Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2322A Calculus III for Engineers
Final exam
December 9, 2003, 9:30-12:30

Name: Solution

Student number: ____________________

No texbooks, notes, graphing calculators allowed
Show all your work!

Marking table:

1	2	3	4	5	6	7	8	9	Total

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1. (10 points) Find the critical points of the function $f(x, y) = x^4 + y^4 - 4xy$ and use the second derivative test to classify them.

$\nabla f = 0$

$f_x \quad 4x^3 - 4y = 0$

$f_y \quad 4y^3 - 4x = 0$

$\Rightarrow x = y^3, \ (y^3)^3 = y$

$y^9 = y$

$y = 0 \quad y = 1 \quad y = -1$

$x = 0 \quad x = 1 \quad x = -1$

3 crit. pts: $(0,0), \ (1,1), \ (-1,-1)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16$$

	D	f_{xx}	
$(0,0)$	$-16 < 0$		saddle
$(1,1)$	$128 > 0$	$12 > 0$	local min
$(-1,-1)$	$128 > 0$	$12 > 0$	local min

2. (10 points) Find the points on the cone $z^2 = x^2 + y^2$ which are closest to the point $(1, 1, 0)$.

$$\text{dist}^2 = \cancel{f(x,y}\ (x-1)^2 + (y-1)^2 + z^2 \to \min$$

$$\text{subj. to} \quad x^2+y^2 = z^2$$

replace:

$$(x-1)^2 + (y-1)^2 + x^2 + y^2 \to \min$$

$$f(x,y) = 2x^2 - 2x + 2y^2 - 2y + 2 \to \min$$

Crit. pts:

$$4x - 2 = 0$$

$$4y - 2 = 0$$

$$x = \tfrac{1}{2} \quad y = \tfrac{1}{2}$$

$$z = \pm\sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \pm \frac{1}{\sqrt{2}}$$

2pts: $\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{\sqrt{2}}\right)$

3. (10 points) Find the mass of the solid bounded by the xz-plane, the yz-plane, the plane $z = 1$, and the plane $x + y + z = 2$, if the density of the solid is given by $\delta(x, y, z) = 12(z-1)^2$.

$$m = \int_0^1 \int_0^{1-x} \int_1^{2-x-y} 12(z-1)^2 \, dz \, dy \, dx =$$

$$= \int_0^1 \int_0^{1-x} 4(z-1)^3 \Big|_{z=1}^{z=2-x-y} dy \, dx =$$

$$= \int_0^1 \int_0^{1-x} 4(1-x-y)^3 \, dy \, dx =$$

$$= \int_0^1 -(1-x-y)^4 \Big|_{y=0}^{y=1-x} dx = \int_0^1 0 + (1-x)^4 \, dx =$$

$$= -\frac{1}{5}(1-x)^5 \Big|_0^1 = 0 - \left(-\frac{1}{5}\right) = \boxed{\frac{1}{5}}$$

$P \qquad Q$

4. (10 points) Consider the vector field $\vec{F} = (2x + aye^{-x})\vec{i} + e^{-x}\vec{y}$,

(a) For which value of a does $\vec{F}$ have a potential function?

(b) Find the potential function for the vector field in part (a).

(c) Calculate $\int_C \vec{F} \cdot d\vec{r}$, where C is given by:

$$x = \cos t, \; y = \sin t, \qquad \frac{\pi}{4} \le t \le \pi.$$

(a) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

$-e^{-x} = ae^{-x}, \quad a = -1$

(b) $f_x = 2x \, \cancel{} - ye^{-x} \Rightarrow$

$\Rightarrow f = x^2 + ye^{-x} + g(y)$

$\boxed{x^2 + ye^{-x}} \leftarrow f(x,y)$

(c) $\int_C \vec{F} \cdot dr \overset{(b)}{=} f(-1,0) - f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) =$

$= 1 - \frac{1}{2} - \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}} = \frac{1}{2} - \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}$

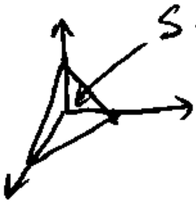
5. (10 points) Let C be the contour of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,2)$, oriented counter-clockwise if viewed from above. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = (e^{x^2} + z)\vec{i} + (\sin y + x)\vec{j} + (\ln z + y)\vec{k}.$$

Use Stokes' Theorem to evaluate the integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \langle 1, 1, 1 \rangle$$



