Cubical Assemblies and the Independence of the Propositional Resizing Axiom

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Overview

I present a model of cubical type theory [Cohen et al., 2018] in the category of internal cubical objects in the category of assemblies. This model

- has a univalent and *impredicative* universe;
- does NOT satisfy the propositional resizing axiom.

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Homotopy Type Theory

Homotopy type theory [Univalent Foundations Program, 2013] is an extension of Martin-Löf's dependent type theory inspired by homotopy-theoretic interpretations of type theory.

Features

- Identity type $a = b$ is thought of as *path space* from a to b.
- Univalence axiom: $(A \simeq B) \simeq (A = B)$ for all types A and B.
- Higher inductive types

Cubical Type Theory

Cubical type theory [Cohen et al., 2018] is a type theory intended for constructive justifications of the univalence axiom and higher inductive types.

- Direct manipulation of n-dimensional cubes.
- The univalence axiom is provable.
- Canonicity [Huber, 2017].
- Semantics in cubical sets.

Impredicative Universes

We say a universe $\mathcal{U}$ is *impredicative* if, for any type A and type family $B : A \to \mathcal{U}$, the dependent product type $\prod_{x:A} B(x)$ belongs to $\mathcal{U}$.

Examples

- $(\prod_{X:\mathcal{U}} X \to X) : \mathcal{U}$
- $(\prod_{X:\mathcal{U}} X \to (X \to X) \to X) : \mathcal{U}$
- $(\prod_{X:\mathcal{U}} \prod_{x:X} x = x \to X) : \mathcal{U}$

Impredicative Encodings of (Higher) Inductive Types

An interesting use of such an impredicative universe is to represent data types as polymorphic function types [Shulman, 2011].

- $\mathbb{N} :\equiv \prod_{X:\mathcal{U}} X \to (X \to X) \to X$
- $\mathbb{S}^1 :\equiv \prod_{X:\mathcal{U}} \prod_{x:X} x = x \to X$
- $\|A\| :\equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$ where hProp is the universe of homotopy propositions in $\mathcal{U}$.

They have constructors and recursors in the sense of the HoTT Book, but do not satisfy the induction principle in general. Refinements of impredicative encodings are studied by Awodey, Frey and Speight [Awodey et al., 2018].

Propositional Resizing

- A type A is a *homotopy proposition* if $\prod_{x,y:A} x = y$ is inhabited.
- Propositional resizing axiom: for any homotopy proposition A, there exist a homotopy proposition A' in $\mathcal{U}$ and an equivalence $A \simeq A'$.
- Originally proposed by Voevodsky [Voevodsky, 2012].
- A form of impredicativity for homotopy propositions.

If $\mathcal{U}$ is impredicative, we have a candidate for propositional resizing. For a homotopy proposition A, define
$A^* :\equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$ and $\eta_A :\equiv \lambda a.\lambda X h.h a : A \to A^*$.

Proposition

A homotopy proposition A admits propositional resizing if and only if the function $\eta_A : A \to A^*$ is an equivalence.

Outline

Overview

Internalizing cubical set models
[Bezem et al., 2014, Cohen et al., 2018] in the category **Asm** of
assemblies.

- ▶ **Asm** is a model of extensional dependent type theory with propositional truncation.
- ▶ **Asm** has an impredicative universe **PER**.
- ▶ We define an internal category □ of cubes in **Asm**.
- ▶ Let **CAsm** be the category of internal presheaves on □ which we call *cubical assemblies*.

Theorem
CAsm is a model of cubical type theory with an impredicative universe.

The Orton-Pitts Axioms

CAsm can be seen as an instance of more general construction of models of cubical type theory given by Orton and Pitts [Orton and Pitts, 2016]. They construct a model of cubical type theory in an elementary topos with

- ▶ an interval object $\mathbb{I}$; and
- ▶ an object $\mathrm{Cof} \rightarrowtail \Omega$ of *cofibrant propositions*

satisfying several axioms. In that model,

- ▶ a context Γ is an object of the topos;
- ▶ a type $\Gamma \vdash A$ is a morphism $A \to \Gamma$ equipped with a *composition structure*; and
- ▶ a term $\Gamma \vdash a : A$ is a section of the morphism $A \to \Gamma$.

The axioms can be written in extensional dependent type theory with propositional truncation but without the subobject classifier. So we can use their construction for making **CAsm** a model of cubical type theory, although **CAsm** is not an elementary topos.

