Fibrations and Proofs - Tutorial for FMCS 2008, Halifax

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1 Basic Theory and Examples

The categorical notion of fibration arose from algebraic geometry as a language for formulating and studying descent problems. However, fibrations are useful in many situations, both in mathematics and in computer science. They are an appropriate setting for modeling situations where one kind of object depends on, or is *indexed*, or *fibred* over another kind of object.

Definition 1.1. Let $p : \mathbb{E} \longrightarrow \mathbb{B}$ be a functor.

(i) A morphism v in $\mathbb{E}$ is *vertical* if $p(v) = 1$

(ii) For an object I in $\mathbb{B}$, the *fibre of* p *over* I is the category with objects X for which $pX = I$ and with vertical morphisms between them. We denote this category by $\mathbb{E}_p(I), \mathbb{E}(I)$, or sometimes $p^{-1}(I)$.

If $pX = I$ we say that X is an object over I. Similarly, if $pf = g$, we call f a morphism over g.

Definition 1.2 (Cartesian morphism; fibration)**.**

(i) Let $X \xrightarrow{m} Y$ be a morphism over $pm = f$. We say that m is *cartesian* (over f) if every other map $n : Z \longrightarrow Y$ over f factors through m via a unique vertical map v.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! v"'] \arrow[dr, "n"] & \\ X \arrow[r, "m"] & Y \end{tikzcd}$$

$$I \xrightarrow{pm=f} pY$$

In this case, we call m a *cartesian lifting* of f.

(ii) The functor $p : \mathbb{E} \longrightarrow \mathbb{B}$ is a *fibration* if every map $I \xrightarrow{f} pY$ has a cartesian lifting with codomain Y, and if cartesian morphisms compose.

It should be remarked that one often strengthens the notion of cartesianness and builds in closure under composition; it is readily verified that this gives the same notion of fibration.

Exercise 1.3. Show that for a fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$, every morphism in $\mathbb{E}$ factors as a vertical map followed by a cartesian map and that this factorization is essentially unique.

Exercise 1.4. Show that a map which is both vertical and cartesian is an isomorphism.

Definition 1.5. If $\mathbb{E} \xrightarrow{p} \mathbb{B}$ and $\mathbb{E}' \xrightarrow{p'} \mathbb{B}$ an functors, then a functor $\mathbb{E} \longrightarrow \mathbb{E}'$ over $\mathbb{B}$ is called *cartesian* if it preserves cartesian maps. One also speaks of a *fibred functor*.

Fibrations over $\mathbb{B}$, cartesian functors over $\mathbb{B}$ and fibrewise (i.e. vertical) natural transformations form a 2-category, which we will denote $\mathsf{Fib}(\mathbb{B})$. There is a forgetful 2-functor $\mathsf{Fib}(\mathbb{B}) \longrightarrow \mathsf{Cat}/\mathbb{B}$.

1.1 Examples.

Here are a number of standard examples. Examples pertaining specifically to logic will be presented in section 2.

1. Any projection $\mathbb{B} \times \mathbb{C} \xrightarrow{\pi_{\mathbb{B}}} \mathbb{B}$ is a fibration: given a map $I \xrightarrow{f} J$ in $\mathbb{B}$ and an object (J, C) over J, a cartesian lift of f is $(I, C) \xrightarrow{(f, 1_C)} (J, C)$; such a fibration may be called *constant* (since all fibre categories are isomorphic to $\mathbb{C}$).

2. Let $0 \longrightarrow K \longrightarrow E \xrightarrow{p} G \longrightarrow 0$ be an exact sequence of groups. Then $E \xrightarrow{p} G$ is a fibration if we regard E, G as one-object categories. Note that every map in E is cartesian!

3. The category Mod has as objects pairs (R, M) where R is a commutative ring and M is a left R-module. A map $(R, M) \longrightarrow (S, N)$ is a ring homomorphism $f : R \longrightarrow S$ together with a map of R-modules $m : R \longrightarrow f^*S$, where f^*S is the R-module with underlying group S and action $r \cdot x = f(r) \cdot x$, for $r \in R, x \in S$. The projection on the category CRng of commutative rings is a fibration.

