels and a generalization of the Snake Lemma.

Contents

1. Introduction

It is well-known that a functor $F: \mathbb{A} \to \mathbb{B}$ between groupoids is a fibration if for every object A in $\mathbb{A}$ and every morphism $g: B \to F(A)$ in $\mathbb{B}$, there is a morphism $f: A' \to A$ in $\mathbb{A}$ such that $F(f) = g$. Since we are considering groupoids, this is equivalent to the dual notion of opfibration. If $\mathcal{A}$ is a regular category, we can internalize this concept (see for instance [6]) in order to get the notion of a fibration $F: \mathbb{A} \to \mathbb{B}$ between internal groupoids $\mathbb{A}$ and $\mathbb{B}$ in $\mathcal{A}$. The first aim of this paper is to characterize such internal fibrations using strong homotopy pullbacks in the 2-category $\mathbf{Grpd}(\mathcal{A})$ of internal groupoids in $\mathcal{A}$. More specifically, let T be the comparison between the pullback and the strong homotopy pullback of F along the embedding of B_0 (the object of objects of $\mathbb{B}$) into $\mathbb{B}$; we show that F is a fibration if and only if this T is a weak equivalence.

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The weaker notion of a $*$-fibration $F\colon \mathbb{A} \to \mathbb{B}$ makes sense when $\mathbb{A}$ and $\mathbb{B}$ are pointed groupoids, or more generally, internal groupoids in a pointed regular category $\mathcal{A}$. The idea is that now only maps $g\colon 0 \to F(A)$ are required to have a lifting. In a similar way than for fibrations, we characterize $*$-fibrations as functors F for which the comparison map $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ from the kernel to the strong homotopy kernel of F is a weak equivalence. In particular, if $\mathbb{K}(F)$ is proper, this implies that $\pi_0(J)$ and $\pi_1(J)$ are isomorphisms, where π_0 and π_1 are respectively the connected components functor and the automorphisms functor (see [6]). This fact has an interesting application: given a functor $F\colon \mathbb{A} \to \mathbb{B}$ between internal groupoids in a pointed regular category $\mathcal{A}$ with reflexive coequalizers such that $\mathbb{K}(F)$, $\mathbb{A}$ and $\mathbb{B}$ are proper, the (internal version of the) Gabriel-Zisman sequence [7, 14] is an exact sequence

$$\pi_1(\mathbb{K}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

involving the π_0 and π_1's of the strong homotopy kernel, domain and codomain of F. If moreover F is assumed to be a fibration (or merely a $*$-fibration), the isomorphisms $\pi_0(J)$ and $\pi_1(J)$ allow us to deduce from it the Brown exact sequence [4, 14]

$$\pi_1(\mathbb{K}\mathrm{er}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}\mathrm{er}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

involving now the kernel of F. The internal version of the Brown sequence was established from the Gabriel-Zisman sequence in [14] up to the proof that J is a weak equivalence, which is now done here.

The normalization process, introduced in [2], has been widely studied. For instance, it is well-known that in a semi-abelian context, it induces an equivalence between $\mathbf{Grpd}(\mathcal{A})$ and the category $\mathbf{XMod}(\mathcal{A})$ of internal crossed-modules [11]. It is also known that the normalized version of Brown and Gabriel-Zisman sequences are, respectively, the snake and snail sequences, studied in the context of pointed regular protomodular categories in [2, 18, 12]. For a finitely complete pointed category $\mathcal{A}$, normalization induces a functor

$$\mathcal{N}\colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$$

to the category (with null-homotopies) of arrows in $\mathcal{A}$. In the last section of the paper, we study the appropriate notions of fibrations, $*$-fibrations and weak equivalences in $\mathbf{Arr}(\mathcal{A})$. In particular, $*$-fibrations are defined as morphisms (f, f_0) for which the comparison map $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ from the kernel to the strong homotopy kernel of (f, f_0) is a weak equivalence. Since the notion of strong homotopy pullbacks in general is not available in $\mathbf{Arr}(\mathcal{A})$, we cannot use a similar definition for fibrations. We rather define them, in a pointed regular context, as morphisms (f, f_0) for which f is a regular epimorphism, inspired by the corresponding condition in the Snake Lemma. We then show that, under some conditions on $\mathcal{A}$, the normalization functor $\mathcal{N}$ preserves and reflects these notions. We also prove that for a pointed regular category $\mathcal{A}$, the expected implication "fibration $\Rightarrow$ $*$-fibration" is in fact equivalent to the condition that $\mathcal{A}$ is protomodular (see [1]). Finally, we show that for a regular pointed protomodular category

$\mathcal{A}$ with cokernels, a weak equivalence $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$ with b proper induces isomorphisms $\mathrm{K}(f, f_0)\colon \mathrm{Ker}(a) \to \mathrm{Ker}(b)$ and $\mathrm{C}(f, f_0)\colon \mathrm{Coker}(a) \to \mathrm{Coker}(b)$. From that fact, we show that to obtain the Snake Lemma from the Snail Lemma [18], one only needs to consider a $*$-fibration in $\mathbf{Arr}(\mathcal{A})$ instead of a fibration as classically stated.

For the reader's convenience, we recall in full detail all the notions of strong homotopy pullback, strong homotopy kernel, fibration and $*$-fibration involved in the paper, both in $\mathbf{Grpd}(\mathcal{A})$ and in $\mathbf{Arr}(\mathcal{A})$. This is because the terminology is not so well established in the literature and we want to avoid any possible confusion. Finally, the proof style deserves a comment. On one hand, some of our results can be proved using embedding theorems for categories with finite limits or for regular categories; on the other hand, we are interested in providing direct proofs using the internal approach. As a compromise, we decided to give a quite complete diagrammatic proof of Proposition 3.2 and to omit several other proofs.

Notation: the composition of two arrows

$$\xrightarrow{\ f\ }\ \xrightarrow{\ g\ }$$

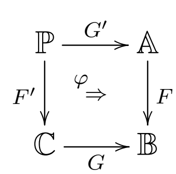
will be denoted by $f \cdot g$.

2. Strong pullbacks and strong h-pullbacks

2.1. . Let $\underline{\mathcal{B}}$ be a 2-category with invertible 2-cells, and $\mathcal{B}$ its underlying category. We adopt the following terminology:
1. A 1-cell $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{\mathcal{B}}$ is *fully faithful* if, for any $\mathbb{X}$ in $\underline{\mathcal{B}}$, the induced functor

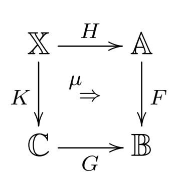
$$- \cdot F\colon \underline{\mathcal{B}}(\mathbb{X}, \mathbb{A}) \to \underline{\mathcal{B}}(\mathbb{X}, \mathbb{B})$$

is fully faithful in the usual sense.
2. Consider 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in $\underline{\mathcal{B}}$. A *strong homotopy pullback* (strong h-pullback, for short) of F and G is a diagram of the form



satisfying the following universal property:

(a) For any diagram of the form



there exists a unique 1-cell $T\colon \mathbb{X} \to \mathbb{P}$ such that $T \cdot G' = H, T \cdot F' = K$ and $T \cdot \varphi = \mu$.

(b) Given 1-cells $L, M\colon \mathbb{X} \rightrightarrows \mathbb{P}$ and 2-cells $\alpha\colon L \cdot F' \Rightarrow M \cdot F'$ and $\beta\colon L \cdot G' \Rightarrow M \cdot G'$, if

$$\begin{array}{ccc} L \cdot F' \cdot G & \overset{\alpha \cdot G}{\Longrightarrow} & M \cdot F' \cdot G \\ {\scriptstyle L\cdot\varphi}\Downarrow & & \Downarrow{\scriptstyle M\cdot\varphi} \\ L \cdot G' \cdot F & \underset{\beta \cdot F}{\Longrightarrow} & M \cdot G' \cdot F \end{array}$$

commutes, then there exists a unique 2-cell $\mu\colon L \Rightarrow M$ such that $\mu \cdot F' = \alpha$ and $\mu \cdot G' = \beta$.

3. We say that a pullback $\mathbb{C} \times_{G,F} \mathbb{A}$ of $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in the category $\mathcal{B}$ is *strong* (in $\underline{\mathcal{B}}$) if

$$\begin{array}{ccc} \mathbb{C} \times_{G,F} \mathbb{A} & \xrightarrow{\widehat{G}} & \mathbb{A} \\ {\scriptstyle \widehat{F}}\downarrow & \overset{\mathrm{id}}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathbb{C} & \underset{G}{\longrightarrow} & \mathbb{B} \end{array}$$

satisfies condition (b) above.

2.2. . Another way to express the universal property of the strong h-pullback is first to fix an object $\mathbb{X}$ in $\underline{\mathcal{B}}$ and to construct the comma-square of groupoids (which is precisely the strong h-pullback in the 2-category of groupoids).

$$\begin{array}{ccc} (- \cdot G \downarrow - \cdot F) & \longrightarrow & \underline{\mathcal{B}}(\mathbb{X}, \mathbb{A}) \\ \downarrow & \cong & \downarrow{\scriptstyle - \cdot F} \\ \underline{\mathcal{B}}(\mathbb{X}, \mathbb{C}) & \underset{- \cdot G}{\longrightarrow} & \underline{\mathcal{B}}(\mathbb{X}, \mathbb{B}) \end{array}$$

Then the universal property of the strong h-pullback means that, for any $\mathbb{X}$, the canonical comparison functor

$$\underline{\mathcal{B}}(\mathbb{X}, \mathbb{P}) \to (- \cdot G \downarrow - \cdot F)$$

is bijective on objects (condition a) and fully faithful (condition b), that is, it is an isomorphism of categories. This makes evident that a strong h-pullback is determined by its universal property up to isomorphism.