Fibrations

Fibrations of cubical sets are defined by the lifting property:

$$A \times \mathbb{I} \cup B \times \{\varepsilon\} \longrightarrow X$$

$$B \times \mathbb{I} \longrightarrow Y$$

for a cofibration $A \to B$ and $\varepsilon = 0, 1$. In the internal language of cubical sets, it is an operator

$$\mathsf{fill}(Y, X) : (y : \mathbb{I} \to Y)(\varepsilon : \{0, 1\})(\varphi : \mathsf{Cof})(\bar{x} : \varphi \to (i : \mathbb{I}) \to X(yi))$$
$$(x_0 : X(y\varepsilon)) \to (\forall_{u:\varphi} \to \bar{x}u\varepsilon = x_0) \to$$
$$(i : \mathbb{I}) \to \{x : X(yi) \mid (\forall_{u:\varphi}\bar{x}ui = x) \wedge (i = \varepsilon \to x = x_0)\}.$$

Universes of Fibrant Types

Followins Licata, Orton, Pitts and Spitters [Licata et al., 2018], we use the adjunction $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ to construct universes of fibrant types. Recall

$$\text{fill}(Y, X) : (y : \mathbb{I} \to Y) \to F(\lambda i.X(yi))$$

for some $F : \mathcal{U}^{\mathbb{I}} \to \mathcal{U}$. Then we get a universe of fibrant types as the pullback

$$
\begin{array}{ccc}
\mathcal{U}^{F} & \longrightarrow & (\sum_{X:\mathcal{U}} X)_{\mathbb{I}} \\
\downarrow & & \downarrow \\
\mathcal{U} & \xrightarrow{\ \tilde{F}\ } & \mathcal{U}_{\mathbb{I}}.
\end{array}
$$

Outline

Propositional Resizing

Recall:

Proposition

In cubical type theory with an impredicative universe, a homotopy proposition $\Gamma \vdash A$ admits propositional resizing if and only if the function $\Gamma \vdash \eta_A : A \to A^*$ is an equivalence, where $A^* :\equiv \prod_{X:\mathsf{hProp}} (A \to X) \to X$.

Theorem

In the cubical assembly model, there exists a homotopy proposition $\Gamma \vdash A$ such that A^* is inhabited but A is not.

Overview

1. Find a family $\Gamma \vdash A$ of assemblies that is uniform and well-supported but not inhabited.
2. The codiscrete presheaf $\Delta\Gamma \vdash \nabla A$ over the constant presheaf $\Delta\Gamma$ is uniform and well-supported but not inhabited.
3. ∇A is a homotopy proposition.
4. It follows from the uniformity and well-supportedness that $(\nabla A)^*$ is inhabited.

Assemblies

- An *assembly* is a set A equipped with a non-empty set $E_A(a)$ of natural numbers for each $a \in A$. Elements of $E_A(a)$ are called *realizers* of a.

- A morphism $f : A \to B$ of assemblies is a function $f : A \to B$ between the underlying sets such that there exists a partial recursive function e such that, for any $a \in A$ and $n \in E_A(a)$, the application en is defined and belongs to $E_B(f(a))$.

- The category $\mathbf{Asm}$ of assemblies has an impredicative universe $\mathbf{PER}$ of partial equivalence relations on $\mathbb{N}$.

- A morphism $f : A \to \mathbf{PER}$ corresponds to a *modest family* $B \to A$: for any $a \in A$ and $b, b' \in B_a$, if $E(b) \cap E(b') \neq \emptyset$ then $b = b'$.

Uniform Objects

- An assembly A is *uniform* if $\bigcap_{a \in A} E(a)$ is non-empty.
- In a regular category, an object A is *well-supported* if the unique morphism $A \to 1$ is regular epi.

Proposition

For assemblies A and X, if A is uniform and well-supported and X is modest, then the morphism

$$\lambda x a.x : X \to (A \to X)$$

is an isomorphism.

Uniform Presheaves

Uniform and modest internal presheaves are defined pointwise.

Proposition

For internal presheaves A and X, if A is uniform and well-supported and X is modest, then the morphism

$$\lambda x a.x : X \to (A \to X)$$

is an isomorphism.

Corollary

If a type $\Gamma \vdash A$ in $\mathbf{CAsm}$ is uniform and well-supported, then $A^* \equiv \prod_{X:\mathsf{hProp}}(A \to X) \to X$ is inhabited.

Codiscrete Cubical Assemblies

For an assembly Γ, let $\Delta\Gamma$ denote the constant presheaf. For a family $\Gamma \vdash A$ of assemblies, we can define the *codiscrete cubical assembly* $\Delta\Gamma \vdash \nabla A$ in such a way that

- the points of ∇A is elements of A; and
- any two points are connected by a unique path.

Proposition

- If A is uniform then so is ∇A.
- If A is well-supported then so is ∇A.
- If ∇A is inhabited then so is A.