4. A *presheaf* on $\mathbb{B}$ is a contravariant functor $\mathbb{B} \xrightarrow{F} \mathsf{Set}$. Define, given such F, a category $\int_{\mathbb{B}} F$ with objects (I, c) where $I \in Ob(\mathbb{B})$, $c \in F(I)$; a map $(I, c) \longrightarrow (J, d)$ is a map $f : I \longrightarrow J$ in $\mathbb{B}$ for which $F(f)(d) = c$. Then the projection $\int_{\mathbb{B}} F \xrightarrow{\pi} \mathbb{B}$, $(I, c) \mapsto I$ is a fibration. It is called the *Grothendieck construction* on F. Note that every morphism in $\int_{\mathbb{B}} F$ is cartesian. Such a fibration is called *discrete*: the fibres are sets (discrete categories).

   Exercise 1.6. Show the converse: every discrete fibration over $\mathbb{B}$ gives a presheaf on $\mathbb{B}$. Express this as an equivalence of categories $\mathsf{DisFib}(\mathbb{B}) \simeq \mathsf{Set}^{\mathbb{B}^{op}}$.

5. Assume $\mathbb{B}$ has binary products. Define the *simple fibration* over $\mathbb{B}$ as follows: objects of $\mathcal{S}(\mathbb{B})$ are pairs (I, X) with $I, X \in Ob(\mathbb{B})$. Morphisms $(I, X) \longrightarrow (J, Y)$ are pairs of the form (f, F) where $f : I \longrightarrow J, F : I \times X \longrightarrow Y$. The composite of $(I, X) \xrightarrow{(f,F)} (J, Y) \xrightarrow{(g,G)} (K, Z)$ is

$$(I, X) \xrightarrow{(gF, G<f\pi_I, F>)} (K, Z)$$

6. Let $\mathbb{B} = \mathsf{Set}$. Define, for a set I, a category of I-indexed families of sets : objects are families $\{X_i\}_{i \in I}$, and a morphism is an I-indexed family $\{f_i : X_i \longrightarrow Y_i\}_{i \in I}$. This category is the fibre over I of the *family fibration* $\mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$. A general morphism $\{X_i\}_{i \in I} \longrightarrow \{Y_j\}_{j \in J}$ consists of a map $f : I \longrightarrow J$ in and an I-indexed family of maps $\{m_i : X_i \longrightarrow Y_{f(i)}\}_{i \in I}$.

7. Let $\mathbb{C}$ be a category. $\mathsf{Fam}(\mathbb{C})$ is the category with objects $\{X_i\}_{i \in I}$ where the X_i are objects of $\mathbb{C}$. A morphism $\{X_i\}_{i \in I} \longrightarrow \{Y_j\}_{j \in J}$ is a pair $(f, \{m_i\}_{i \in I})$ with $f : I \longrightarrow J$ and each $m_i : X_i \longrightarrow Y_{f(i)}$ a morphism in $\mathbb{C}$. The projection $\mathsf{Fam}(\mathbb{C}) \xrightarrow{\pi} \mathsf{Set}, \{X_i\}_{i \in I} \mapsto I$ is a fibration.

   Exercise 1.7. Show that $\mathsf{Fam}(\mathbb{C})$ is the free *coproduct completion* of $\mathbb{C}$.

   Exercise 1.8. Let $\mathbb{C} = (\begin{tikzcd} C_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & C_0 \end{tikzcd})$ be a category.

(i) Show that an I-indexed family of objects $\{X_i\}_{i\in I}$ of $\mathbb{C}$ is the same as a function $I \xrightarrow{x} C_0$.

(ii) Show that a morphism $(f, \{m_i\}_{i\in I}) = \{X_i\}_{i\in I} \longrightarrow \{Y_j\}_{j\in J}$ is the same as a function $I \xrightarrow{m} C_1$ for which

$$\begin{tikzcd} I \arrow[rr, "x"] \arrow[dr, "m"] \arrow[dd, "f"'] & & C_0 \\ & C_1 \arrow[ur, "d_0"'] \arrow[dr, "d_1"] & \\ J \arrow[rr, "y"] & & C_0 \end{tikzcd} \tag{1}$$

commutes.

