Positive Opetopes with Contractions form a Test Category

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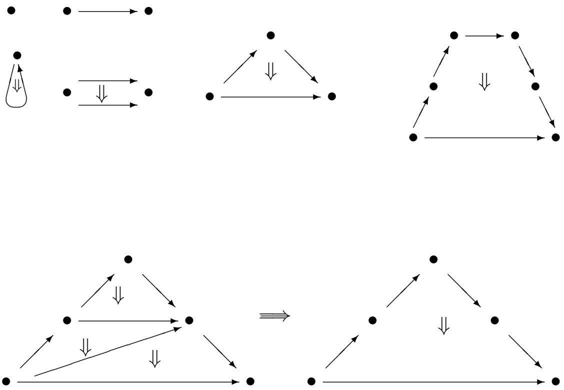
This talk is dedicated to
Jaap van Oosten and **Thomas Streicher**
on the occasion of their sixtieth birthday.

Plan of the talk

1. Opetopes and contraction morphisms
2. Test categories
3. Main Theorem
4. Description of the product $I \times P$

Opetopes - informal introduction

Opetopes were identified by J. Baez and J. Dolan as reasonable shapes that can serve as a base for a notion of a (virtual, weak) ∞-dimensional category.
They are are shapes like these...



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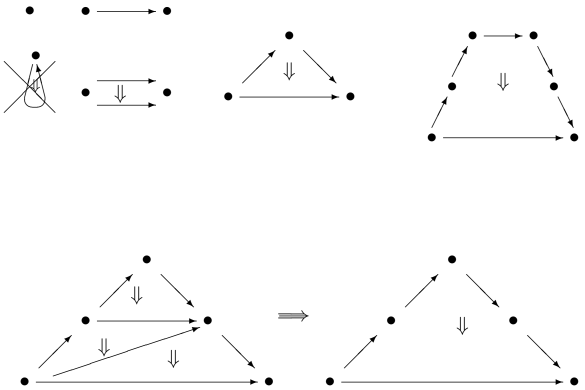
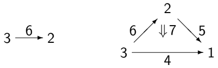
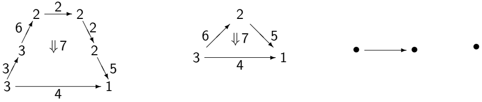


Morphisms of Opetopes - informal introduction

Morphisms of opetopes send faces to faces of the same dimension preserving domains and codomains (face maps only) **Ope**:



Contraction morphisms of opetopes send faces to faces of **at most** the same dimension preserving domains and codomains (some degeneracies allowed) $\mathbf{pOpe}_\iota$:

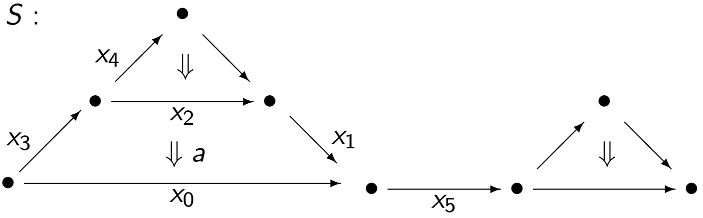


Positive Opetopes

Positive opetopes have easier combinatorics and from now on I will talk only about **positive opetopes** and positive opetopic sets, positive opetopic sets with contractions, and the like.

Positive Opetopic Cardinals
primitive notions

An example of a positive 2-dimensional opetopic cardinal



Primitive notions:

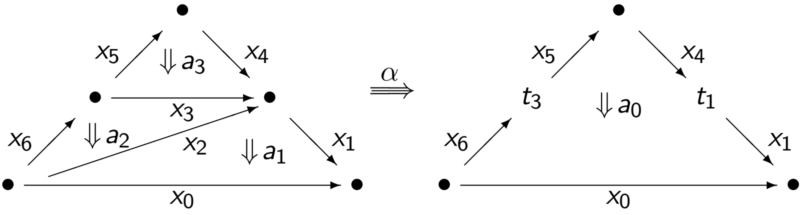
1. γ - **the codomain operation**: $\gamma(a) = x_0$
2. δ - **the domain operation**: $\delta(a) = \{x_1, x_2, x_3\}$ (positive = non-empty set)

Derived notions:

1. $<^-$ - **lower order**: $x_3 <^- x_2 <^- x_1 <^- x_5$; $(\gamma(x_3) \in \delta(x_2))$
2. $<^+$ - **upper order**: $x_4 <^+ x_2 <^+ x_0$; ($x_2 \in \delta(a)$ and $\gamma(a) = x_0$)

