A simplicial framework for de Rham cohomology in a tangent category

Geoff Cruttwell
(joint work with Rory Lucyshyn-Wright)
Mount Allison University

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Overview

- Tangent categories provide an abstract framework to develop many concepts in differential geometry.
- Many key concepts and results from differential geometry have already been developed in this framework (Lie bracket, vector bundles, connections). But differential forms and de Rham cohomology have proven elusive.
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- Tangent categories provide an abstract framework to develop many concepts in differential geometry.
- Many key concepts and results from differential geometry have already been developed in this framework (Lie bracket, vector bundles, connections). But differential forms and de Rham cohomology have proven elusive.
- In this talk we’ll look at variants of the notion of differential form in tangent categories.
- In particular, we’ll look at sector forms, and show that they have very rich structure. Our results about this structure appear to be new, even in ordinary differential geometry.
- From the sector forms, we’ll get a definition of de Rham cohomology in a tangent category as a simple corollary.
A tangent category consists of a category $\mathbf{X}$ with:

- an endofunctor $T : \mathbf{X} \to \mathbf{X}$;
- a natural transformation $p : T \to 1_{\mathbf{X}}$;
- for each $M$, the pullback of $n$ copies of $p_M : TM \to M$ along itself exists (and is preserved by each $T^m$), call this pullback $T_nM$;
- for each $M \in \mathbf{X}$, $p_M : TM \to M$ has the structure of a commutative monoid in the slice category $\mathbf{X}/M$, in particular there are natural transformations $+: T_2 \to T$, $0 : 1_{\mathbf{X}} \to T$;

($TM$ represents the “tangent bundle” of an object $M$.)
Definition

- (canonical flip) there is a natural transformation $c : T^2 \to T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \to T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for $\ell$ and $c$;
- (universality of vertical lift) “an element of $T^2M$ which has $T(p) = 0$ is uniquely given by an element of $T_2M$”.
Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category (includes all Fermat theories by a result of MacAdam).

(iv) The infinitesimally linear objects in a model of synthetic differential geometry (SDG).

(v) Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.

(vi) The category of \( C^\infty \)-rings.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.
Differential objects

Definition

A **differential object** in a tangent category consists of a commutative monoid $E$ with a map $\hat{p} : TE \to E$ such that

$$
\begin{array}{ccc}
E & \xrightarrow{\hat{p}} & TE \\
\xleftarrow{\hat{p}} & & \xrightarrow{p_E} \ & \ & \ & \ & \ & E
\end{array}
$$

is a product diagram, and such that $\hat{p}$ satisfies various coherences with the tangent structure.

Examples:

- $\mathbb{R}^n$’s in the category of smooth manifolds.
- Convenient vector spaces in the category of convenient manifolds.
- Euclidean $R$-modules in models of SDG.
Differential objects also have a map

$$\lambda : E \rightarrow TE$$

which will be useful when defining “linear” maps to these objects.

If $E$ is a differential object, any map

$$X \xrightarrow{f} E$$

has an associated “derivative” $D(f) : TX \rightarrow E$ given by

$$TX \xrightarrow{Tf} TE \xrightarrow{\hat{p}} E$$
The classical notion of differential $n$-form on a smooth manifold $M$ is a smooth map

$$T_n M \xrightarrow{\omega} \mathbb{R}$$

which is multilinear and alternating (switching two of the inputs gives the negative).

In a tangent category, we have the objects $T_n M$, can replace $\mathbb{R}$ with a differential object $E$, and give a suitable definition of multilinear and alternating to get “classical” differential forms as multilinear alternating maps

$$T_n M \xrightarrow{\omega} E$$
But the exterior derivative of a classical form $\omega$ is problematic.

Classically, the exterior derivative is defined locally (not possible in an arbitrary tangent category!) by an alternating sum of various derivatives of $\omega$.

In a tangent category, if we have a classical form

$$T_n(M) \xrightarrow{\omega} E$$

then its derivative is

$$T(T_n M) \xrightarrow{D(\omega)} E$$

which is not the right type.

An arbitrary $M$ does not have a canonical choice of map

$$T_{n+1}(M) \rightarrow T(T_n(M))$$

to get a classical $(n + 1)$-form.
In SDG, one instead considers *singular* forms: maps

$$T^n(M) \xrightarrow{\omega} E$$

suitably multilinear and alternating.