$S: \quad z = 2 - 2x - 2y$

Use f-la $\iint_S \vec{F} \cdot d\vec{A} = \iint_R -P_1 f_x - P_2 f_y + P_3 d$

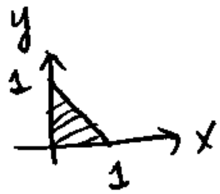
$f_x = -2$

$f_y = -2$

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_R -1(-2) - 1(-2) + 1 \, dx\,dy = \iint_R 5 \, dx\,dy$$

$$= 5 \cdot \text{area}(R) = \frac{5}{2}$$

R:



6. (10 points) Consider a particle whose motion is given by the vector function

$$\vec{r}(t) = (4t^2 - 1)\vec{i} + 2\vec{j} + (3t^2 + 2)\vec{k}.$$

(a) Find the velocity, speed, and acceleration of the particle at the moment of time $t = 1$.

(b) This particle moved from the point $(3, 2, 5)$ to the point $(35, 2, 29)$ along its trajectory. Find the distance traveled by the particle.

(a) $\vec{r}\,'(t) = \langle 8t, 0, 6t \rangle$

$\vec{r}\,'(1) = \langle 8, 0, 6 \rangle \longleftarrow$ vel.

$\|\vec{r}\,'(1)\| = \sqrt{8^2 + 6^2} = 10 \longleftarrow$ speed

$\vec{a}(t) = \vec{r}\,''(t) = \langle 8, 0, 6 \rangle \longleftarrow$ acc.

(b) distance $= \displaystyle\int_{t_1}^{t_2} \|r'(t)\|\,dt$

$(3,2,5) \to t_1 = 1$

$(35,2,29) \to t_2 = 3$

$\displaystyle\int_1^3 10t\,dt = 40$

7. (10 points) Let $\vec{F} = (x + \cos y)\vec{i} + (y + \sin z)\vec{j} + (z + e^x)\vec{k}$, and let W be the solid bounded by the planes $z = 0$, $y = 0$, $y = 2$ and the parabolic cylinder $z = 1 - x^2$. If the surface S, which is the boundary of W, is oriented towards the exterior of the solid, compute

$$\int_S \vec{F} \cdot d\vec{A}.$$

Use the Div. Theorem

$$\operatorname{div} \vec{F} = 1+1+1 = 3$$

$$\iint_S \vec{F} \cdot d\vec{A} = \iiint_W 3\, dV = \int_{y=0}^{2} \int_{x=-1}^{1} \int_0^{1-x^2} 3 \, dz\, dx\, dy =$$

$$= 3\int_0^2 \int_{-1}^{1} (1-x^2)\, dx\, dy = 3\int_0^2 x - \frac{x^3}{3}\Big|_{-1}^{1} dy = 3\int_0^2 \left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)$$

$$= 3\int_0^2 \tfrac{4}{3}\, dy = 4 \cdot \int_0^2 dy = 8$$

8. (10 points) Find the flux of the vector field $\vec{F} = x\vec{i} + y\vec{j} + 3\vec{k}$ through the surface S consisting of the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$. The surface S has the upward orientation.

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_R -P_1 f_x - P_2 f_y + P_3 \, dx\,dy$$

$$f(x,y) = x^2+y^2$$

$$= \iint_R -x(2x) - y(2y) + 3 \, dx\,dy =$$

$$= \iint_R 3 - 2(x^2+y^2) \, dx\,dy =$$

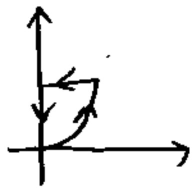
$$= \int_0^{2\pi}\int_0^2 (3-2r^2)r \, dr\,d\theta = \int_0^{2\pi} 3\frac{r^2}{2} - \frac{r^4}{2}\Big|_0^2 \, d\theta =$$

$$= \int_0^{2\pi} 6 - 8 \, d\theta = (-2)\, 2\pi = -4\pi$$

9. (10 points) Use Green's Theorem to evaluate $\oint_C \vec F \cdot d\vec r$ where

$$\vec F = \overset{P}{(x + xy^2)}\vec i + \overset{Q}{2(x^2 y - y^2 \sin y)}\vec j,$$

and C is the curve from the origin to $(1, 1)$ along $y = x^2$, from $(1, 1)$ to $(0, 1)$ along $y = 1$ and from $(0, 1)$ to the origin along $x = 0$ in that order.



$$\oint F \cdot d\vec r = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$$= \iint_D 4xy - 2xy \; dx\,dy =$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt y} 2xy \; dx\,dy = \int_0^1 x^2 y \Big|_{x=0}^{x=\sqrt y} dy = \int_0^1 y^2\,dy =$$

$$= \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$