Positive Opetopic Cardinals
globularity axiom

An example of a 3-dimensional positive opetope



$$\gamma(\alpha) = a_0, \quad \delta(\alpha) = \{a_1, a_2, a_3\}$$

$$\gamma\gamma(\alpha) = x_0, \qquad \delta\gamma(\alpha) = \{x_1, x_4, x_5, x_6\}$$

$$\gamma\delta(\alpha) = \{x_0, x_2, x_3\}, \qquad \delta\delta(\alpha) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

Globularity. For any face α of dimension ≥ 2

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \qquad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

A family of sets $S = \{S_k\}_{k\in\omega}$ (almost all empty), S_k - faces of dimension k, together with operations γ, δ and relations $<^+$ and $<^-$ is *a positive opetopic cardinal* iff it satisfies

Globularity. For any face $\alpha \in S_{\geq 2}$

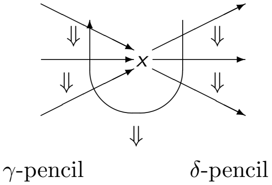
$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \qquad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

Strictness. The relation $<^+$ on each set S_k is a strict order (transitive irreflexive). The relation $<^+$ on S_0 is a linear order.

Disjointness. $<^+ \cap <^- = \emptyset$.

Pencil linearity. The sets of cells with common codomain (γ-*pencil*) and the sets of cells that have the same distinguished cell in the domain (δ-*pencil*) are linearly ordered by $<^+$.

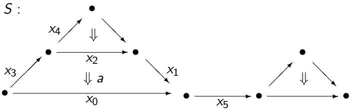
Pencil order $\prec_x$ over face x



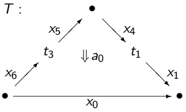
The **size of a positive opetopic cardinal** S is defined as an infinite sequence of natural numbers

$$size(S) = \{size(S)_k\}_{k\in\omega} = \{|S_k - \delta(S_{k+1})|\}_{k\in\omega}$$

(almost all equal 0).
A positive opetopic cardinal S is a **positive opetope** iff $size(S)_k \leq 1$ for $k \in \omega$.
Example.



$size(S) = (1, 3, 3, 0, \ldots)$,



$size(T) = (1, 1, 1, 0, \ldots)$.

A *morphism of positive opetopic cardinals* $f : T \to S$ is a family of functions $f_k : T_k \to S_k$, for $k \in \omega$, that commutes with γ's and δ's, i.e. for any $a \in T_{\geq 1}$,

$$\gamma(f(a)) = f(\gamma(a)) \text{ and } f_a : \delta(a) \longrightarrow \delta(f(a))$$

is a bijection, where f_a is the restriction of f to $\delta(a)$.

Thus we have a category **pOpe** of positive opetopes and the above morphisms (face maps).

We have an embedding functor

$$(-)^* : \mathbf{pOpe} \longrightarrow \omega\mathbf{Cat}$$

$$S \mapsto S^*$$

k-cells in S^* are sub-opetopic cardinals T of S with $dim(T) \leq k$.

This embedding is full on isomorphisms and hence we can think of morphisms in **pOpe** as some ω-functors, those that send generators to generators.

The category $\mathbf{pOpe}_\iota$ of positive opetopes with contractions is the category whose objects are positive opetopes and whose morphisms are ω-functors that send generators to either generators or (iterated) identities on generators.

Test categories were introduced by A. Grothendieck.
They are supposed to be small categories that are 'as good as the simplicial category Δ' to do homotopy theory.
But for some particular purposes other test categories could be even more suitable then Δ.

Let $\mathcal{N} : \mathbf{Cat} \longrightarrow \widehat{\Delta}$ be the usual nerve functor. We say that a functor $f : C \to D$ in **Cat** is a weak equivalence iff its nerve $\mathcal{N}(f)$ is a weak homotopy equivalence of simplicial sets. We denote by $\mathcal{W}_\infty$ the class of weak equivalences of categories.

Test categories
definition

For any small category $\mathcal{A}$, we have an adjunction $(\int_{\mathcal{A}} \dashv \mathcal{N}_{\mathcal{A}})$

$$\widehat{\mathcal{A}} \underset{\mathcal{N}_{\mathcal{A}}}{\overset{\int_{\mathcal{A}}}{\rightleftarrows}} \mathbf{Cat}$$

where $\widehat{\mathcal{A}}$ is the category of presheaves on $\mathcal{A}$, $\int_{\mathcal{A}}(F)$ is the category of elements of presheaf F in $\widehat{\mathcal{A}}$, and $\mathcal{N}_{\mathcal{A}}(C)(a) = \widehat{\mathcal{A}}(A_{/a}, C)$ for C in **Cat** and $a \in \mathcal{A}$.

