Instructions:

- No notes or books.
- Do all of the questions 1-4. From questions 5-8, choose 3. Indicate clearly which questions you have selected. The last question is for bonus marks.
- Unless specifically stated otherwise, you should give precise motivation for your answers.
- Time: 3 hours.
- Good luck!
Question 1. True or False?  Only write T/F; no motivation required.

1. There exist r.e. sets which are neither recursive nor creative.
2. The class of r.e. sets is closed under countable unions.
3. If two sets $A, B$ are both arithmetical then so is their union $A \cup B$.
4. There are simple sets whose complement is also simple.
5. If $A$ is creative, then its complement is infinite.
6. Any finite function is recursive (a function is called finite if its domain is a finite set).
7. If $A \leq_T B$ and $B$ is m-complete, then so is $A$.
8. For any set $A$, we have $A \leq_m A'$ (where $A'$ is the jump of $A$).
9. If $A$ is r.e. in $B$ then also $A \leq_T B$.
10. If $A \leq_T B$ then also $A \leq_m B$. 
Question 2.

1. Define the classes of sets of natural numbers $\Sigma_n, \Pi_n$ and $\Delta_n$.

2. Give an example of a set which is in $\Pi_n$ but not in $\Sigma_n$. You don’t have to motivate your answer.

3. Use the Tarski-Kuratowski algorithm to prove that the set

$$A = \{n | \phi_n \text{ has infinite range}\}$$

is in $\Pi_2$.

4. Show that $A$ is in fact $\Pi_2$-complete. Hint: consider the set

$$\text{Tot} = \{e | \phi_e \text{ is total}\},$$

which is known to be $\Pi_2$-complete, and show that $\text{Tot} \leq_m A$. Why is this sufficient?
Question 3.

1. State the definition of a creative set. Explain in a few sentences what the relevance of this concept is.

2. Show directly that $K$, the halting set, is creative.

3. Show that if $f$ is a total recursive injective function, then $f(K)$, the image of the halting set under $f$, is again creative.
**Question 4.** A partial function $f$ is called *extendible* if there exists a total recursive $g$ which extends $f$, i.e. whenever $x \in \text{dom}(f)$, then $f(x) = g(x)$.

1. Show that there exist partial recursive functions which are not extendible. Hint: consider $\phi_x(x) + 1$.

2. Let $\text{Ext} = \{e | \phi_e \text{ is extendible}\}$. Prove that this set is not recursive.
**Question 5.**

If $f$ is a total function, then we write $F_f = \{ e | \phi_e = \phi_{f(e)} \}$ for the set of fixed points of $f$.

1. Explain why the fixed point theorem implies that for any total recursive $f$, the set $F_f$ is non-empty.

2. Prove that for any total recursive $f$, the set $F_f$ is in fact infinite. Hint: suppose $F_f$ were finite. Then let $e$ be a code such that $\phi_e$ is different from $\phi_i$ for all $i \in F_f$. Then define $g$ by

$$g(x) \cdot y = \begin{cases} \phi_e(y) & \text{if } x \in F_f \\ f(x) \cdot y & \text{otherwise.} \end{cases}$$

Explain why $g$ is total recursive, and prove that $g$ does not have a fixed point, contradicting the fixed point theorem.
**Question 6.** Given a countable family of sets \( \{A_i | i \in \mathbb{N}\} \), we write \( \oplus_i A_i \) for the set \( \{[a, i] | a \in A_i\} \). (Here, \([-,-]\) is the recursive bijective pairing function.) Recall also that \( A^{(n)} \) stands for the n-fold jump of the set \( A \).

1. Show that for all \( i \) we have \( A_i \leq_T \oplus_i A_i \).

2. We define \( A^{(\omega)} = \oplus_i A^{(i)} \), and call it the infinitary jump of \( A \). Prove that for each \( n \), \( A^{(n)} \leq_T A^{(\omega)} \) and that for no \( n \) we have \( A^{(\omega)} \leq_T A^{(n)} \).

3. Using the above result, explain why this gives an explicit construction of a non-arithmetic set.
Question 7. When $T$ is a first-order theory which can represent primitive recursive predicates and functions, we write $\text{Con}_T$ for the sentence expressing the consistency of $T$.

1. Explain briefly how such a sentence is formulated, and why this requires the assumption on $T$.

2. If $\alpha : \mathbb{N} \rightarrow \{0, 1\}$, then we define a sequence of extensions of Peano Arithmetic by

$$ PA_0^\alpha = PA \quad PA_{n+1}^\alpha = \begin{cases} PA_n^\alpha + \text{Con}_{PA_n} & \text{if } \alpha(n) = 0 \\ PA_n^\alpha + \neg\text{Con}_{PA_n} & \text{if } \alpha(n) = 1. \end{cases} $$

Finally, let $PA^\alpha = \bigcup_n PA_n^\alpha$. Prove that $PA^\alpha$ is a consistent extension of $PA$.

3. Prove that the assignment $\alpha \mapsto PA^\alpha$ is injective, and therefore that $PA$ has uncountably many consistent extensions.
Question 8. In this question we write $T(e, x, y)$ for the T-predicate, which states that $y$ is the code of a computation of the function with code $e$ on input $x$; we also write $W_e$ for the domain of the function with code $e$. We define:

$$W_{e,s} = \{x|\exists y \leq s.T(e, x, y)\}.$$

1. Show that for each $e, s$ the set $W_{e,s}$ is finite.
2. Prove that $W_e = \bigcup_{s \in \mathbb{N}} W_{e,s}$.
3. Define, for two r.e. sets $W_e, W_f$:

$$W_e \searrow W_f = \{x|\exists s.(x \in W_{e,s} \land x \not\in W_{f,s})\}.$$

Prove that this set is r.e., and that $(W_e \searrow W_f) \cap (W_f \searrow W_e) = \emptyset$. 


**Bonus Question.** A man is sentenced to death by hanging. The judge tells him on Saturday: “Some day between now and next Saturday you will be executed at 10am, and you will not know in advance on which day.” The man reasons as follows: “The last possible day for my hanging is next Saturday. So if I’m still alive on Friday at 10.05am, then I know in advance when I will be executed. Therefore, the execution cannot be next Saturday. So Friday at the latest. But then I would already know it in advance on Thursday, 10.05am, so Friday is also not possible.” Continuing this line of reasoning, the man concludes that he cannot be executed, and falls asleep relieved in his cell. However, Thursday morning 10am the man was executed and he didn’t know it in advance.

Briefly explain the paradox and explain how it relates to the course.