In smooth manifolds, giving such a map is equivalent to giving a classical form (!).

One can similarly define singular forms in tangent categories, and define an appropriate exterior derivative for such singular forms in a tangent category, as the derivative of $\omega$

$$T^{n+1}(M) \xrightarrow{D(\omega)} E$$

has the correct type (the exterior derivative is then defined as an alternating sum of permutations of this derivative).
When calculating with singular forms, it becomes natural to consider maps

\[ T^n(M) \stackrel{\omega}{\to} E \]

which are merely multilinear (not necessarily alternating).

These are known as “sector forms”, and have been investigated only briefly in differential geometry in a book by J.E. White.

These will be the main object of interest for us.
Comparison of forms

For comparison:

- $T_n(M)$: $n$ (first-order) tangent vectors on $M$.
- $T^n(M)$: $n$th order tangent vector on $M$.
- There is a canonical map $T^n(M) \rightarrow T_n(M)$.
- Thus sector forms generalize classical forms, singular forms, and covariant tensors:

<table>
<thead>
<tr>
<th>Alternating</th>
<th>Not Alternating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain $T_n$</td>
<td>$T^n$</td>
</tr>
<tr>
<td>Differential form</td>
<td>Covariant tensor</td>
</tr>
<tr>
<td>Singular form</td>
<td>Sector form</td>
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</tbody>
</table>
Definition of sector forms in a tangent category

**Definition**

A **sector $n$-form** on $M$ with values in $E$ is a morphism $\omega : T^n M \to E$ such that for each $i \in \{1, ..., n\}$, $\omega$ is *linear in the $i$th variable*; that is, the following diagram commutes:

$$
\begin{array}{ccc}
T^n M & \xrightarrow{\omega} & E \\
\downarrow a^n_i & & \downarrow \lambda \\
T^{n+1} M & \xrightarrow{T(\omega)} & TE
\end{array}
$$

(where $a^n_1 = \ell$, $a^n_2 = cT(\ell)$, $a^n_3 = cT(c)T^2(\ell)$, etc.)

The set of sector $n$ forms on $M$ with values in $E$ will be denoted by $\Psi_n(M; E)$; we will often abbreviate this to $\Psi_n(M)$. 
Fundamental derivative of a sector form

- There is an operation
  \[ \delta_1 : \Psi_n(M) \to \Psi_{n+1}(M) \]
  given by sending a sector \( n \)-form
  \[ \omega : T^n M \to E \]
  to the sector \( (n + 1) \)-form
  \[ D(\omega) : T^{n+1} M \to E \]

- **Note**: even if \( \omega \) is alternating, \( \delta_1(\omega) := D(\omega) \) need not be.
- But there are actually \( n \) other related “derivatives”...
Symmetry operations

- For any $n \geq 2$, pre-composing a sector $n$-form $\omega$ with the canonical flip again gives an $n$-form:

$$T^n M \xrightarrow{c_{T^{n-2}M}} T^n M \xrightarrow{\omega} E$$

 giving an operation

$$\sigma_1 : \Psi_n M \rightarrow \Psi_n M$$

- And for higher $n$, pre-composing with $T(c_{T^{n-3}M})$, $T^2(c_{T^{n-4}M})$, etc. gives $n - 1$ different symmetry operations

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1} : \Psi_n M \rightarrow \Psi_n M$$
Derivative/coface operations

- By post-composing the fundamental derivative

\[ \delta_1 : \Psi_n(M) \rightarrow \Psi_{n+1}(M) \]

with the first symmetry

\[ \sigma_1 : \Psi_{n+1}(M) \rightarrow \Psi_{n+1}(M) \]

we get a new “derivative”

\[ \delta_2 : \Psi_n(M) \rightarrow \Psi_{n+1}(M) \]

- Post-composing this with \( \sigma_2 \) gives \( \delta_3 \), then \( \delta_4 \), etc...continuing in this way we get \((n+1)\) total ways to get an \((n+1)\)-form from an \(n\)-form, notated as

\[ \delta_1, \delta_2, \delta_3, \ldots \delta_{n+1} : \Psi_n M \rightarrow \Psi_{n+1} M \]

which we refer to as the \textit{co-face} operations.
Codegeneracy operations

- For a sector $n$-form $\omega : T^n M \to E$, pre-composing with the lift $\ell$ gives an $(n - 1)$-form:

$$T^{n-1} M \xrightarrow{\ell_{T^{n-2} M}} T^n M \xrightarrow{\omega} E$$


- Similarly, for higher $n$, pre-composing with $T(\ell_{T^{n-3} M}), T^2(\ell_{T^{n-4} M})$, etc. gives $n - 1$ different codegeneracy operations

