Omitting types theorem, conceptual completeness and definability for infinitary logic

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These shaped the model theory studied in the third-quarter of the XXth century.

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A type p is called isolated if there is $\psi \in p$, $\psi \neq \bot$, such that $\psi(\mathbf{x}) \vdash_{\mathbf{x}} \phi(\mathbf{x})$ for every $\phi \in p$.

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A type p is called isolated if there is $\psi \in p$, $\psi \neq \perp$, such that $\psi(\mathbf{x}) \vdash_{\mathbf{x}} \phi(\mathbf{x})$ for every $\phi \in p$.

Theorem

Given a countable theory $\mathcal{T}$, every non-isolated type p is omitted by some model. Moreover, every countable set of non-isolated types is simultaneously omitted by some model.

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Consider the syntactic category $\mathcal{C}_{\mathcal{T}}$ of the theory $\mathcal{T}$, and consider the Grothendieck coverage τ generated by:

- finite epimorphic families
- families of the form $\{[\mathbf{x}, \neg\phi_i] \longrightarrow [\mathbf{x}, \top]\}_{i \in I}$, for every type $p = \{\phi_i(\mathbf{x}) : i \in I\}$ in a countable set of types we want to omit.

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Observe that a model omitting the countable set of types is the same as a flat functor $\mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}et$ which is continuous for this topology, i.e., a point of the topos $\mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$.

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Next, observe that $\mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$ is a *separable* topos: a topos for which there is a site on a category with countably many objects and morphisms, and where the Grothendieck topology is generated by countably many covering families.

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The topos $\mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$ could, however, be degenerate. In fact, this is the case if the theory is complete and at least one type is isolated.
But this is the only contraint. If every type $p = \{\phi_i(\mathbf{x}) : i \in I\}$ is non-isolated, then $\bigwedge_{i \in I} \phi_i = \bot$ in the classifying topos $\mathcal{S}et[\mathcal{T}]$, and hence $\neg\neg \bigvee_{i \in I} \neg\phi_i = \top$ there. This is enough to guarantee that $[-, 0]$ is a sheaf for τ, and therefore, $0 \neq 1$ in $\mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$.

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Definition

A κ-topos is a topos of sheaves on a site with κ-small limits in which the covers of the topology satisfy in addition the transfinite transitivity property (a transfinite version of the transitivity property), i.e., transfinite composites of covering families form a covering family.

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A κ-topos is κ-separable if the underlying category of the site has at most κ many objects and morphisms, and where the Grothendieck topology is generated by at most κ many covering families.

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A κ-topos is κ-separable if the underlying category of the site has at most κ many objects and morphisms, and where the Grothendieck topology is generated by at most κ many covering families.
A κ-point of a κ-topos is a point whose inverse image preserves all κ-small limits.

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Theorem

(E.) Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then a κ-separable κ-topos has enough κ-points.

This is an infinitary version of Deligne completeness theorem. When κ is strongly compact (e.g., $\kappa = \omega$), we recover the usual version: a κ-coherent topos has enough κ-points.

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$$\frac{\begin{array}{ll} \phi_f \vdash_{\mathbf{y}_f} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \exists \mathbf{x}_g \phi_g & \beta < \kappa, f \in \gamma^\beta \\ \phi_f \dashv\vdash_{\mathbf{y}_f} \bigwedge_{\alpha < \beta} \phi_{f|_\alpha} & \beta < \kappa, \text{ limit } \beta, f \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{f \in B} \exists_{\beta < \delta_f} \mathbf{x}_{f|_{\beta+1}} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

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Theorem

(E.) Let κ *be a regular cardinal such that* $\kappa^{<\kappa} = \kappa$ *(resp.* κ *is weakly compact). Let* F *be a* κ*-fragment of* $\mathcal{L}_{\kappa^+,\kappa}$ *(resp.* $\mathcal{L}_{\kappa,\kappa}$*) and let* $\{p_i : i < \kappa\}$ *be a set of non-isolated types. Then there is a model that simultaneously omits all the types.*

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This theorem can be expressed in an entirely semantical way:

Theorem

(E.) Let $\mathcal{T}$ be a satisfiable theory in a κ-fragment of $\mathcal{L}_{\kappa^+,\kappa}$ (resp. $\mathcal{L}_{\kappa,\kappa}$) and let p_i, for each $i < \kappa$, be a set of formulas of the fragment. Suppose that whenever ψ is such that $\mathcal{T} \cup \exists \mathbf{x} \psi$ is satisfiable, there is ϕ in p_i such that $\mathcal{T} \cup \exists \mathbf{x}(\psi \wedge \neg \phi)$ is satisfiable. Then the theory:

$$\mathcal{T} \cup \bigwedge_{i<\kappa} \forall \mathbf{x} \bigvee_{\phi \in p_i} \neg \phi(\mathbf{x})$$

is satisfiable.

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Proposition

(Infinitary Vaught) The following are equivalent:

1. $\mathcal{T}$ *has a prime model.*
2. $\mathcal{T}$ *is atomic.*

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Let now κ be a weakly compact cardinal. Let $\mathcal{T}$ be a complete theory in $\mathcal{L}_{\kappa,\kappa}$ with at most κ axioms that has models of cardinality κ.