A weaker universal property consists in asking that the canonical comparison functors $\underline{\mathcal{B}}(\mathbb{X}, \mathbb{P}) \to (- \cdot G \downarrow - \cdot F)$ are equivalences of groupoids. In this way one gets what is sometimes called a *bipullback*, which is determined only up to equivalence.

Intermediate situations between strong homotopy pullbacks and bipullbacks are considered in the literature. For example, in [8] the comparison functors are required to be bijective on objects but not fully faithful (the name of h-pullback is used in this case), and in [7] the comparison functors are required to be surjective on objects and full.

2.3. . Among strong h-pullbacks, the following one, already appearing in [10], plays a special role.

$$\begin{array}{ccc} \vec{\mathbb{B}} & \xrightarrow{\gamma} & \mathbb{B} \\ {\scriptstyle\delta}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathbb{B} & \xrightarrow[\mathrm{Id}]{} & \mathbb{B} \end{array}$$

Indeed, if for a category $\mathcal{A}$ we denote by $\mathbf{Arr}(\mathcal{A})$ the category having arrows of $\mathcal{A}$ as objects and commutative squares as arrows, then the universal property of $\vec{\mathbb{B}}$ gives an isomorphism of categories

$$\underline{\mathcal{B}}(\mathbb{X}, \vec{\mathbb{B}}) \to \mathbf{Arr}(\underline{\mathcal{B}}(\mathbb{X}, \mathbb{B}))$$

so that to give a 1-cell $\mathbb{X} \to \vec{\mathbb{B}}$ is the same as giving a 2-cell $\mathbb{X} \underset{\to}{\overset{\to}{\Downarrow}} \mathbb{B}$. We refer the reader to Section 5 for a more detailed treatment of the category $\mathbf{Arr}(\mathcal{A})$.

2.4. . Pasting together two strong h-pullbacks, in general one does not get a strong h-pullback. The main interest of the notion of strong pullback relies on the following fact: given a diagram in $\underline{\mathcal{B}}$ of the form

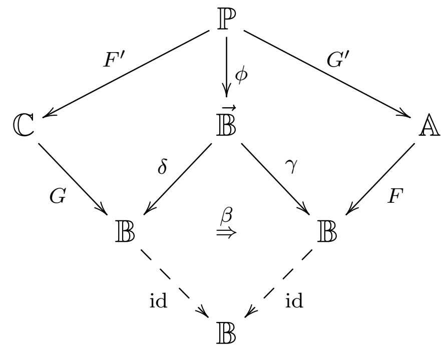
$$\begin{array}{ccccc} \mathbb{P}' & \xrightarrow{\widehat{H}} & \mathbb{P} & \xrightarrow{G'} & \mathbb{A} \\ {\scriptstyle\widehat{F'}}\downarrow & \overset{\mathrm{id}}{\Rightarrow} & \downarrow{\scriptstyle F'} & \overset{\varphi}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathbb{D} & \xrightarrow[H]{} & \mathbb{C} & \xrightarrow[G]{} & \mathbb{B} \end{array}$$

if the right-hand part is a strong h-pullback, then the total diagram is a strong h-pullback if and only if the left-hand part is a strong pullback.

This fact has an interesting consequence on the existence of strong h-pullbacks: Assume that $\underline{\mathcal{B}}$ has strong pullbacks and that the strong h-pullback

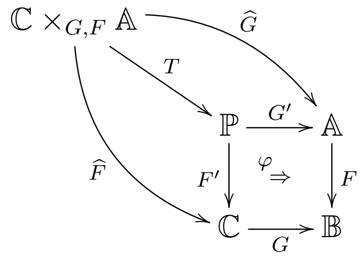
$$\begin{array}{ccc} \vec{\mathbb{B}} & \xrightarrow{\gamma} & \mathbb{B} \\ {\scriptstyle\delta}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathbb{B} & \xrightarrow[\mathrm{Id}]{} & \mathbb{B} \end{array}$$

exists in $\underline{\mathcal{B}}$ for any object $\mathbb{B}$. Then, for any pair of 1-cells $F\colon \mathbb{A} \to \mathbb{B}, G\colon \mathbb{C} \to \mathbb{B}$, a strong h-pullback of F and G exists and can be obtained by the following limit of solid arrows in $\mathcal{B}$



together with $\varphi = \phi \cdot \beta \colon F' \cdot G = \phi \cdot \delta \Rightarrow \phi \cdot \gamma = G' \cdot F$, see [10].

2.5. . Later, we will use the following fact on strong pullbacks and strong h-pullbacks: if the pullback in $\mathcal{B}$ and the strong h-pullback in $\underline{\mathcal{B}}$ of F and G exist,

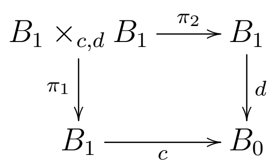


then the pullback $\mathbb{C} \times_{G,F} \mathbb{A}$ is strong if and only if the canonical comparison T is fully faithful.

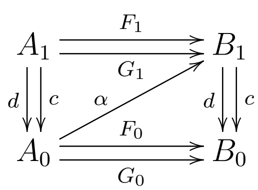
2.6. . Now we specialize the previous discussion taking as $\underline{\mathcal{B}}$ the 2-category $\mathbf{Grpd}(\mathcal{A})$ of groupoids, functors and natural transformations internal to a category $\mathcal{A}$ with pullbacks. The notation for a groupoid $\mathbb{B}$ in $\mathcal{A}$ is

$$\mathbb{B} = (\ B_1 \times_{c,d} B_1 \xrightarrow{m} B_1 \underset{c}{\overset{d}{\rightrightarrows}} B_0 \ , \quad B_1 \xrightarrow{i} B_1\ )$$

where



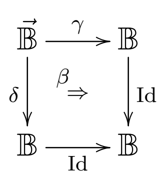
is a pullback. The notation for a natural transformation $\alpha \colon F \Rightarrow G \colon \mathbb{A} \rightrightarrows \mathbb{B}$ is



2.7. . Following 2.4, to prove that $\mathbf{Grpd}(\mathcal{A})$ has strong h-pullbacks we need two ingredients. The first one is easy, the second one is the standard construction of the groupoid of 'arrows', and we recall it from [17] or [15].

1. Since pullbacks in $\mathbf{Grpd}(\mathcal{A})$ are constructed level-wise, it is straightforward to check that they are strong.

2. For every internal groupoid $\mathbb{B}$, the strong h-pullback



exists, and it can be described as follows:

$$\vec{\mathbb{B}} = (\ \vec{B}_1 \times_{\vec{c},\vec{d}} \vec{B}_1 \xrightarrow{\vec{m}} \vec{B}_1 \overset{\vec{d}}{\underset{\vec{c}}{\rightrightarrows}} B_1 \ (\text{with } \vec{e}\colon B_1 \to \vec{B}_1)\ ,\quad \vec{B}_1 \xrightarrow{\vec{i}} \vec{B}_1\ )$$

where

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{m_2} & B_1 \times_{c,d} B_1 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

is a pullback, and $\vec{d}$, $\vec{c}$ and $\vec{e}$ are defined by

$$\vec{d}\colon\ \vec{B}_1 \xrightarrow{m_1} B_1 \times_{c,d} B_1 \xrightarrow{\pi_1} B_1 \qquad \vec{c}\colon\ \vec{B}_1 \xrightarrow{m_2} B_1 \times_{c,d} B_1 \xrightarrow{\pi_2} B_1$$

$$\begin{array}{l} B_1 \xrightarrow{\vec{e}} \vec{B}_1, \quad m_2 \cdot \vec{e} = \langle d\cdot e, \mathrm{id}\rangle\colon B_1 \to B_1 \times_{c,d} B_1, \quad m_1 \cdot \vec{e} = \langle \mathrm{id}, c\cdot e\rangle\colon B_1 \to B_1 \times_{c,d} B_1 \\[4pt] \begin{array}{ccc} \vec{B}_1 & \xrightarrow{m_2} & B_1 \times_{c,d} B_1 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array} \end{array}$$

The groupoid $\vec{\mathbb{B}}$ is equipped with two functors $\delta\colon \vec{\mathbb{B}} \to \mathbb{B}$ and $\gamma\colon \vec{\mathbb{B}} \to \mathbb{B}$ given by

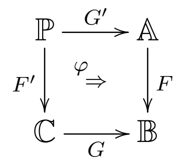
$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\delta_1 = m_2\cdot\pi_1} & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\delta_0 = d]{} & B_0 \end{array} \qquad \begin{array}{ccc} \vec{B}_1 & \xrightarrow{\gamma_1 = m_1\cdot\pi_2} & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\gamma_0 = c]{} & B_0 \end{array}$$