1. The category $\mathcal{A}$ is a **weak test category** iff for any category C, the counit of adjunction $\varepsilon_C : \int_{\mathcal{A}} \mathcal{N}_{\mathcal{A}}(C) \to C$ is in $\mathcal{W}_\infty$.
2. The category $\mathcal{A}$ is a **test category** iff $\mathcal{A}$ and every slice $\mathcal{A}_{/a}$ is a weak test category.

Theorem

$\mathcal{A}$ - test category. The classes of monos and $\mathcal{W}_{\mathcal{A}} = (\int_{\mathcal{A}})^{-1}(\mathcal{W}_\infty)$ are cofibrations and weak equivalences of a proper model structure on $\widehat{\mathcal{A}}$. The adjunction $\int_{\mathcal{A}} \dashv \mathcal{N}_{\mathcal{A}}$ induces an equivalence of categories between homotopy categories

$$\mathcal{W}_{\widehat{\mathcal{A}}}^{-1}\widehat{\mathcal{A}} \underset{\bar{\mathcal{N}}_{\mathcal{A}}}{\overset{\bar{\int}_{\mathcal{A}}}{\rightleftarrows}} \mathcal{W}_\infty^{-1}\mathbf{Cat}$$

In particular,

1. *the homotopy category $\mathcal{W}_{\mathcal{A}}^{-1}\widehat{\mathcal{A}}$ is equivalent to Hot, the category of CW-complexes and homotopy classes of continuous functions;*
2. *$\widehat{\mathcal{A}}$ provides a model for the intensional Martin-Löf Type Theory.*

Main Theorem

Theorem

The category $\mathbf{pOpe}_\iota$ *of positive opetopes with contractions is a test category.*

Since $\mathbf{pOpe}_\iota$ has terminal object it suffices to show that the one dimensional opetope I:

$$- \xrightarrow{a} +$$

is locally aspherical. And for this it is enough to show, that for any positive opetope P, the product $I \times P$ is aspherical (see G. Maltsiniotis book, La théorie de l'homotopie de Grothendieck, for much more). I will show that

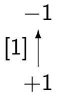
$$I \times P = \bigcup_{\vec{x} \in \mathbf{Flags}(P)} P^{\vec{x}}$$

where $\mathbf{Flags}(P)$ is the set of flags in P and $P^{\vec{x}}$ is an opetope.

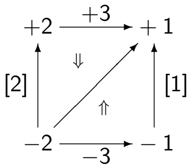
Product $I \times P$
examples

Some examples of products

$P = 1$:

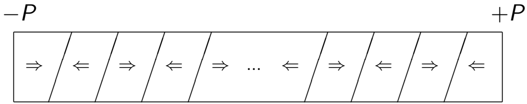


$P = 2 \xrightarrow{3} 1$:



Product $I \times P$
informal description

General picture of the product $I \times P$:



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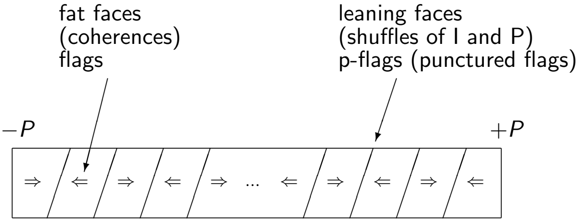

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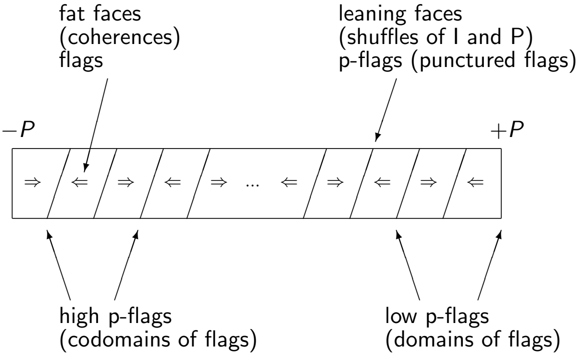


We have an embedding $\kappa : \mathbf{pOpe} \longrightarrow \mathbf{pOpe}_\iota$ and hence a left Kan extension

$$\kappa_! : \widehat{\mathbf{pOpe}} \longrightarrow \widehat{\mathbf{pOpe}_\iota}$$

We shall describe the object $\mathbf{Cyl}(P)$ in $\widehat{\mathbf{pOpe}}$ that is aspherical already in $\widehat{\mathbf{pOpe}}$ so that $\kappa_!(\mathbf{Cyl}(P))$ is the product of I and P in $\widehat{\mathbf{pOpe}_\iota}$.