So our goal is to find a family $\Gamma \vdash A$ of assemblies that is uniform and well-supported but not inhabited.

The Counterexample

Define a family $\Gamma \vdash A$ of assemblies as follows.

- ▶ $\Gamma = (\mathbb{N}, n \mapsto \{m \in \mathbb{N} \mid m > n\})$
- ▶ $A(n) = (\{m \in \mathbb{N} \mid m > n\}, m \mapsto \{n, m\})$

Then A is uniform and well-supported but not inhabited, and thus $\Delta\Gamma \vdash \nabla A$ is a homotopy proposition in $\mathbf{CAsm}$ that does not admit propositional resizing.

Conclusion

We have got a model of cubical type theory in the category of cubical assemblies that

- ▶ has a univalent and impredicative universe; but
- ▶ does not satisfy the propositional resizing axiom.

Related and Future Work I

- Is there a model of type theory with a univalent and impredicative universe that satisfies the propositional resizing axiom?
  - Benno van den Berg [van den Berg, 2018] has constructed a model of type theory with a univalent and impredicative universe of 0-types that satisfies the propositional resizing axiom, but in his model computational rules for identity types and dependent product types hold only up to propositional equality.
- Higher inductive types via W-types with reductions [Swan, 2018b], including propositional truncation.
- Can we say that $\mathbf{CAsm}$ is a "realizability higher category"?
  - In $\mathbf{Asm}$ every proposition (monomorphism) is modest.
  - Church's Thesis: $\forall_{f:\mathbb{N}\to\mathbb{N}}\exists_{e:\mathbb{N}}\varphi_e = f$, where φ_e is the e-th partial recursive function. It does not hold in $\mathbf{CAsm}$ because the type of points is $\prod_{f:\mathbb{N}\to\mathbb{N}}\sum_{e:\mathbb{N}}\varphi_e = f$ in $\mathbf{Asm}$. But there is a full subcategory of $\mathbf{CAsm}$ that satisfies Church's Thesis (joint work with Andrew Swan).

Related and Future Work II

- Model structures
  - Frumin and van den Berg [Frumin and van den Berg, 2018] has constructed a model structure on a full subcategory of an elementary topos that satisfies similar conditions to those of Orton and Pitts. Their construction will work for cubical assemblies.
  - Trivial cofibration-fibration factorization on **CAsm** using Swan's small object argument over codomain fibrations [Swan, 2018a, Swan, 2018b].
- How to relate Stekelenburg's constructive simplicial homotopy [Stekelenburg, 2016]?

References I

📑 Awodey, S., Frey, J., and Speight, S. (2018).
Impredicative Encodings of (Higher) Inductive Types.
In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 76–85, New York, NY, USA. ACM.

📑 Bezem, M., Coquand, T., and Huber, S. (2014).
A Model of Type Theory in Cubical Sets.
In Matthes, R. and Schubert, A., editors, *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 107–128, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

References II

Cohen, C., Coquand, T., Huber, S., and Mörtberg, A. (2018).
Cubical Type Theory: A Constructive Interpretation of the
Univalence Axiom.
In Uustalu, T., editor, *21st International Conference on Types
for Proofs and Programs (TYPES 2015)*, volume 69 of *Leibniz
International Proceedings in Informatics (LIPIcs)*, pages
5:1–5:34, Dagstuhl, Germany. Schloss
Dagstuhl–Leibniz-Zentrum fuer Informatik.

Frumin, D. and van den Berg, B. (2018).
A homotopy-theoretic model of function extensionality in the
effective topos.

Huber, S. (2017).
Canonicity for Cubical Type Theory.

References III

Licata, D. R., Orton, I., Pitts, A. M., and Spitters, B. (2018).
Internal Universes in Models of Homotopy Type Theory.
In Kirchner, H., editor, *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*, volume 108 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 22:1–22:17, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

Orton, I. and Pitts, A. M. (2016).
Axioms for Modelling Cubical Type Theory in a Topos.
In Talbot, J.-M. and Regnier, L., editors, *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 24:1–24:19, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

Shulman, M. (2011).
Higher inductive types via impredicative polymorphism.

References IV

📄 Stekelenburg, W. P. (2016).
Constructive Simplicial Homotopy.

📄 Swan, A. W. (2018a).
Lifting Problems in Grothendieck Fibrations.

📄 Swan, A. W. (2018b).
W-Types with Reductions and the Small Object Argument.

📄 Univalent Foundations Program (2013).
Homotopy Type Theory: Univalent Foundations of Mathematics.
Institute for Advanced Study.

📄 van den Berg, B. (2018).
Univalent polymorphism.

📄 Voevodsky, V. (2012).
A universe polymorphic type system.