(iii) Express composition diagrammatically. (Hint: use $C_2 = C_1 \times_{C_0} C_1$, the object of composable pairs in $\mathbb{C}$.)

8. Let $\mathbb{B}$ be a category with pullbacks, and let $\mathbb{C} = (\begin{tikzcd} C_1 \arrow[r, shift left, bend left, "d_0"] \arrow[r, shift right, bend right, "d_1"'] & C_0 \end{tikzcd})$ be an internal category in $\mathbb{B}$.

   Define $\mathsf{Fam}(\mathbb{C})$ to have objects $I \xrightarrow{x} \mathbb{C}_0$ and maps as in (1). As in the previous exercise this gives a category fibred over $\mathbb{B}$. $\mathsf{Fam}(\mathbb{C}) \longrightarrow \mathbb{B}$ is called the *externalization* of $\mathbb{C}$. Fibrations of this form are called *small*.

9. For any category $\mathbb{B}$, let $\mathbb{B}^{\longrightarrow}$ be the arrow category whose objects are maps $X \xrightarrow{f} Y$ and whose morphisms are commutative squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"] & Y \arrow[d, "n"'] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

   The codomain map $(X \xrightarrow{f} Y) \mapsto Y$ is a functor $\mathbb{B}^{\longrightarrow} \xrightarrow{d_1} \mathbb{B}$. This is a fibration precisely when $\mathbb{B}$ has pullbacks. This is called the *codomain fibration* on $\mathbb{B}$.

10. For the *subobject fibration* of a category with pullbacks let $\mathsf{Sub}_{\mathbb{B}}(I)$ be the category whose objects are subobjects of I (equivalence classes of monics $M \longrightarrow I$). These are partially ordered, so $\mathsf{Sub}_{\mathbb{B}}(I)$ is a poset, hence a category. The $\mathsf{Sub}_{\mathbb{B}}(I)$, for varying I, can be glued together as to obtain a category $\mathsf{Sub}(\mathbb{B})$. In this category, a map $M \longrightarrow N$ is an $f : I \longrightarrow J$ in the base for which $M \leq f^*N$. The projection $\mathsf{Sub}(\mathbb{B}) \longrightarrow \mathbb{B}$ sending $M \in Sub(I)$ to I is a fibration whose fibres are posets.

    Exercise 1.9. Show that, at least in the case $\mathbb{B} = \mathsf{Set}$, this is an instance of externalization of an internal category.

We end by giving two constructions for obtaining new fibrations from old:

- If $p : \mathbb{E} \longrightarrow \mathbb{F}$ and $q : \mathbb{F} \longrightarrow \mathbb{G}$ are fibrations, then so is qp.
- Fibrations are stable under pullback, i.e. if

$$\begin{tikzcd} \mathbb{F} \arrow[r] \arrow[d, "q"'] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[r] & \mathbb{B} \end{tikzcd}$$

  is a pullback in Cat and p is a fibration, then so is q.

1.2 Split and cloven fibrations

The reader may have noticed that being a fibration is a property of a functor, as opposed to additional structure. The notion of an indexed category is structural, and may be regarded as giving a *presentation* of a fibration.

Definition 1.10. A $\mathbb{B}$-indexed category is a pseudofunctor $\mathbb{B}^{op} \xrightarrow{F} \mathsf{Cat}$. Explicitly, this gives for each object I of $\mathbb{B}$ a category $F(I)$, for each arrow $I \xrightarrow{f} J$ a functor $F(f) : F(J) \longrightarrow F(I)$, together with coherence isomorphisms $F(f)F(g) \cong F(gf)$ and $F(1) \cong 1$, subject to coherence axioms. Moreover, F is *split* if these are identities. We write f^* for $F(f)$.