The notion of a flag is due to T. Palm (2003).
A **flag** in P is a sequence of faces in P

$$\vec{x} = \begin{bmatrix} x_k \\ x_{k-1} \\ \dots \\ x_0 \end{bmatrix}$$

so that x_i is a face of dimension i and $x_i \in \delta(x_{i+1}) \cup \gamma(x_{i+1})$, $i = 0, \dots, k-1$. A flag is **maximal** if x_k is the top face of P, i.e. $dim(P) = k$.

Sign of a flag

$$\mathbf{sgn}(x_k, x_{k-1}, \dots, x_1) = \begin{cases} 1 & \text{if } k = 1 \\ \mathbf{sgn}(x_{k-1}, \dots, x_0) & \text{if } x_{k-1} = \gamma(x_k) \\ (-1) \cdot \mathbf{sgn}(x_{k-1}, \dots, x_0) & \text{if } x_{k-1} \in \delta(x_k) \end{cases}$$

Formal definition of $\mathbf{Cyl}(P)$

order on flags

$\mathbf{Flags}_P$ - is the set of maximal flags in P.

Flag order $\lhd$ on $\mathbf{Flags}_P$, the set of maximal flags in P. Let $\vec{x}, \vec{y} \in \mathbf{Flags}_P$ be two different flags. Let $k = \min\{j | x_j \neq y_j\}$. We put $\vec{x} \lhd \vec{y}$ iff

1. $k = 0$ and $y_0 <^+ x_0$,
2. or $k > 0$, $\mathbf{sgn}(x_{k-1}, \ldots, x_l) = 1$ and $x_k \prec_{x_{k-1}} y_k$
3. or $k > 0$, $\mathbf{sgn}(x_{k-1}, \ldots, x_l) = -1$ and $y_k \prec_{x_{k-1}} x_k$.

Clearly, the flag order is strict and linear. Its reflexive closure will be denoted by $\unlhd$.

Lemma

Two consecutive flags differ by exactly one face.

'Intersection' of two consecutive flags is a p-flag (plug 0 - a dummy face) in place flags differ. They are of form

high p-flag $(\vec{x}_{high})$ **low p-flag** $(\vec{x}_{low})$

$$\begin{bmatrix} x_k \\ 0 \\ x_{k-2} \\ \dots \\ x_0 \end{bmatrix} \qquad \begin{bmatrix} x_k \\ \gamma(x_k) \\ \dots \\ \gamma^{(l+2)}(x_k) \\ t \\ 0 \\ x_{l-1} \\ \dots \\ x_0 \end{bmatrix}$$

where $t \in \delta\gamma^{(l+2)}(x_k)$.

Faces in the opetopic set **Cyl**(P) in $\widehat{\mathbf{pOpe}}$:

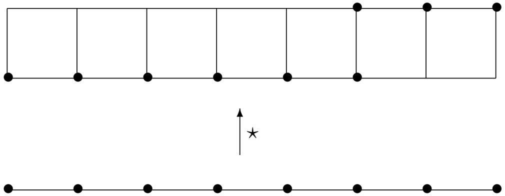
1. Flat faces $\{-\} \times P$ and $\{+\} \times P$;
2. All flags of all faces of P;
3. All p-flags of all faces in P.

The dimension of a face $-p$ or $+p$ is the dimension of p. The dimension of a flag or p-flag is the number of non-zero faces in the sequence. $\mathbf{Cyl}(P)_k$ is the set of all faces of **Cyl**(P) of dimension k.

