Practice problems: Maclaurin series

For each of the following functions, express it as a powerseries.

1. \( f(x) = \frac{3}{1-2x} \)

   **Solution.** Use \( \frac{1}{1-x} = \sum_{n=1}^{\infty} x^n \). Replace \( x \) by \( 2x \) and multiply by 3:
   \[
   \frac{3}{1-2x} = \sum_{n=0}^{\infty} 3(2x)^n = \sum_{n=0}^{\infty} 3 \cdot 2^n x^n.
   \]

2. \( f(x) = \frac{1}{2-x} \)

   **Solution.** Use \( \frac{1}{1-x} = \sum_{n=1}^{\infty} x^n \). Divide by two:
   \[
   \frac{1}{2-x} = \frac{1}{1-x/2} = \sum_{n=0}^{\infty} \frac{1}{2} (x/2)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n
   \]

3. \( f(x) = \frac{1}{(1-2x)^2} \)

   **Solution.** Note first that \( f(x) \) is the derivative of the function \( g(x) = \frac{1}{2(1-2x)} \), which has Maclaurin series \( g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} x^n \). We differentiate this series and get
   \[
   f(x) = \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n} x^{n-1}.
   \]

4. \( f(x) = \frac{1}{(1-2x)^3} \)

   **Solution.** This function is the derivative of \( h(x) = \frac{1}{4(1-2x)^2} \), which (by the previous problem) has Maclaurin series \( h(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x^{n-1} \). We differentiate this series and get
   \[
   f(x) = \sum_{n=2}^{\infty} n \cdot \frac{1}{4} 2^n x^{n-2}.
   \]

5. \( f(x) = \ln |1-x| \)

   **Solution.** This is the antiderivative of \( \frac{1}{1-x} \). Thus we integrate the series \( \sum_{n=1}^{\infty} x^n \) and get
   \[
   \ln |1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{x+1}.
   \]

6. \( f(x) = 3e^2x \)

   **Solution.** This is a polynomial so already is a powerseries (the only nonzero coefficient is \( c_1 = 3e^2 \)).
7. \( f(x) = e^{x^2} \)

**Solution.** Use \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Replace \( x \) by \( x^2 \) to get

\[
e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.
\]

8. \( f(x) = xe^{x^2} \)

**Solution.** This is the derivative of \( g(x) = \frac{1}{2} e^{x^2} \), which has series \( \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} \). Differentiating gives

\[
x e^{x^2} = \sum_{n=1}^{\infty} \frac{2n}{2n!} x^{2n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{2n-1}.
\]

9. \( f(x) = -3x^2e^{x^3} \)

**Solution.** The Maclaurin series for \( e^{x^3} \) is \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \). Differentiate this and multiply by \(-1\):

\[-3x^2e^{x^3} = \sum_{n=1}^{\infty} \frac{3n}{n!} x^{3n-1}.
\]

10. \( f(x) = \int e^{x^2} \, dx \)

**Solution.** Integrate \( \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \) to get

\[
\int e^{x^2} \, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}.
\]

11. \( f(x) = x^2e^{x^3} \)

**Solution.** As for \( f(x) = -3x^2e^{x^3} \).