Finally, the natural transformation $\beta\colon \delta \Rightarrow \gamma$ is simply $\beta = \mathrm{id}_{B_1}\colon B_1 \to B_1$.
To help intuition, let us point out that when $\mathcal{A}$ is the category of sets, an element S of the object $\vec{B}_1$ involved in the above description of the strong h-pullback is a commutative square

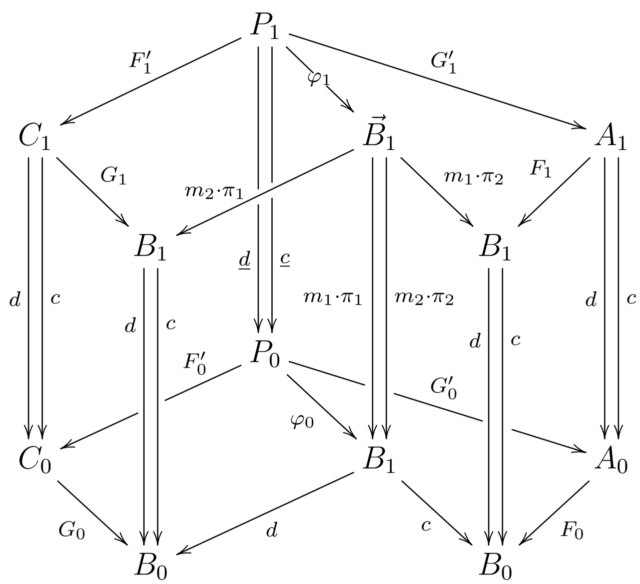
$$\begin{array}{ccc} \cdot & \xrightarrow{g_0} & \cdot \\ {\scriptstyle b_1}\downarrow & S & \downarrow{\scriptstyle b_2} \\ \cdot & \xrightarrow[f_0]{} & \cdot \end{array}$$

with $m_1(S) = \langle b_1, f_0\rangle, m_2(S) = \langle g_0, b_2\rangle, \vec{d}(S) = b_1, \vec{c}(S) = b_2, \delta_1(S) = g_0, \gamma_1(S) = f_0$.

2.8. . Putting together 2.4 and 2.7, we can conclude that the 2-category **Grpd**$(\mathcal{A})$ has strong h-pullbacks. Moreover, a strong h-pullback

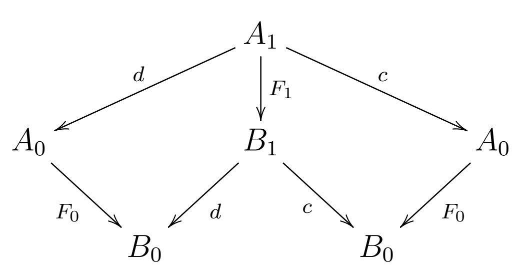


of $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in **Grpd**$(\mathcal{A})$ is described by the following diagram in $\mathcal{A}$, where the top and bottom faces are limit diagrams:



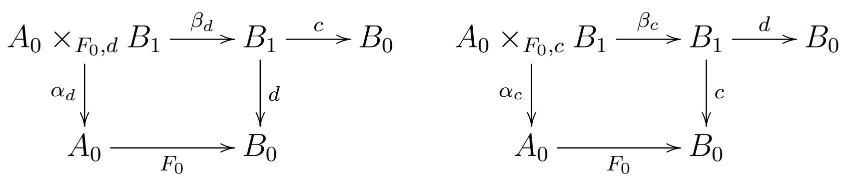
2.9. . In **Grpd**$(\mathcal{A})$, as in any 2-category, the notion of equivalence makes sense. Moreover, in **Grpd**$(\mathcal{A})$ we have also available the notion of weak equivalence. From [5, 17], recall that a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$ is:

1. fully faithful if and only if the following diagram is a limit diagram;



2. an equivalence if and only if it is fully faithful and, moreover, the first row in one (equivalently, in both) of the following diagrams is a split epimorphism (the squares

are pullbacks).



The functors δ and γ defined in 2.7 are examples of equivalences.

If $\mathcal{A}$ is a regular category, one can say that a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$ is:

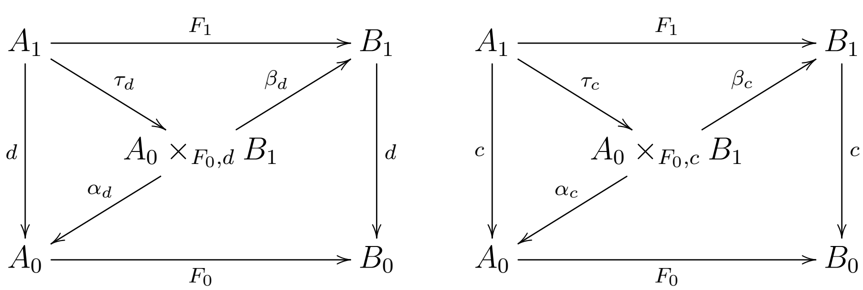
3. *essentially surjective* if $\beta_d \cdot c$ (equivalently $\beta_c \cdot d$) of the above diagrams is a regular epimorphism;

4. a *weak equivalence* if it is fully faithful, and essentially surjective.

3. Fibrations and strong h-pullbacks

In this section, we assume that the base category $\mathcal{A}$ is regular.

3.1. . Let us recall the terminology for fibrations (= opfibrations) between groupoids (compare with [6, Definition 5.1] for the notion of $\mathcal{E}$-fibrations between internal categories, w.r.t. a class $\mathcal{E}$ of morphisms of $\mathcal{A}$). Consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the induced factorizations through the pullbacks as in the following diagrams.



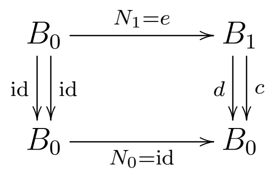
1. F is a *fibration* when τ_d (equivalently, τ_c) is a regular epimorphism.

2. F is a *split epi fibration* when τ_d (equivalently, τ_c) is a split epimorphism.

3. F is a *discrete fibration* when τ_d (equivalently, τ_c) is an isomorphism.

In [16], Street defined 0-fibrations in the more general context of a representable 2-category. As Chevalley criterion [9, 13, 16], he characterized 0-fibrations as those F for which the canonical functor $S\colon \vec{\mathbb{A}} \to (F \downarrow \mathbb{B})$ (where $(F \downarrow \mathbb{B})$ is the comma object of F over $\mathbb{B}$, which coincides with the strong h-pullback of F and $\mathrm{Id}_{\mathbb{B}}$ in $\mathbf{Grpd}(\mathcal{A})$) has a left adjoint weak right inverse, i.e., the unit of the adjunction is an isomorphism. One can

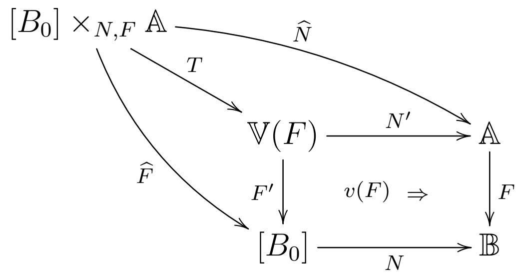
show that a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$ is a split epi fibration if and only if the comparison functor S has a left adjoint right inverse, i.e., the unit of the adjunction is an identity. Therefore, our notion of split epi fibration is a bit stronger than Street's notion of 0-fibrations.
In the next characterization of fibrations and split epi fibrations, we use the canonical embedding $N\colon [B_0] \to \mathbb{B}$ of the discrete groupoid of objects of $\mathbb{B}$ into $\mathbb{B}$. Explicitly,



The discrete groupoid 2-functor is denoted as

$$[\ \ ]\colon \mathcal{A} \to \mathbf{Grpd}(\mathcal{A}).$$

3.2. Proposition. *Consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the comparison functor T from the pullback to the strong h-pullback, as in the following diagram.*



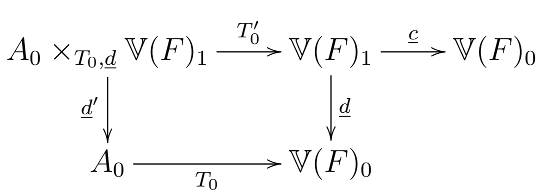
1. *F is a fibration if and only if T is a weak equivalence.*
2. *F is a split epi fibration if and only if T is an equivalence.*

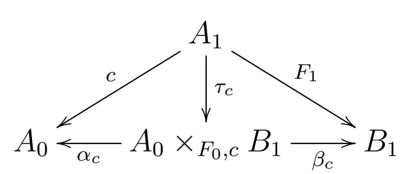
In fact, we are going to prove a more precise statement: the arrow attesting that T is essentially surjective is the same arrow τ_c attesting that F is a fibration.
(To help intuition, it is worth providing a description of the groupoid $\mathbb{V}(F)$ when the base category $\mathcal{A}$ is the category of sets. In this case, one has

$$\mathbb{V}(F) = \coprod_{b \in \mathbb{B}} (b \downarrow F)$$

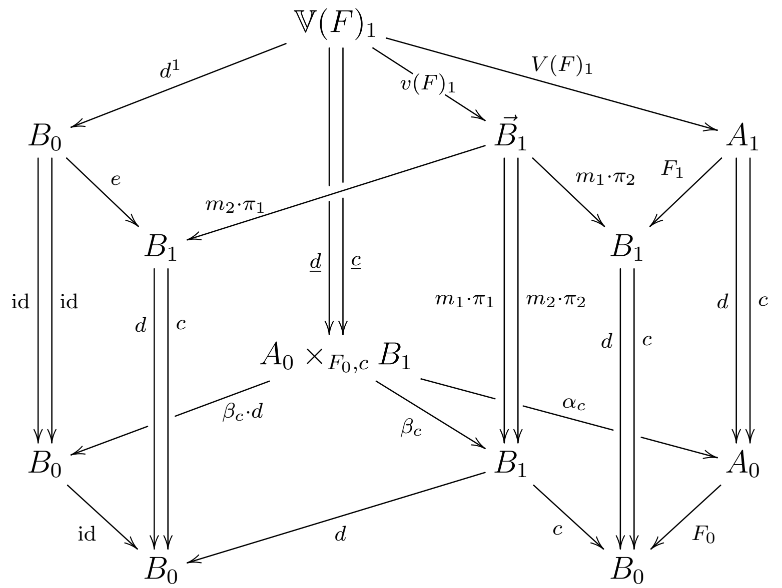
i.e., the disjoint union of the comma categories $(b \downarrow F)$.)