We have a 'projection' function $\pi_P : \mathbf{Cyl}(P) \longrightarrow P$ such that for a face $\varphi \in \mathbf{Cyl}(P)$

$$\pi_P(\varphi) = \begin{cases} x & \text{if } \varphi \in \{-x, +x\}, \\ x_k & \text{if } \varphi = x_k, \ldots, x_0 \end{cases}$$

Intuitions from symplicial sets $\triangle_1 \times \triangle_7$:



We define an 'inverse' operation to projection on flags

$$\star : P \times \mathbf{Flags}_P \to \mathbf{Cyl}(P)$$

so that, for $\vec{x} = [x_n, \ldots, x_0] \in \mathbf{Flags}_P$ and $p \in P_k$, we have

$$p \star \vec{x} = \begin{cases} \vec{x}_{\lceil k} & \text{if } p = x_k; \\ & \text{(flag)} \\ [p, 0, \vec{x}_{\lceil k-2}] & \text{otherwise, if } k > 0 \text{ and } x_k <^+ p; \\ & \text{(high p-flag)} \\ [p, t, 0, \vec{x}_{\lceil k-2}] & \text{otherwise, if } k > 1,\ x_{k-1} \leq^+ t \in \delta(p); \\ & \text{(low p-flag)} \\ [p, \gamma(p) \star \vec{x}] & \text{otherwise, if } k > 2,\ \gamma(p) \star \vec{x} \text{ is a p-flag}; \\ & \text{(induction, low p-flag)} \\ -p & \text{otherwise, if } \gamma^{(0)}(p) \leq^+ x_0; \\ & \text{(bottom flat face)} \\ +p & \text{otherwise, if } x_0 <^+ \gamma^{(0)}(p). \\ & \text{(top flat face)} \end{cases}$$

Formal definition of $\mathbf{Cyl}(P)$
domains and codomains of flags

Let $\vec{x} = x_k, \ldots, x_0$ be a flag in $\mathbf{Cyl}(P)$. We put

$$\gamma(\vec{x}) = \vec{x}_{high}$$

and

$$\delta(\vec{x}) = \{\vec{x}_{\lceil k-1}, \vec{x}_{low}\}$$

$$\delta(\vec{x}) = \begin{cases} \{\vec{x}_{\lceil k-1}, -x_k\} & \text{if } \vec{x} \text{ is the first flag,} \\ \{\vec{x}_{\lceil k-1}, +x_k\} & \text{if } \vec{x} \text{ is the last flag,} \\ \{\vec{x}_{\lceil k-1}, \vec{x}_{low}\} & \text{otherwise.} \end{cases}$$

Let $\vec{x} = x_k, \ldots, \widehat{x_i}, \ldots, x_0$ be a p-flag in P. We put

$$\gamma(\vec{x}) = \gamma(x_k) \star \vec{x} = \begin{cases} \begin{bmatrix} \gamma(x_k) \\ 0 \\ x_{n-3} \\ \ldots \\ x_0 \end{bmatrix} & \text{if } i = k-1, k-2, \\ \begin{bmatrix} x_{k-1} \\ x_{k-2} \\ \ldots \\ \widehat{x_i} \\ \ldots \\ x_0 \end{bmatrix} & \text{otherwise.} \end{cases}$$

$$\delta(\vec{x}) = \begin{cases} \{p \star \vec{x} \mid p \in \delta(x_k)\} & \text{if } \vec{x} \text{ is a low p-flag,} \\ \{p \star \vec{x} \mid p \in \delta(x_k)\} \cup \{\vec{x}_{\lceil k-2}\} & \text{if } \vec{x} \text{ is a high p-flag.} \end{cases}$$

For $p \in P_{\geq 1}$, we have

$$\gamma(-p) = -\gamma(p), \quad \gamma(+p) = +\gamma(p)$$

and

$$\delta(-p) = \{-q : q \in \delta(p)\}, \quad \delta(+p) = \{+q : q \in \delta(p)\}.$$

Let $\vec{x}$ be a flag or p-flag in P.
By $P^{\vec{x}}$ we denote the least subset of faces of $\mathbf{Cyl}(P)$ containing the face $\vec{x}$ and closed under γ's and δ's.

Lemma

$P^{\vec{x}}$ is an opetope.

Theorem

1. *Let $\vec{x}' \lhd \vec{x}$ be two consecutive flags. Then, in $\widehat{\mathbf{pOpe}}$, we have*
$$\Big(\bigcup_{\vec{y} \unlhd \vec{x}'} P^{\vec{y}}\Big) \cap P^{\vec{x}} = P^{\vec{x}' \cap \vec{x}};$$
2. $\mathbf{Cyl}(P) = \bigcup_{\vec{x} \in \mathbf{Flags}_P} P^{\vec{x}}$ *in* $\widehat{\mathbf{pOpe}}$;
3. $I \times P = \mathbf{Cyl}_\iota(P) := \kappa_!(\mathbf{Cyl}(P))$ *in* $\widehat{\mathbf{pOpe}_\iota}$;
4. $I \times P$ *is aspherical in* $\widehat{\mathbf{pOpe}_\iota}$.

Thank You for Your Attention!