Construction 1.11. For a $\mathbb{B}$-indexed category F, let $\int_{\mathbb{B}} F$ be the category with objects (I, x) where $x \in Ob(F(I))$. A morphism $(I, x) \longrightarrow (J, j)$ is a pair (F, m) where $f : I \longrightarrow J$ and $m : x \longrightarrow f^*(y)$. The projection $\int_{\mathbb{B}} F \longrightarrow \mathbb{B}$ is a fibration, the maps f^* endow if with a *choice of cartesian liftings*. Such fibrations are called *cloven* (and a choice of cartesian liftings is called a *cleavage*). If F is split, then the cleavage is functorial.

Exercise 1.12. Show the converse: every cloven fibration p gives an indexed category $I \longrightarrow p^{-1}(I)$. This sets up an equivalence between the category $\mathsf{CFib}(\mathbb{B})$ of cloven fibrations and that of $\mathbb{B}$-indexed categories. Similarly, split fibrations are equivalent to split indexed categories.

Exercise 1.13. Express pullback of fibrations and composition of fibrations in the indexed language. (Note: the latter is worth two bottles of (good) champagne!)

While we may always choose a cleavage, not every fibration is split! (Example: take a non-split exact sequence of groups, and look at Example 2.)

Theorem 1.14 (Giraud)**.** *Every fibration is equivalent to a split fibration.*

Proof. (sketch) Let $\mathbb{E} \xrightarrow{p} \mathbb{B}$ be a fibration. For $x \in Ob(\mathbb{B})$, let $\mathbb{B}/I$ be the slice category, and consider $\mathbb{B}/I \xrightarrow{d_0} \mathbb{B}$, the domain(!) projection.

Exercise 1.15.

(i) Show that $\mathbb{B}/I \xrightarrow{d_0} \mathbb{B}$ is a discrete fibration. (Fibrations of this form are also called *representable.*)

(ii) Show that the mapping

$$I \longmapsto \left(\mathbb{B}/I \xrightarrow{d_0} \mathbb{B} \right)$$

can be extended to a functor $\mathbb{B} \xrightarrow{Y} \mathsf{Fib}(\mathbb{B})$.

Now consider a fibred functor

$$\begin{tikzcd} \mathbb{B}/I \arrow[rr, "H"] \arrow[dr, "d_0"'] & & \mathbb{E} \arrow[dl, "p"] \\ & \mathbb{B} & \end{tikzcd}$$

It sends an $X \xrightarrow{m} I$ to $H(X \xrightarrow{m} I)$, an object of $p^{-1}(X)$. In particular, $H(I \xrightarrow{1} I)$ is an object in the fibre over I. Thus H may be viewed as selecting cartesian maps in $\mathbb{E}$.

Exercise 1.16.

(i) Show that we actually get a functor

$$\Phi_I : \mathsf{Fib}(\mathbb{B}) \left(\mathbb{B}/I \xrightarrow{d_0} \mathbb{B} \, , \, \mathbb{E} \xrightarrow{p} \mathbb{B} \right) \longrightarrow \mathbb{E}(I).$$

(ii) Prove that ϕ_I is an equivalence of categories. (This is the *Fibred Yoneda Lemma.*)

Now let $\mathsf{Split}(p)$ be the split indexed category

$$I \longmapsto \mathsf{Fib}(\mathbb{B})\left(\mathbb{B}/I \xrightarrow{d_0} \mathbb{B}\ ,\ \mathbb{E} \xrightarrow{p} \mathbb{B} \right).$$

Exercise 1.17. Verify that this is indeed a split indexed category (what is reindexing?) and conclude that we have an equivalence of fibrations $p \simeq \mathsf{Split}(p)$.

□

In fact, the assignment $p \mapsto \mathsf{Split}(p)$ is right adjoint to the inclusion of split fibrations into fibrations.

1.3 Fibrations as algebras

Recall that there is a 2-functor $\mathsf{Fib}(\mathbb{B}) \longrightarrow \mathsf{Cat}/\mathbb{B}$, which forgets that a functor is a fibration.

Theorem 1.18. *The functor* $\mathsf{Fib}(\mathbb{B}) \longrightarrow \mathsf{Cat}/\mathbb{B}$ *has a right 2-adjoint. Even better,* $\mathsf{Fib}(\mathbb{B})$ *is 2-monadic over* $\mathsf{Cat}/\mathbb{B}$.