Proof. Since pullbacks in $\mathbf{Grpd}(\mathcal{A})$ are strong (2.7), we can apply 2.5 and we know that T is fully faithful. Now we have to compare $T_0' \cdot \underline{c}$ with τ_c.

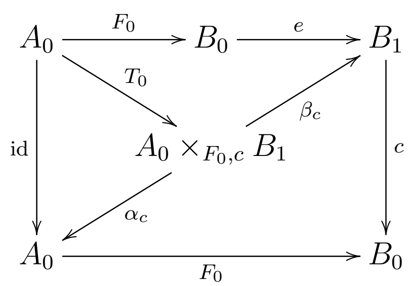




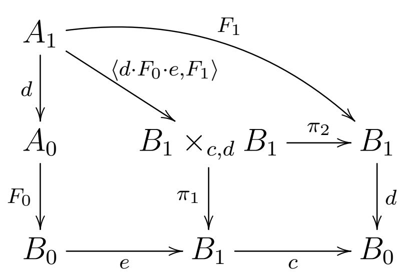
The diagram giving the strong h-pullback $\mathbb{V}(F)$ is

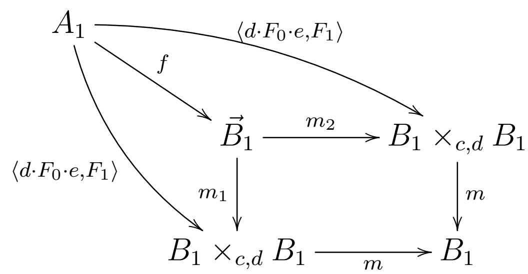


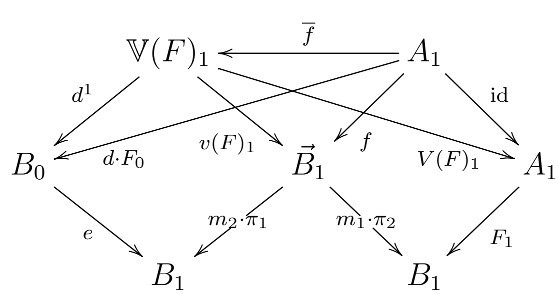
so that T_0 is the factorization through the pullback as in the following diagram.



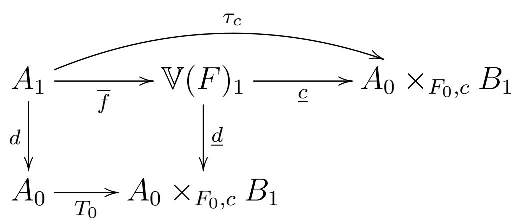
Now we construct an arrow $\overline{f} \colon A_1 \to \mathbb{V}(F)_1$ in three steps:



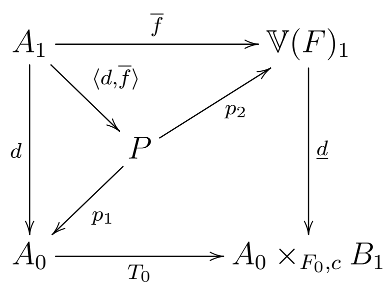




Finally, we get the following diagram



and we have to check that it is commutative and that the square is a pullback. Once this done, the commutativity of the upper region immediately gives both statements of the proposition. We construct an isomorphism between A_1 and the pullback of T_0 and $\underline{d}$ and we leave to the reader to check the various commutativities. The needed isomorphism is given by the factorization of the previous square through the pullback

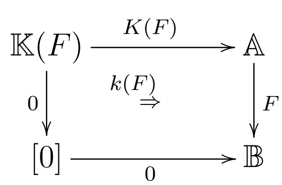


and by the arrow $p_2 \cdot V(F)_1 \colon P \to \mathbb{V}(F)_1 \to A_1$. ∎

4. $*$-Fibrations and strong h-kernels

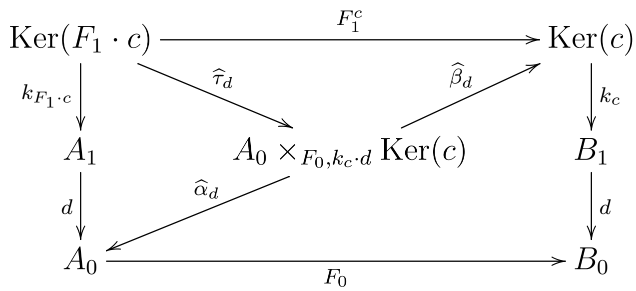
In this section, we assume that the base category $\mathcal{A}$ is regular and pointed.

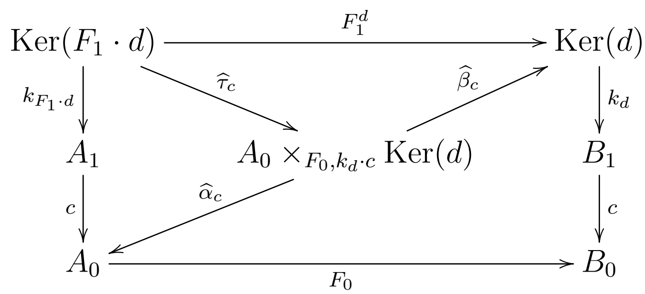
4.1. . Since $\mathcal{A}$ is pointed, as a special case of 2.8 we get a description of the strong h-kernel of a functor $F \colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$.



Now we introduce $*$-fibrations and split epi $*$-fibrations. Consider a functor $F \colon \mathbb{A} \to \mathbb{B}$

between groupoids in $\mathcal{A}$, and the induced factorizations $\widehat{\tau}_d$ and $\widehat{\tau}_c$ through the pullbacks

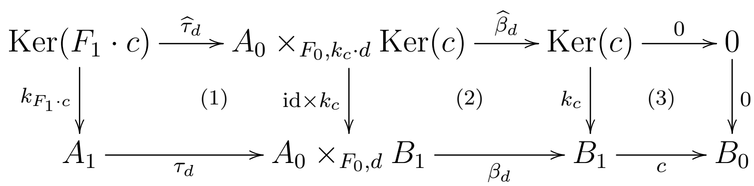




where F_1^c and F_1^d are determined by the conditions $F_1^c \cdot k_c = k_{F_1 \cdot c} \cdot F_1$ and $F_1^d \cdot k_d = k_{F_1 \cdot d} \cdot F_1$.

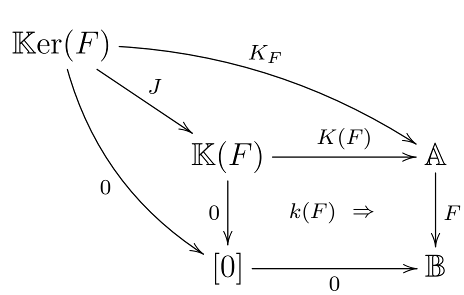
1. F is a $*$-*fibration* when $\widehat{\tau}_d$ (equivalently, $\widehat{\tau}_c$) is a regular epimorphism.
2. F is a *split epi* $*$-*fibration* when $\widehat{\tau}_d$ (equivalently, $\widehat{\tau}_c$) is a split epimorphism.

4.2. . Since in the diagram



part (2) and part (3) are pullbacks and the whole is a pullback (because $\tau_d \cdot \beta_d = F_1$), it follows that part (1) also is a pullback. This proves that any fibration is a $*$-fibration and any split epi fibration is a split epi $*$-fibration.

4.3. Proposition. *Consider a functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in $\mathcal{A}$, and the comparison J from its kernel to its strong h-kernel as in the following diagram.*



1. *F is a $*$-fibration if and only if J is a weak equivalence.*

2. *F is a split epi $*$-fibration if and only if J is an equivalence.*

Similarly to what we did in Proposition 3.2, one can prove a more precise statement: the arrow attesting that J is essentially surjective is the same arrow $\widehat{\tau}_c$ attesting that F is a $*$-fibration.