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Proposition

(Infinitary Ryll-Nardzewski / Blass-Ščedrov) The following are equivalent:

1. $\mathcal{T}$ *is* κ*-categorical.*
2. *For each context* $\mathbf{x}$*, the Boolean algebra of (equivalence classes of) formulas with free variables among* $\mathbf{x}$ *has cardinality less than* κ*.*
3. *The* κ*-classifying topos of* $\mathcal{T}$ *is Boolean.*

Applications of the omitting types theorem

It is a well-known fact of set theory that a cardinal κ is weakly compact if and only if the structure $(V_\kappa, \in, R)$ (where R is a unary predicate) has an elementary end extension, i.e., there is a transitive set W properly extending V_κ and some $R^* \subseteq W$ such that $(V_\kappa, \in, R) \prec (W, \in, R^*)$.

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V_κ is a model of ZF_κ of cardinality κ, but it's not the only one: there are 2^κ many! Techniques from inner model theory and forcing can be used in the infinitary case.

Duality, definability and descent

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Theorem

(E.) The λ-classifying topos of $\mathcal{T}$, considered as a theory in $\mathcal{L}_{\lambda^+,\lambda}$, is precisely the presheaf topos $\mathcal{S}et^{Mod_p(\mathcal{T})}$.

Duality, definability and descent

The first consequence is a positive result in definability theorem for infinitary logic. If $\mathcal{C}_{\mathcal{T}}$ is the syntactic category of $\mathcal{T}$ considered in $\mathcal{L}_{\lambda^+,\lambda}$, we have that

$$ev : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}et^{Mod_p(\mathcal{T})}$$

can be identified with Yoneda embedding

$$Y : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$$

where the coverage τ consists of λ^+-small jointly epic families of arrows.

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Theorem

(Infinitary Beth) Let $\phi(R)$ be a formula in $\mathcal{L}_{\kappa^+,\kappa}$ over the language $\mathcal{L} \cup R$ containing the predicate R. If every $\mathcal{L}$-structure has a unique expansion to a model of $\phi(R)$ and the interpretation of R in each such model is preserved by $\mathcal{L}$-homomorphisms, then there is an $\mathcal{L}$-formula ψ of $\mathcal{L}_{\lambda^+,\lambda}$ such that $R \dashv\vdash_{\mathbf{x}} \psi$.

Duality, definability and descent

Another consequence is the conceptual completeness theorem for $\mathcal{L}_{\kappa^+,\kappa}$:

Theorem

(Infinitary conceptual completeness) If a λ*-coherent functor* $I : \mathcal{P} \longrightarrow \mathcal{S}$*, where* $\mathcal{P}$ *is a* λ^+*-pretopos, induces an equivalence between their categories of models* $I^* : Mod(\mathcal{S}) \longrightarrow Mod(\mathcal{P})$*, then* I *is itself an equivalence.*

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Theorem

(Infinitary Joyal) If $\mathcal{T}$ is intuitionistic first-order, the functor:

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This version of Joyal's theorem provides a proof of completeness with respect to Kripke models for theories in $\mathcal{L}_{\kappa^+,\kappa,\kappa}$.

Duality, definability and descent

To state the duality theorem, we introduce the following:

Definition

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between λ-accessible categories is λ^+-coherent if the induced functor $F^* : FC_\lambda(D, \mathcal{S}et) \longrightarrow FC_\lambda(C, \mathcal{S}et)$ preserves λ^+-coherent objects.

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Theorem

(Infinitary Stone duality) Let $\lambda > \kappa$ be inaccessible. There is a (bi-)equivalence (given by homming into $\mathcal{S}et$) between the following categories:

1. *$\mathcal{A}$: λ^+-pretopos completion of (syntactic categories of) theories in $\mathcal{L}_{\lambda^+,\lambda}$ with less than λ axioms in $\mathcal{L}_{\lambda,\lambda}$; λ^+-pretopos morphisms; natural transformations.*
2. *$\mathcal{B}$: μ-accessible categories for $\mu < \lambda$; λ-accessible, λ^+-coherent functors preserving λ-presentable objects; natural transformations.*

Corollary

The category of $\mathcal{A}$-morphisms between two objects $\mathcal{T}$ and $\mathcal{S}$, in $\mathcal{A}$, is equivalent to the category of $\mathcal{B}$-morphisms between $Mod(\mathcal{S})$ and $Mod(\mathcal{T})$.

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Definability, conceptual completeness and Kripke completeness results all imply their versions when κ is weakly compact and the theories are taken in $\mathcal{L}_{\kappa,\kappa}$. In particular, they work for $\kappa = \omega$, which gives the usual corresponding finitary results.

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It turns out that the previous duality theorem is flexible enough to cast Zawadowski's argument for the descent theorem, which simplifies his proof. We get:

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The category of $\mathcal{A}$-morphisms between two objects $\mathcal{T}$ and $\mathcal{S}$, in $\mathcal{A}$, is equivalent to the category of $\mathcal{B}$-morphisms between $Mod(\mathcal{S})$ and $Mod(\mathcal{T})$.

Definability, conceptual completeness and Kripke completeness results all imply their versions when κ is weakly compact and the theories are taken in $\mathcal{L}_{\kappa,\kappa}$. In particular, they work for $\kappa = \omega$, which gives the usual corresponding finitary results.
It turns out that the previous duality theorem is flexible enough to cast Zawadowski's argument for the descent theorem, which simplifies his proof. We get:

Theorem

(Infinitary Zawadowski) If κ is weakly compact, conservative κ-pretopos morphisms between κ-pretoposes are of effective descent.

Thank you!