Proof. We sketch the constructions. Take a functor $\mathbb{E} \xrightarrow{p} \mathbb{B}$, and consider the comma category $\mathbb{B}/p$ whose objects are of the form $I \xrightarrow{x} pX$ and whose maps are pairs (f, m) making

$$\begin{tikzcd} I \arrow[r, "x"] \arrow[d, "f"'] & pX \arrow[d, "pm"] \\ J \arrow[r, "y"'] & pY \end{tikzcd}$$

commute.

There are evident projection functors $\mathbb{B}/p \xrightarrow{\pi_{\mathbb{B}}} \mathbb{B}$ and $\mathbb{B}/p \xrightarrow{\pi_{\mathbb{E}}} \mathbb{E}$.

Exercise 1.19.

(i) Show that there is a natural transformation

$$\begin{tikzcd} \mathbb{B}/p \arrow[r, "\pi_{\mathbb{E}}"] \arrow[d, "\pi_{\mathbb{B}}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B} \arrow[r, "1"] & \mathbb{B} \end{tikzcd}$$

which is universal, in the sense that any 2-cell $G \to pH$ (for $G : \mathbb{D} \longrightarrow \mathbb{B}, H : \mathbb{D} \longrightarrow \mathbb{E}$) factors through it in a unique way. This means that the above square is a *comma square.*

(ii) From the universal property, show that $p \mapsto \mathbb{B}/p$ is an endofunctor on $\mathsf{Cat}/\mathbb{B}$.

(iii) Using the universal property, construct a unit and a multiplication and show that $p \mapsto \mathbb{B}/p$ is a 2-monad.

(iv) Show that $\mathbb{B}/-$ is a KZ-monad (algebras are adjoint to units).

(v) Show that $\mathbb{B}/p \longrightarrow \mathbb{B}$ is a fibration (called the *free fibration* on p).

(vi) Show that an algebra structure for $\mathbb{B}/-$ is essentially the same thing as a fibration over $\mathbb{B}$. (Hint: it is helpful to find out how to factor a map in the comma category as a vertical map followed by a fibration.)

□

2 Fibrations and Logic

We now look at fibrations through the eyes of a logician. Table 1 gives the viewpoint.

Table 1: Fibrations from a logical point of view

What it is	What the logician sees
General fibration	proofs
Preorder fibration	provability
Object X in the base	Type (context) X
Object α over X	Predicate $\alpha(x)$, or formula with free variable of type X
Object α over 1	proposition
Morphism $Y \xrightarrow{f} X$ in $\mathbb{B}$	Term, change-of-context
Morphism $\alpha \xrightarrow{\phi} \beta$ in fibre over X	proof $\phi(x) : \alpha(x) \longrightarrow \beta(x)$
Vertical morphism $\alpha \longrightarrow \beta$ (preorder case)	Entailment $\alpha(x) \vdash_X \beta(x)$
Reindexing along $Y \xrightarrow{f} X$	substitution $x := f(y)$ e.g. $f^*(\alpha(x)) = \alpha(f(y))$

2.1 Propositional structure

We first consider some examples of fibrations which are best seen in this light, and which will also give us an indication of how to handle propositional structure.

Example 2.1. Let H be a Heyting algebra (poset with $\top, \bot, \wedge, \vee, \longrightarrow$). Define a Set-indexed poset by

$$X \longmapsto \mathsf{Set}(X, H)$$

Where $\mathsf{Set}(X, H)$ is a poset under the pointwise ordering. Reindexing is given by composition. All the logical structure on H lifts to the posets $\mathsf{Set}(X, H)$ and is preserved by reindexing. In case H is a locale (complete Heyting algebra), the indexed structure $\mathsf{Set}(-, H)$ is the *tripos of H-valued sets.*

Example 2.2. Define a Set-indexed preorder by

$$X \longmapsto \mathsf{Set}(X, \mathcal{P}\mathbb{N})$$

where, for $\phi, \psi : X \longrightarrow \mathcal{P}\mathbb{N}$, we put $\phi \vdash_X \psi \Leftrightarrow \bigcap_{x \in X}\{n \in \mathbb{N} | \forall m \in \phi(x).n \bullet m \in \psi(x)\} \neq \emptyset$

Exercise 2.3. Verify that this is transitive and reflexive.