Proof. The proof is similar to that of Proposition 3.2 and we omit it. ■

From 4.2 and Proposition 4.3, we get the following result, announced in Proposition 4.4 of [14]:

4.4. Corollary. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a fibration between internal groupoids. The canonical comparison $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ from the kernel to the strong h-kernel is a weak equivalence. If F is a split epi fibration, then J is an equivalence.*

4.5. . Thanks to Proposition 4.3, we can slightly improve Proposition 4.6 in [14]: assume that $\mathcal{A}$ is a pointed regular category with reflexive coequalizers and consider a $*$-fibration $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Grpd}(\mathcal{A})$, with $\mathbb{A}, \mathbb{B}$ and $\mathbb{K}(F)$ proper (in Proposition 4.6 of [14], F is assumed to be a fibration and not just a $*$-fibration). There exists an exact sequence (called the Brown sequence)

$$\pi_1(\mathbb{K}\mathrm{er}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}\mathrm{er}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

where $\pi_1(\mathbb{A})$ is the internal group given by the joint kernel of domain and codomain maps of $\mathbb{A}$ and $\pi_0(\mathbb{A})$ is the object of connected components of $\mathbb{A}$, i.e. the coequalizer of the above maps. (Here, the exactness at B of

$$A \xrightarrow{f} B \xrightarrow{g} C$$

means that f factors as a regular epimorphism followed by the kernel of g). Indeed, since $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ is a weak equivalence, the arrows $\pi_0(J)\colon \pi_0(\mathbb{K}\mathrm{er}(F)) \to \pi_0(\mathbb{K}(F))$ and $\pi_1(J)\colon \pi_1(\mathbb{K}\mathrm{er}(F)) \to \pi_1(\mathbb{K}(F))$ are isomorphisms (Lemma 4.5 in [14]). Therefore, the above exact sequence immediately follows from the Gabriel-Zisman exact sequence

$$\pi_1(\mathbb{K}(F)) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \to \pi_0(\mathbb{K}(F)) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

established in Section 3 of [14].

4.6. . Corollary 4.4 can be obtained also from Proposition 3.2 without using the notion of $*$-fibration (4.2 and 4.3). Indeed, it is possible to get $J\colon \mathbb{K}\mathrm{er}(F) \to \mathbb{K}(F)$ as the pullback of $T\colon [B_0] \times_{N,F} \mathbb{A} \to \mathbb{V}(F)$ along a discrete fibration and then use the following general fact, easy to be proved by using Barr embedding theorem for regular categories.

4.7. Lemma. *Consider a pullback in* $\mathbf{Grpd}(\mathcal{A})$

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\widehat{F}} & \mathbb{C} \\ {\scriptstyle \widehat{G}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

and assume that G is a discrete fibration.

1. *If F is a weak equivalence, then $\widehat{F}$ is a weak equivalence.*
2. *If F is an equivalence, then $\widehat{F}$ is an equivalence.*

(It is a general fact on strong pullbacks that, if F is fully faithful, then $\widehat{F}$ is fully faithful.)

5. Normalized fibrations and normalized $*$-fibrations

5.1. . From [8], recall that a *category with null-homotopies* $\underline{\mathcal{B}}$ is given by

- a category $\mathcal{B}$,
- for each morphism $f\colon A \to B$ in $\mathcal{B}$, a set $\mathcal{H}(f)$ (the set of null-homotopies on f),
- for each triple of composable morphisms $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$, a map

$$f \circ - \circ h\colon \mathcal{H}(g) \to \mathcal{H}(f \cdot g \cdot h),\ \mu \mapsto f \circ \mu \circ h.$$

(If $f = \mathrm{id}_B$ or $h = \mathrm{id}_C$, we write $\mu \circ h$ or $f \circ \mu$ instead of $f \circ \mu \circ h$.)
These data have to satisfy

1. the identity condition: given a morphism $f\colon A \to B$, one has $\mathrm{id}_A \circ \mu \circ \mathrm{id}_B = \mu$ for all $\mu \in \mathcal{H}(f)$,
2. the associativity condition: given morphisms

$$A' \xrightarrow{f'} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{h'} D'$$

one has $(f' \cdot f) \circ \mu \circ (h \cdot h') = f' \circ (f \circ \mu \circ h) \circ h'$ for any $\mu \in \mathcal{H}(g)$.

5.2. . For what concerns the present work, a relevant example of category with null-homotopies is the category $\mathbf{Grpd}(\mathcal{A})$ of internal groupoids in a pointed category $\mathcal{A}$, with the natural transformations $0 \Rightarrow F$ playing the role of null-homotopies.

5.3. . The structure of category with null-homotopies is not rich enough to express the notion of strong h-pullback, but still, following [8, 18], we can express the notion of strong homotopy kernel. Let $\underline{\mathcal{B}}$ be a category with null-homotopies and let $f \colon A \to B$ be a morphism in $\mathcal{B}$. A triple

$$(\mathrm{Ker}(f),\ \mathrm{K}(f)\colon \mathrm{Ker}(f) \to A,\ k(f) \in \mathcal{H}(\mathrm{K}(f) \cdot f))$$

1. is a *homotopy kernel* (h-kernel, for short) of f if for any triple

   $$(D, g\colon D \to A, \mu \in \mathcal{H}(g \cdot f)),$$

   there exists a unique morphism $g' \colon D \to \mathrm{Ker}(f)$ in $\underline{\mathcal{B}}$ such that $g' \cdot \mathrm{K}(f) = g$ and $g' \circ k(f) = \mu$,
2. is a *strong homotopy kernel* (strong h-kernel, for short) of f if it is a h-kernel of f and, moreover, for any triple $(D, h\colon D \to \mathrm{Ker}(f), \mu \in \mathcal{H}(h \cdot \mathrm{K}(f)))$ such that $\mu \circ f = h \circ k(f)$, there exists a unique $\lambda \in \mathcal{H}(h)$ such that $\lambda \circ \mathrm{K}(f) = \mu$.

Notice that in [8], the identity condition in the definition of a category with null-homotopies has been omitted. We think it should not, since it allows to prove that h-kernels and strong h-kernels are determined up to isomorphism by their universal properties.
Finally, let us remark that the definition of strong h-kernels given in 4.1 is consistent with the one given here, applied to category with null-homotopies $\mathbf{Grpd}(\mathcal{A})$ for a finitely pointed complete category $\mathcal{A}$.

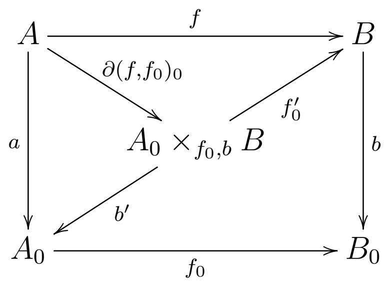
5.4. . For a category $\mathcal{A}$, we consider the arrow category $\mathbf{Arr}(\mathcal{A})$: the objects are the arrows $a \colon A \to A_0$ in $\mathcal{A}$ and the morphisms $(f, f_0) \colon a \to b$ are commutative squares of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

From [18], recall that $\mathbf{Arr}(\mathcal{A})$ is a category with null-homotopies: a null-homotopy for an arrow (f, f_0) is a diagonal, that is an arrow $d \colon A_0 \to B$ such that $a \cdot d = f$ and $d \cdot b = f_0$. If $\mathcal{A}$ has finite limits and a zero object, then $\mathbf{Arr}(\mathcal{A})$ has kernels and strong h-kernels. The kernel of (f, f_0) is just the level-wise kernel.

$$\begin{array}{ccccc} \mathrm{Ker}(f) & \xrightarrow{k_f} & A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{K}(a)}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathrm{Ker}(f_0) & \xrightarrow[k_{f_0}]{} & A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

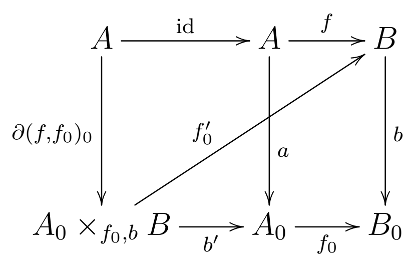
To construct the strong h-kernel of (f, f_0), consider the factorization through the pullback



The strong h-kernel is then given by the triple

$$(\partial(f, f_0)_0\,, \quad (\mathrm{id}, b')\,, \quad f_0')$$

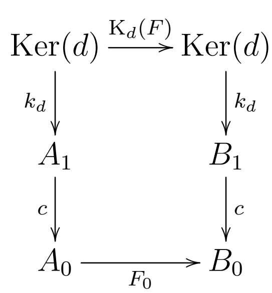
conveniently described by the following diagram.