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1} : \psi_n M \to \psi_{n-1} M$$
Symmetric cosimplicial objects

Definition (Grandis/Barr)

An (augmented) **symmetric cosimplicial object** in a category \( \mathbf{X} \) consists of a sequence of objects

\[ C_0, C_1, C_2, \ldots, C_n, \ldots \]

with, for each \( n \), maps

\[ \delta^n_i : C_n \rightarrow C_{n+1} \text{ for each } i = 1 \ldots n + 1; \text{ (Cofaces)} \]

\[ \varepsilon^n_i : C_n \rightarrow C_{n-1} \text{ for each } i = 1 \ldots n - 1; \text{ (Codegeneracies)} \]

\[ \sigma^n_i : C_n \rightarrow C_n \text{ for each } i = 1 \ldots n - 1 \text{ (Symmetries)} \]

satisfying 15 equations relating these maps, for example, for \( i < j \),

\[ \varepsilon_j \delta_i = \delta_i \varepsilon_{j-1}. \]

Such an object is equivalent to giving a functor

\[ C : \text{finCard} \rightarrow \mathbf{X}. \]
Main result

**Theorem**

Let $\mathfrak{X}$ be a tangent category with a differential object $E$. 

- Each object $M$ has an associated symmetric cosimpicial monoid $\Psi(M)$, where $\Psi_n(M)$ is the set of of sector $n$-forms, and cofaces, codegeneracies, and symmetries are as described previously.
- This assignment is contravariantly functorial.
Main result

Theorem

Let $\mathbb{X}$ be a tangent category with a differential object $E$.

- Each object $M$ has an associated symmetric cosimplicial monoid $\Psi(M)$, where $\Psi_n(M)$ is the set of sector $n$-forms, and cofaces, codegeneracies, and symmetries are as described previously.
- This assignment is contravariantly functorial.

Corollary

For each function $f : n \to m$ between finite cardinals there is an associated map between sector forms

$$\Psi_f : \Psi_n(M) \to \Psi_m(M).$$

These appear to be new results in the category of smooth manifolds.
Now suppose that the differential object $E$ is “subtractive”; that is, it’s underlying monoid is in fact a group.

In this case, each $\Psi(M)$ is actually a symmetric cosimpicial group.

Any cosimpicial group $\Psi$ has an associated map $\delta^n : \Psi_n \to \Psi_{n+1}$ given by

$$\partial^n(\omega) := \sum_{i=1}^{n+1} (-1)^{i-1} \delta^n_i(\omega)$$

which has the property that $\delta^{n+1}(\delta^n(\omega)) = 0$.

**Corollary**

If $E$ is subtractive, each $\Psi(M; E)$ can be given the structure of a cochain complex.

This also appears to be a new result for smooth manifolds.
Recall that singular forms are alternating sector forms.

It is easy to show that the above operation $\partial$ restricts to singular forms.

**Corollary**

If $E$ is subtractive, the singular forms on $M$ with values in $E$ have the structure of a cochain complex.

In the category of smooth manifolds, this cochain complex is the de Rham complex.
Conclusions

- Sector forms in tangent categories have a very rich structure which has not previously been fully described, even in the canonical category of smooth manifolds.

- As a consequence, tangent categories support a notion of generalization of de Rham cohomology (and in fact possess a possibly distinct cohomology of sector forms).
Conclusions

- Sector forms in tangent categories have a very rich structure which has not previously been fully described, even in the canonical category of smooth manifolds.

- As a consequence, tangent categories support a notion of generalization of de Rham cohomology (and in fact possess a possibly distinct cohomology of sector forms).

- (J. E. White) If $g : T_2M \to \mathbb{R}$ is a Pseudo-Riemannian metric on $M$ (in particular, a covariant 2-tensor) quantities like the cycle

$$\delta_1 g + \delta_2 g - \delta_3 g,$$

and balance

$$\delta_1 g - \delta_2 g$$

of $g$ are sector forms which are not themselves tensors; thus general results about sector forms may further understanding of such invariants.
References