One may show that $\mathcal{P}\mathbb{N}$ is not just a preorder, but is a Heyting prealgebra, i.e. has all propositional connectives. Use the following definitions (for $A, B \in \mathcal{P}\mathbb{N}$):

- $T = \mathbb{N}$
- $A \wedge B = \{j(a, b) | a \in A, b \in B\}$ where j is recursive pairing
- $A \vee B = (A \wedge \{0\}) \cup (B \wedge \{1\})$
- $A \Rightarrow B = \{e | \forall n \in A.e \bullet n \in B\}$
- $\bot = \emptyset$

This structure lifts to the fibres $\mathbf{Set}[X, \mathcal{P}(\mathbb{N}]$, each of which becomes a Heyting prealgebra as well; moreover, reindexing trivially preserves the structure. This fibration is called the *effective tripos*, with total category denoted Eff.

Example 2.4. There are a couple of fibrations related to the effective tripos which we mention here (all over Set): the *Partitioned assembly fibration*: objects of PAsm are morphisms $\alpha : X \longrightarrow \mathbb{N}$, where X is a set. A morphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ is a function $f : A \longrightarrow B$ which is *tracked*, in the sense that there is a partial recursive v such that $v(\alpha(x)) = \beta(f(x))$, all $x \in X$. The projection $(X, \alpha) \mapsto X$ (normally called global sections) is a preorder fibration.

This may be viewed as a more elementary structure than the effective tripos: the construction of meets and top element work, but other connectives fail. The singleton map $\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ induces a morphism of fibrations.

Example 2.5. The *Assembly fibration* is obtained by replacing the object $\mathcal{P}\mathbb{N}$ by $\mathcal{P}_{+}\mathbb{N}$, the nonempty subsets. This gives another subfibration of the effective tripos, denoted Asm. We have inclusions

$$\begin{tikzcd} \mathsf{PAsm} \arrow[r] \arrow[dr] & \mathsf{Asm} \arrow[r] \arrow[d] & \mathsf{Eff} \arrow[dl] \\ & \mathsf{Set}. & \end{tikzcd}$$

Later we will discuss the precise nature of these inclusions.

Since we wish to do logic, we need to say how to handle the propositional connectives in a fibration. As the above examples suggest, we wish that each of the fibres have the relevant structure and that this is preserved by reindexing. Let us assume first that we are in the case of a preorder fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$.

Definition 2.6. We say that p has *fibred meets (joins, implication, top, bottom)* when each fibre has meets (joins, ...) and when reindexing preserves the structure.

Let us look at another example: the subobject fibration $\mathsf{Sub}(\mathbb{B}) \longrightarrow \mathbb{B}$. This is a posetal fibration; it has a fibrewise top element (maximal subobject). If $\mathbb{B}$ has finite limits, then the subobject fibration has fibred meets; if $\mathbb{B}$ has an initial object, then we have a fibred bottom element. And so on.

To move from preordered to general fibrations, we note that the same type of definition works for the general case: if P is a categorical property (products, coproducts, exponents, ...) then a fibration is said to have fibred P if every fibre has P preserved by reindexing.

Exercise 2.7. Show that a small category $\mathbb{C}$ has finite products if and only if the family fibration $\mathsf{Fam}(\mathbb{C}) \longrightarrow \mathsf{Set}$ has fibred finite products. Show similar statements for other properties P. (Part of the point of fibred category theory is that this enables us to discuss such properties P for internal categories via their externalization!)

There is a generic theorem of the following form:

Theorem 2.8 (Typical theorem). *The category $\mathbb{C}$ has the ordinary categorical property P if and only if the family fibration $\mathsf{Fam}(\mathbb{C}) \longrightarrow \mathbb{B}$ has property P in the fibred sense.*

2.2 Quantification

We treat $\exists$, leaving $\forall$ to the reader.