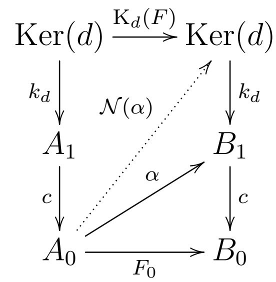
5.5. . For a finitely complete pointed category $\mathcal{A}$, the examples presented in 5.2 and in 5.4 are related by a functor, introduced in [2] and called the *normalization functor*

$$\mathcal{N}\colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$$

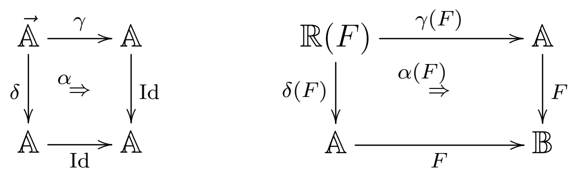
which sends an internal functor $F\colon \mathbb{A} \to \mathbb{B}$ to the commutative diagram



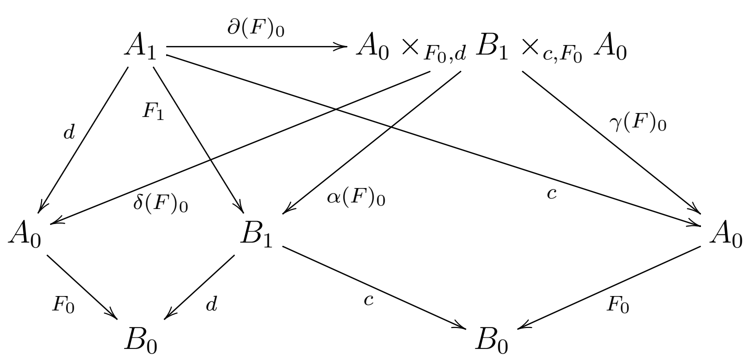
where $\mathrm{K}_d(F)$ is defined by $\mathrm{K}_d(F) \cdot k_d = k_d \cdot F_1$. Moreover, if $\alpha\colon 0 \Rightarrow F$ is a null-homotopy in $\mathbf{Grpd}(\mathcal{A})$, it gives rise to a null-homotopy $\mathcal{N}(\alpha)$ of $\mathcal{N}(F)$. Indeed, the natural transformation α is represented by an arrow $\alpha\colon A_0 \to B_1$. Since in particular $\alpha \cdot d = 0$, it factorizes as $\alpha = \mathcal{N}(\alpha) \cdot k_d$ and gives rise to the following commutative diagram



5.6. . In order to define fully faithful morphisms in $\mathbf{Arr}(\mathcal{A})$, let us look more carefully to the situation in $\mathbf{Grpd}(\mathcal{A})$. For a category $\mathcal{A}$ with pullbacks and a functor $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Grpd}(\mathcal{A})$, consider the following strong h-pullbacks



and the unique functor $\partial(F)\colon \vec{\mathbb{A}} \to \mathbb{R}(F)$ such that $\partial(F)\cdot\delta(F) = \delta$, $\partial(F)\cdot\gamma(F) = \gamma$ and $\partial(F)\cdot\alpha(F) = \alpha\cdot F$. The 0-level of the functor $\partial(F)$ is precisely the unique arrow making commutative the following diagram.



Therefore, we can say that the functor $F\colon \mathbb{A} \to \mathbb{B}$ is:

1. *faithful* if $\partial(F)_0$ is a monomorphism;
2. *full* if $\partial(F)_0$ is a regular epimorphism and in the context where $\mathcal{A}$ is a regular category;
3. fully faithful if $\partial(F)_0$ is an isomorphism (accordingly to 2.9).

5.7. . Now we imitate the previous argument using strong h-kernels in $\mathbf{Arr}(\mathcal{A})$. For a morphism $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$, consider the following strong h-kernels, together with the induced comparison arrow $\partial(f, f_0)$:

$$\begin{array}{ccccc} \mathbb{K}(\mathrm{Id}) & \longrightarrow & a & \xrightarrow{\mathrm{Id}} & a \\ {\scriptstyle \partial(f,f_0)}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle (f,f_0)} \\ \mathbb{K}(f, f_0) & \longrightarrow & a & \xrightarrow[(f,f_0)]{} & b \end{array}$$

The 0-level of $\partial(f, f_0)$ is precisely the factorization $\partial(f, f_0)_0 \colon A \to A_0 \times_{f_0,b} B$ through the pullback, as in 5.4 (while the 'domain level' of $\partial(f, f_0)$ is just the identity arrow on A). This suggests part of the following terminology.

5.8. . For a finitely complete pointed category $\mathcal{A}$, consider an arrow $(f, f_0) \colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$ together with the induced factorization $\partial(f, f_0)_0 \colon A \to A_0 \times_{f_0,b} B$ through the pullback, as in the description of the strong h-kernel in 5.4. The arrow (f, f_0) is:

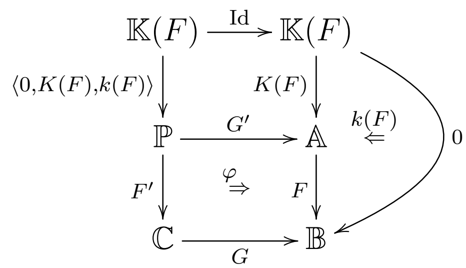
1. *faithful* if $\partial(f, f_0)_0$ is a monomorphism;
2. *fully faithful* if $\partial(f, f_0)_0$ is an isomorphism;
3. *full* if $\partial(f, f_0)_0$ is a regular epimorphism and in the context where $\mathcal{A}$ is a regular category;
4. *essentially surjective* if f_0 and b are jointly strongly epimorphic;
5. a *weak equivalence* if it is fully faithful and essentially surjective;
6. a *fibration* if f is a regular epimorphism and in the context where $\mathcal{A}$ is a regular category.

In order to compare the above terminology with the terminology for internal functors (2.9, 3.1 and 5.6), we need some intermediate steps, collected in the next lemma, easy to be proved by using an appropriate embedding theorem. Indeed, what is really needed to prove Proposition 5.10 is the points 3 and 5 of the lemma, but the conscientious reader will realise that point 1 can be used to prove point 2, point 2 can be used to prove point 3 and point 4 is needed to prove point 5. (Point 4 is the version for strong h-pullbacks of the elementary fact that two parallel arrows in a pullback diagram have isomorphic kernels.)

5.9. Lemma. *Consider a finitely complete pointed category $\mathcal{A}$ and let $F \colon \mathbb{A} \to \mathbb{B}$ be a functor between internal groupoids in $\mathcal{A}$.*

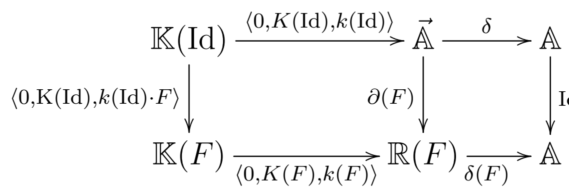
1. *If F is fully faithful, then its normalization $\mathcal{N}(F)$ is a pullback in $\mathcal{A}$.*
2. *The strong h-kernel $K(F) \colon \mathbb{K}(F) \to \mathbb{A}$ is a discrete fibration.*
3. *The normalization functor $\mathcal{N} \colon \mathbf{Grpd}(\mathcal{A}) \to \mathbf{Arr}(\mathcal{A})$ preserves kernels and strong h-kernels.*

4. *Consider the following diagram in* $\mathbf{Grpd}(\mathcal{A})$*, with the bottom square being a strong h-pullback, the right region being a strong h-kernel and the functor* $\langle 0, K(F), k(F)\rangle$ *determined by the universal property of the strong h-pullback with respect to the triple* $(0, K(F), k(F))$*.*



Then the left column is a kernel.

5. *Consider the following diagram (notation as in 5.6).*



Then the rows are kernels and the left-hand square is a pullback.

We are ready to compare the terminology in $\mathbf{Grpd}(\mathcal{A})$ and in $\mathbf{Arr}(\mathcal{A})$. (Recall from [1], that a finitely complete pointed category is *protomodular* when the Split Short Five Lemma holds. When also regular, such a category is called *homological*. Compare also with [6] for points 9 and 10 of the following result.)

5.10. Proposition. *Let* $\mathcal{A}$ *be a finitely complete pointed category and* $F\colon \mathbb{A} \to \mathbb{B}$ *be a functor between groupoids in* $\mathcal{A}$*.*

1. *If* F *is faithful, then* $\mathcal{N}(F)$ *is faithful.*
2. *If* F *is fully faithful, then* $\mathcal{N}(F)$ *is fully faithful.*
3. *If* $\mathcal{A}$ *is regular and* F *is full, then* $\mathcal{N}(F)$ *is full.*
4. *If* $\mathcal{A}$ *is protomodular and* $\mathcal{N}(F)$ *is faithful, then* F *is faithful.*
5. *If* $\mathcal{A}$ *is protomodular and* $\mathcal{N}(F)$ *is fully faithful, then* F *is fully faithful.*
6. *If* $\mathcal{A}$ *is regular and protomodular and* $\mathcal{N}(F)$ *is full, then* F *is full.*
7. *If* $\mathcal{A}$ *is regular and protomodular and* F *is essentially surjective, then* $\mathcal{N}(F)$ *is essentially surjective.*

8. If $\mathcal{A}$ is regular and $\mathcal{N}(F)$ is essentially surjective, then F is essentially surjective.

9. If $\mathcal{A}$ is regular and F is a fibration, then $\mathcal{N}(F)$ is a fibration.

10. If $\mathcal{A}$ is regular and protomodular and $\mathcal{N}(F)$ is a fibration, then F is a fibration.

Proof. From 1 to 6. By point 3 of Lemma 5.9, the functor $\langle 0, K(\mathrm{Id}), k(\mathrm{Id}) \cdot F\rangle \colon \mathbb{K}(\mathrm{Id}) \to \mathbb{K}(F)$ of point 5 of that lemma is sent by $\mathcal{N}$ to

$$\partial(\mathcal{N}(F)) \colon \mathcal{N}(\mathbb{K}(\mathrm{Id})) = \mathbb{K}(\mathrm{Id}_{\mathcal{N}(\mathbb{A})}) \to \mathbb{K}(\mathcal{N}(F)) = \mathcal{N}(\mathbb{K}(F)).$$