Definition 2.9. An arrow $X \xrightarrow{m} Y$ over $I \xrightarrow{f} J$ is *cocartesian* if for every $X \xrightarrow{n} Z$ over f there is a unique vertical $Y \xrightarrow{v} Z$ with $vm = n$.

$$\begin{tikzcd} & Z \\ X \arrow[ur, "n"] \arrow[r, "m"'] & Y \arrow[u, dashed, "\exists! v"'] \end{tikzcd}$$

$$I \xrightarrow{f} J$$

Definition 2.10. A fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$ has existential quantification (sums, coproducts) if for every $pX \xrightarrow{f} J$ there is a cocartesian lift of f with domain X. Moreover, we require the Beck-Chevalley Condition: for every commutative square

$$\begin{tikzcd} X' \arrow[r, "m'"] \arrow[d, "n'"'] & Y' \arrow[d, "n"] \\ X \arrow[r, "m"] & Y \end{tikzcd} \quad \text{in } \mathbb{E} \text{ over a pullback} \quad \begin{tikzcd} I' \arrow[r, "f'"] \arrow[d, "g'"'] & J' \arrow[d, "g"] \\ I \arrow[r, "f"] & J \end{tikzcd} \quad \text{in } \mathbb{B}$$

with n', n cartesian, cocartesianness of m implies cocartesianness of m'.

In terms of indexed categories, this means that reindexing functors f^* have *left adjoints* $\exists_f$; BCC translates to the requirement that the canonical map $\exists_{f'} g'^* \longrightarrow g^* \exists_f$ is an isomorphism.

Exercise 2.11. Show that for any category with pullbacks, the codomain fibration has existential quantification.

Example 2.12. If H is a complete Heyting algebra then the indexed preorder $\mathsf{Set}[-, H]$ has existential quantification: for $f : X \longrightarrow Y$, the left adjoint to f^* is given by

$$\exists_f(\alpha)(y) = \bigvee_{f(x)=y} \alpha(x).$$

The BCC is immediate.

A fibred preorder which has fibred propositional structure and which interprets quantification along projections is sometimes called a *first-order doctrine*. Indexed preorders which allow for the interpretation of full higher order logic are called *triposes* (tripos stands for topos representing indexed preordered set).

Example 2.13. The effective tripos has existential quantification: for $f : X \longrightarrow Y$, the left adjoint to f^* is given by

$$\exists_f(\alpha)(y) = \bigcup_{f(y)=x} \alpha(x).$$

From the construction, we see that this does not work in partitioned assemblies. In assemblies, it works for surjections f.

Theorem 2.14.

(i) *The effective tripos arises from the partitioned assembly fibration by freely adjoining existential quantification. (More precisely:* Eff *is the free indexed preorder with existential quantification on* PAsm*.)*

(ii) Asm *arises by freely adjoining existential quantification along surjections to* PAsm*.*

(iii) *Finally, the effective tripos arises from assemblies by adjoining a bottom element.*

We conclude by indicating the general process of adding existential quantification to a fibration (not necessarily preordered): for a fibration p consider the pullback

$$\begin{tikzcd} \mathsf{Fam}(p) \arrow[r] \arrow[d, "\pi_p"'] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B} \arrow[r, "d_0"'] & \mathbb{B} \end{tikzcd}$$

Now form the composite $d_1 \circ \pi_p : \mathsf{Fam}(p) \longrightarrow \mathbb{B}$. This is a fibration over $\mathbb{B}$, called the family fibration over p. This construction is monadic (in the appropriate 2-categorical sense), and the algebras are precisely the fibrations with existential quantification.

2.3 Dialectica fibrations

We end with an example of a fibration which is motivated by Gödel's functional interpretation. We fix a cloven fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$, where $\mathbb{B}$ has finite products.

Construction 2.15. The category $\mathsf{dial}(p)$ has

- **Objects:** triples (I, X, α) where $\alpha \in \mathbb{E}(I \times X)$.
- **Arrows:** a map from (I, X, α) to (J, Y, β) is a triple (f, F, ϕ) with $f : I \longrightarrow J$, $F : I \times Y \longrightarrow X$, and $\phi = \phi(i, y) : \alpha(i, F(i, y)) \longrightarrow \beta(f(i), y)$.

Composition works as follows: for composable maps $(f, F, \phi) : (I, X, \alpha) \longrightarrow (J, Y, \beta)$ and $(g, G, \psi) : (J, Y, \beta) \longrightarrow (K, Z, \gamma)$, the composite is the triple (h, H, χ) with

$$h = gf; \qquad H(i, z) = F(i, G(f(i), z)); \qquad \chi(i, z) = \psi(f(i), G(f(i), z) \circ \phi(i, F(i, G(f(i), z))).$$

As an interesting structural property of this category, we mention:

Theorem 2.16. *The category* $\mathsf{dial}(p)$ *is symmetric monoidal if* p *is a product fibration, and is symmetric monoidal closed when* p *is a cartesian closed fibration.*

Normally, one thinks of $\mathsf{dial}(p)$ as a category of propositions. But it may be regarded as fibred over the base again, via the projection $(I, X, \alpha) \mapsto \alpha$. This gives rise to:

Proposition 2.17. *The assignment* $p \mapsto \mathsf{dial}(p)$ *is a pseudomonad on the category* $\mathsf{Fib}(\mathbb{B})$.

The proof is an instructive exercise.

Theorem 2.18. *Algebras for the dialectica monad are cloven fibrations which admit universal quantification along projections.*

A direct proof is possible, but here's a better line: recall that $\mathsf{Fam}(p)$ gives the free fibration with existential quantification on p. Now modify, by using the subcategory $\mathbb{B}^{\pi} \hookrightarrow \mathbb{B}^{\rightarrow}$ on the projections.

Exercise 2.19. Show that the resulting fibration, denoted $\mathsf{S}(p)$, is the free fibration on p with existential quantification along projections.

Now $\mathsf{dial}(p)$ may be regarded as arising in a dual fashion:

Proposition 2.20. *We have a fibred isomorphism* $\mathsf{dial}(p) \cong \mathsf{S}(p^{op})^{op}$.

Here, $(-)^{op}$ stands for the *opposite fibration*: in terms of indexed categories, we define the opposite of $\mathbb{B}^{op} \longrightarrow \mathsf{Cat}$ to be the composite $\mathbb{B}^{op} \longrightarrow \mathsf{Cat} \xrightarrow{(-)^{op}} \mathsf{Cat}$, where the second map takes the ordinary opposite of a category.

Construction 2.21. The *predicate logical dialectica category* over p has

- **Objects:** quadruples (I, X, U, α) where $\alpha \in \mathbb{E}(I \times X \times U)$.
- **Arrows:** a map from (I, X, U, α) to (J, Y, V, β) is a quadruple (f, F, F_2, ϕ) with $f : I \longrightarrow J$, $F : I \times Y \longrightarrow X$, $F_2 : I \times Y \times V \longrightarrow U$ and $\phi = \phi(i, y, v) : \alpha(i, F(i, y), F_2(i, y, v)) \longrightarrow \beta(f(i), y, v)$.

We denote this category by $\mathsf{Dial}(p)$ (note the capital D!).

Again, $\mathsf{Dial}(p)$ is fibred over the base via the first projection.

Proposition 2.22. *There is a fibred equivalence* $\mathsf{Dial}(p) \simeq \mathsf{S}(\mathsf{dial}(p))$. *Thus, the predicate logical version of dialectica arises by freely adding* $\exists$ *along projections to the propositional version.*

This accounts nicely for the $\exists\forall$ format of formulae in the image of the original dialectica interpretation. In case the base category is cartesian closed, $\mathsf{Dial}(p)$ has $\forall$ along projections as well.

Finally, we remark that there are various interesting variations on the dialectica construction: the *Girard category* of p has the same objects of $\mathsf{dial}(p)$ but only those arrows (f, F, ϕ) for which ϕ is an isomorphism. The *Lax Chu category* of p only has those maps (f, F, ϕ) for which $F : I \times Y \longrightarrow X$ factors through the projection $I \times Y \longrightarrow Y$. Finally, there is the intersection of the two, which is the *Strict Chu category* of p.