The 0-level of the diagram in this same point 5 gives thus the following diagram in $\mathcal{A}$:

$$\begin{array}{ccccc}
\mathrm{Ker}(d) & \xrightarrow{\;k_d\;} & A_1 & \xrightarrow{\;d\;} & A_0 \\
\downarrow{\scriptstyle \partial(\mathcal{N}(F))_0} & & \downarrow{\scriptstyle \partial(F)_0} & & \downarrow{\scriptstyle \mathrm{id}} \\
A_0 \times_{F_0, k_d \cdot c} \mathrm{Ker}(d) & \xrightarrow[\langle 0, K(F), k(f)\rangle_0]{} & A_0 \times_{F_0,d} B_1 \times_{c,F_0} A_0 & \xrightarrow[\delta(F)_0]{} & A_0
\end{array}$$

Thanks to Lemma 5.9.5, the rows are kernels and the left-hand square is a pullback. Therefore:
1. If $\partial(F)_0$ is a monomorphism, then $\partial(\mathcal{N}(F))_0$ is also a monomorphism.
2. If $\partial(F)_0$ is an isomorphism, then $\partial(\mathcal{N}(F))_0$ is also an isomorphism.
3. If $\partial(F)_0$ is a regular epimorphism, then $\partial(\mathcal{N}(F))_0$ is also a regular epimorphism because the category $\mathcal{A}$ is regular.
4. If $\partial(\mathcal{N}(F))_0$ is a monomorphism, then $\partial(F)_0$ is also a monomorphism because in a protomodular category, pullbacks reflect monomorphisms (see Lemma 3.13 in [3]).
5. If $\partial(\mathcal{N}(F))_0$ is an isomorphism, then $\partial(F)_0$ is also an isomorphism because in a protomodular category, the Split Short Five Lemma holds (see Lemma 3.10 in [3]).
6. If $\partial(\mathcal{N}(F))_0$ is a regular epimorphism, then $\partial(F)_0$ is also a regular epimorphism by Proposition 8 in [2] (see also Proposition 2.4 in [18]), which can be applied here because $\mathcal{A}$ is regular and protomodular and $d \colon A_1 \to A_0$ is a split and then regular epimorphism.
7 and 8. Consider the following commutative diagram.

$$\begin{array}{ccccc}
A_0 & \xrightarrow{\langle \mathrm{id}, F_0 \cdot e\rangle} & A_0 \times_{F_0,d} B_1 & \xleftarrow{\langle 0, k_d\rangle} & \mathrm{Ker}(d) \\
& {\scriptstyle F_0}\searrow & \downarrow{\scriptstyle \beta_d \cdot c} & \swarrow{\scriptstyle k_d \cdot c} & \\
& & B_0 & &
\end{array}$$

7. If $\beta_d \cdot c$ is a regular epimorphism, it is a strong one. Therefore, it suffices to prove that $\langle \mathrm{id}, F_0 \cdot e\rangle$ and $\langle 0, k_d\rangle$ are jointly strongly epimorphic. This is the case since $\mathcal{A}$ is protomodular and $\langle 0, k_d\rangle$ and $\langle \mathrm{id}, F_0 \cdot e\rangle$ are respectively a kernel and a section of α_d (see [2, 3]).

$$\begin{array}{ccc}
\mathrm{Ker}(d) & \xrightarrow{\langle 0, k_d\rangle} & A_0 \times_{F_0,d} B_1 \\
\downarrow & & {\scriptstyle \alpha_d}\downarrow \uparrow{\scriptstyle \langle \mathrm{id}, F_0 \cdot e\rangle} \\
0 & \longrightarrow & A_0
\end{array}$$

8. If F_0 and $k_d \cdot c$ are jointly strongly epimorphic, $\beta_d \cdot c$ is a strong epimorphism and so a regular epimorphism since $\mathcal{A}$ is regular.
9 and 10. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Ker}(d) & \xrightarrow{k_d} & A_1 & \xrightarrow{d} & A_0 \\ {\scriptstyle \mathrm{K}_d(F)}\downarrow & & \downarrow{\scriptstyle \tau_d} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Ker}(d) & \xrightarrow[\langle 0,k_d\rangle]{} & A_0 \times_{F_0,d} B_1 & \xrightarrow[\alpha_d]{} & A_0 \end{array}$$

Since $\mathrm{id}\colon A_0 \to A_0$ is a monomorphism, the left-hand square is a pullback. Therefore:
9. If τ_d is a regular epimorphism, then $\mathrm{K}_d(F)$ is also a regular epimorphism because the category $\mathcal{A}$ is regular.
10. If $\mathrm{K}_d(F)$ is a regular epimorphism, then τ_d is also a regular epimorphism by Proposition 8 in [2]. ∎

We have not yet discussed $*$-fibrations in $\mathbf{Arr}(\mathcal{A})$. For this, we need a last preparatory step. Given a morphism $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$, the triple

$$(\mathbb{K}\mathrm{er}(f, f_0), \quad k_{(f,f_0)}, \quad 0\colon \mathrm{Ker}(f_0) \to B)$$

induces a canonical comparison $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ through the strong h-kernel of (f, f_0).

5.11. LEMMA. *In the category with null-homotopies* $\mathbf{Arr}(\mathcal{A})$ *for a finitely complete pointed category* $\mathcal{A}$*, kernels are strong, i.e., for any morphism* $(f, f_0)\colon a \to b$*, the canonical comparison* $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ *is fully faithful (see 2.5).*

PROOF. Using the descriptions of the kernel and of the strong h-kernel given in 5.4, the comparison J turns out to be the left-hand square in the following diagram.

$$\begin{array}{ccccc} \mathrm{Ker}(f) & \xrightarrow{k_f} & A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{K}(a)}\downarrow & & \downarrow{\scriptstyle \partial(f,f_0)_0} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Ker}(f_0) & \xrightarrow[\langle k_{f_0},0\rangle]{} & A_0 \times_{f_0,b} B & \xrightarrow[f_0']{} & B \end{array}$$

Since both rows are kernels and $\mathrm{id}\colon B \to B$ is a monomorphism, the left-hand square is a pullback, which means that J is fully faithful. ∎

5.12. . Having in mind Proposition 4.3, we could now define a $*$-*fibration* in $\mathbf{Arr}(\mathcal{A})$ as a morphism $(f, f_0)\colon a \to b$ such that the canonical comparison $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ is essentially surjective (and then, by Lemma 5.11, a weak equivalence). We can now complete Proposition 5.10, with two more points:

11. *If* $\mathcal{A}$ *is regular and protomodular and the functor* F *is a* $*$*-fibration in* $\mathbf{Grpd}(\mathcal{A})$*, then the morphism* $\mathcal{N}(F)$ *is a* $*$*-fibration in* $\mathbf{Arr}(\mathcal{A})$*.*

12. If $\mathcal{A}$ is regular and $\mathcal{N}(F)$ is a $*$-fibration, then F is a $*$-fibration.

Now that we have the notions of fibration and $*$-fibration available in $\mathbf{Arr}(\mathcal{A})$, we can ask if every fibration is a $*$-fibration (this is the case in $\mathbf{Grpd}(\mathcal{A})$, as observed in 4.2). The surprise is that not only the answer is negative, but the expected implication "fibration $\Rightarrow$ $*$-fibration" is in fact equivalent, in the pointed regular context, to the condition of protomodularity.

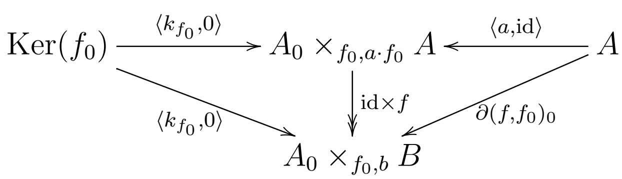
5.13. Proposition. *The following conditions on a pointed regular category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is protomodular (and then homological).*
2. *For every fibration $(f, f_0)\colon a \to b$ in $\mathbf{Arr}(\mathcal{A})$, the canonical comparison*

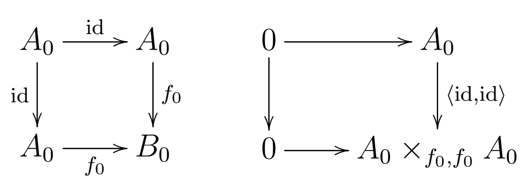
$$J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$$

is a weak equivalence.

Proof. $1 \Rightarrow 2$. Suppose $\mathcal{A}$ is homological and (f, f_0) is a fibration in $\mathbf{Arr}(\mathcal{A})$. Thanks to Lemma 5.11, we already know that J is full and faithful. Consider now the following diagram where $\mathrm{id} \times f$ is a regular epimorphism since so is f.



Thus, in order to prove that J is essentially surjective, it suffices to notice that the protomodularity of $\mathcal{A}$ implies that $\langle k_{f_0}, 0\rangle$ and $\langle a, \mathrm{id}\rangle$ are jointly strongly epimorphic since they are respectively the kernel and a section of the second projection $A_0 \times_{f_0, a\cdot f_0} A \to A$.
$2 \Rightarrow 1$. Firstly, let us prove that if the kernel of a morphism $f_0\colon A_0 \to B_0$ is the zero object, then f_0 is a monomorphism. In order to do so, consider the fibration $(\mathrm{id}, f_0)\colon \mathrm{id} \to f_0$ in $\mathbf{Arr}(\mathcal{A})$.



The diagram on the right represents the canonical comparison $J\colon \mathbb{K}\mathrm{er}(\mathrm{id}, f_0) \to \mathbb{K}(\mathrm{id}, f_0)$. By the assumption, we know that $0 \to A_0 \times_{f_0, f_0} A_0$ and $\langle \mathrm{id}, \mathrm{id}\rangle$ are jointly strongly epimorphic. This is equivalent to the fact that $\langle \mathrm{id}, \mathrm{id}\rangle$ is a regular epimorphism. Since it is also a split monomorphism, it is an isomorphism, which means that f_0 is a monomorphism.

Let us now prove that the Short Five Lemma holds in $\mathcal{A}$. Consider the following diagram where both rows are kernel of regular epimorphisms and $\mathrm{K}(a)$ and b are isomorphisms.

$$
\begin{array}{ccccc}
\mathrm{Ker}(f) & \xrightarrow{k_f} & A & \overset{f}{\twoheadrightarrow} & B \\
{\scriptstyle \mathrm{K}(a)}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
\mathrm{Ker}(f_0) & \xrightarrow[k_{f_0}]{} & A_0 & \underset{f_0}{\twoheadrightarrow} & B_0
\end{array}
$$

Since $\mathrm{K}(a)$ is a monomorphism, its kernel is the zero object. Since b is a monomorphism, the left-hand square is a pullback, hence also the kernel of a is zero. But, by the first part of the proof, this means that a is a monomorphism. So, it remains to prove that it is a regular epimorphism. The morphism $(f, f_0)\colon a \to b$ is a fibration in $\mathbf{Arr}(\mathcal{A})$. Thus, the comparison morphism $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ is a weak equivalence. Since b is an isomorphism, this implies that k_{f_0} and a are jointly strongly epimorphic. But since $\mathrm{K}(a)$ is an isomorphism, k_{f_0} factors through a, so that a is a regular epimorphism. ■

Following [2], we call a morphism in a homological category *proper* when it can be factorized as a regular epimorphism followed by a kernel.

5.14. PROPOSITION. *Let $\mathcal{A}$ be a homological category and $(f, f_0)\colon a \to b$ a morphism in $\mathbf{Arr}(\mathcal{A})$ such that a and b have a cokernel. We denote by $\mathrm{K}(f, f_0)$ (respectively $\mathrm{C}(f, f_0)$) the induced morphism between the kernels (respectively the cokernels) of a and b as in the following commutative diagram.*

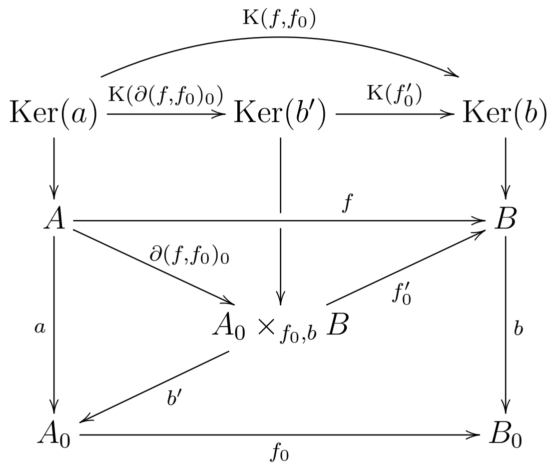
$$
\begin{array}{ccc}
\mathrm{Ker}(a) & \xrightarrow{\mathrm{K}(f,f_0)} & \mathrm{Ker}(b) \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A_0 & \xrightarrow[f_0]{} & B_0 \\
\downarrow & & \downarrow \\
\mathrm{Coker}(a) & \xrightarrow[\mathrm{C}(f,f_0)]{} & \mathrm{Coker}(b)
\end{array}
$$

Then the following implications hold.

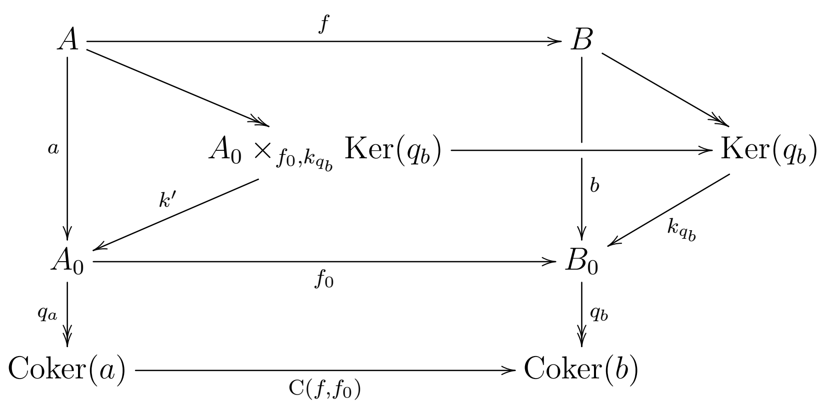
1. *(f, f_0) is faithful if and only if $\mathrm{K}(f, f_0)$ is a monomorphism.*
2. *If (f, f_0) is full, then $\mathrm{K}(f, f_0)$ is a regular epimorphism.*
3. *If (f, f_0) is full and b proper, then $\mathrm{C}(f, f_0)$ is a monomorphism.*
4. *If (f, f_0) is essentially surjective, then $\mathrm{C}(f, f_0)$ is a regular epimorphism.*

In particular, if (f, f_0) is a weak equivalence and b a proper morphism, then $\mathrm{K}(f, f_0)$ and $\mathrm{C}(f, f_0)$ are both isomorphisms.

Proof. Let us consider the factorization through the pullback together with the kernels of a, b' and b.

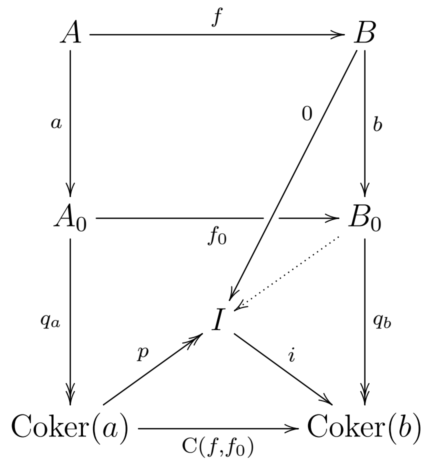


Using Lemma 4.2.4 in [1], we know that $\mathrm{K}(f_0')$ is an isomorphism and the top left trapezium is a pullback. We then deduce points 1 and 2 from the fact that pullbacks preserves and reflects monomorphisms and preserves regular epimorphisms in a homological category. For 3, we know that b factorizes as a regular epimorphism followed by the kernel k_{q_b} of its cokernel. We consider the pullback of k_{q_b} along f_0.



Since (f, f_0) is full, a factorizes as a regular epimorphism followed by k', hence $k' \cdot q_a = 0$. Using the universal properties of $\mathrm{Ker}(q_b)$ and $A_0 \times_{f_0,k_{q_b}} \mathrm{Ker}(q_b)$, it is then not hard to show that k' is the kernel of q_a. According to Lemma 4.2.5 in [1], $\mathrm{C}(f, f_0)$ is then a monomorphism.

For point 4, we consider $p \cdot i$ the (regular epi,mono)-factorization of $\mathrm{C}(f, f_0)$. Since the diagram below is commutative and f_0 and b are jointly strongly epimorphic, q_b factorizes through i.



Therefore i is a regular epimorphism and so is $\mathrm{C}(f, f_0)$. ∎

5.15. . In a similar way than we deduced the Brown exact sequence from the Gabriel-Zisman sequence in 4.5, we can deduce and generalize the Snake Lemma from the Snail Lemma. Let $\mathcal{A}$ be a homological category with cokernels and $(f, f_0)\colon a \to b$ a morphism in $\mathbf{Arr}(\mathcal{A})$. If a, b and $\mathbb{K}(f, f_0) = \partial(f, f_0)_0$ are proper, the Snail Lemma [18] states that the following sequence is exact.

$$\mathrm{Ker}(\mathbb{K}(f, f_0)) \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Coker}(\mathbb{K}(f, f_0)) \to \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

If we assume moreover that (f, f_0) is a $*$-fibration, then by definition, the canonical comparison $J\colon \mathbb{K}\mathrm{er}(f, f_0) \to \mathbb{K}(f, f_0)$ is a weak equivalence. Thus, by the above proposition, $\mathrm{K}(J)$ and $\mathrm{C}(J)$ are isomorphisms and we obtain the snake sequence from the snail sequence:

$$\mathrm{Ker}(\mathbb{K}\mathrm{er}(f, f_0)) \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Coker}(\mathbb{K}\mathrm{er}(f, f_0)) \to \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

Note that classically, the stronger assumption that (f, f_0) is a fibration is assumed.

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