Solution to Sample Final #3

Question 1.
a)

$$\operatorname{div} \vec{F} = yz + 2\,x^2yz^2 + 3\,y^2z^2.$$

b) Since this is not zero, Divergence Test tells us that $\vec{F}$ cannot be the curl of any vector field.

Question 2.

$$\operatorname{grad} f = (2xy - 2x)\vec{\imath} + (x^2 - 2y)\vec{\jmath}.$$

Solve

$$2xy - 2x = 0, \quad x^2 - 2y = 0,$$

to get the critical points:

$$(x, y) = (0, 0) \text{ and } (\pm\sqrt{2}, 1).$$

Now,

$$f_{xx} = 2y - 2, \quad f_{xy} = 2x, \quad f_{yy} = -2.$$

$\boxed{\text{At (0,0):}}$

$$D = (-2)(-2) - 0^2 = 4.$$

Since $f_{xx}(0, 0) = -2 < 0$, (0,0) is a relative maximum.

$\boxed{\text{At } (\sqrt{2}, 1)\text{:}}$

$$D = 0(-2) - (2\sqrt{2})^2 = -8 < 0.$$

Thus, $(\sqrt{2}, 1)$ is a saddle point.

$\boxed{\text{At } (-\sqrt{2}, 1)\text{:}}$

$$D = 0(-2) - (-2\sqrt{2})^2 = -8 < 0.$$

Thus, $(-\sqrt{2}, 1)$ is also a saddle point.

Question 3.

From the limits of integration, we see that the region where the integration takes place is the portion of the intersection between the upper half of the sphere $x^2 + y^2 + z^2 = 18$ and the cone $z = \sqrt{x^2 + y^2}$ where $x > 0$ and $y > 0$.

In spherical coordinates, the same region can be described as:

$$0 \le \theta \le \pi/2, \quad 0 \le \phi \le \pi/4, \quad 0 \le \rho \le \sqrt{18}.$$

The integral is, then

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta = -\frac{486}{5}\,\pi + \frac{486}{5}\,\sqrt{2}\pi$$

Question 4.

This problem is very easy to solve if you notice that $\vec{F}$ is the gradient of $f = 2x^2 + 3y^2$.

Then, by the Fundamental Theorem of Calculus for Line Integrals, we get:

$$\int_C \vec{F} \cdot d\vec{r} = f(0,4) - f(0,0) = 48.$$

For not-so-easy solution, use

$$\vec{r} = 2\sqrt{2}t(\vec{i} + \vec{j}), \quad 0 \le t \le 1,$$

and

$$\vec{r} = 4(\cos t\vec{i} + \sin t\vec{j}), \quad \pi/4 \le t \le \pi/2,$$

and compute the line integrals. See the solution to Question 8 for the sample final exam #1.

Question 5.

Solve

$$2x = 4\lambda x^3, \quad 2y = 4\lambda y^3, \quad 2z = 4\lambda z^3,$$

$$x^4 + y^4 + z^4 = 1.$$

From the first equation, we get $x(1 - 2\lambda x^2) = 0$.

Thus, either $x = 0$ or $x^2 = \frac{1}{2\lambda}$. Likewise, either $y = 0$ or $y^2 = \frac{1}{2\lambda}$, and either $z = 0$ or $z^2 = \frac{1}{2\lambda}$.

The possibilities are:

1. None of x, y, or z is zero.
2. Only one of x, y, or z is zero.
3. Exactly two of x, y, or z are zero.

Case 1. In this case, $x^2 = y^2 = z^2 = \frac{1}{2\lambda} > 0$, so the 4th equation becomes:

$$3 \cdot \frac{1}{(2\lambda)^2} = 1 \quad \Rightarrow \quad \frac{1}{2\lambda} = \frac{1}{\sqrt{3}}.$$

Thus, $x = \pm\frac{1}{\sqrt[4]{3}}$, $y = \pm\frac{1}{\sqrt[4]{3}}$ and $z = \pm\frac{1}{\sqrt[4]{3}}$, with all 8 possible combinations of pluses and minuses.

In any of these cases, $f = 3 \cdot \frac{1}{2\lambda} = \frac{3}{\sqrt{3}} = \sqrt{3}$.

Case 2. In this case, the 4th equation is

$$2 \cdot \frac{1}{(2\lambda)^2} = 1 \quad \Rightarrow \quad \frac{1}{2\lambda} = \frac{1}{\sqrt{2}}.$$

Thus, the possible solutions are:

$$\begin{aligned}(x,y,z) &= (0, \pm\tfrac{1}{\sqrt[4]{2}}, \pm\tfrac{1}{\sqrt[4]{2}}) \quad \text{(with all 4 possible combinations of pluses and minuses)},\\ &= (\pm\tfrac{1}{\sqrt[4]{2}}, 0, \pm\tfrac{1}{\sqrt[4]{2}}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{''}\\ &= (\pm\tfrac{1}{\sqrt[4]{2}}, \pm\tfrac{1}{\sqrt[4]{2}}, 0) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{''}\end{aligned}$$

In any of these cases, $f = 2 \cdot \frac{1}{2\lambda} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Case 3. In this case, either one of x^2, y^2, or z^2 equals 1 while the other two are 0.

So the solutions are: $(x, y, z) = (\pm 1, 0, 0) = (0, \pm 1, 0) = (0, 0, \pm 1)$.

In any of these cases, $f = 1$.

Since the constraint is closed and bounded, we conclude that global maximum $f = \sqrt{3}$ occurs at any of the solutions in Case 1 while global minimum $f = 1$ occurs at any of the solutions in Case 3.

Question 6.

c) Use the Divergence Theorem.

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2.$$

Let W be the cylindrical bar enclosed in the closed cylinder. This can be described in cylindrical coordinates as follows:

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 1.$$

Thus, by the Divergence Theorem:

$$\begin{aligned}\text{flux} &= \int_W \operatorname{div} \vec{F}\, dV = \int_0^1 \int_0^{2\pi} \int_0^1 3(x^2 + y^2)\, dV \\ &= \int_W \operatorname{div} \vec{F}\, dV = \int_0^1 \int_0^{2\pi} \int_0^1 3r^3\, dz\, d\theta\, dr = \frac{3}{2}\pi.\end{aligned}$$

Question 7.

Note that we cannot user Stokes' Theorem here.

We will use the formula for flux through spherical surfaces.

The surface S can be described in spherical coordinates as follows:

$$\rho = 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le 3\pi/4.$$

On S, in spherical coordinates,

$$\vec{F} = \sin\phi \sin\theta \vec{i} - \sin\phi \cos\theta \vec{j} + \cos\phi \vec{k}.$$

Then

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S \vec{F} \cdot \left(\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k} \right) \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{3\pi/4} \cos^2\phi \sin\phi\, d\phi\, d\theta \\ &= 2\pi \left(\frac{\sqrt{2}}{12} + \frac{1}{3} \right)\end{aligned}$$

Question 8.

$\vec{H}$ is the only one whose scalar curl is zero. Thus, by the Curl Test, there must be a potential function f for $\vec{H}$.

$$f = \int (1 + 2xy)dx = x + x^2y + C(y, z),$$
$$f_y = x^2 + C_y(y, z) = x^2 + 2y.$$

Thus, $C_y(y, z) = 2y$, so

$$C(y, z) = \int 2y\,dy = y^2 + C(z).$$

.

Finally, since $f_z = C'(z) = 0$, we have $C(z) = C$.

Thus,

$$f = x + x^2y + y^2 + C$$

is a potential function for any constant C.

Question 9.

The plane in question is a graph of $f = 1 - x - y$.

To determine the domain of f, we compute the intersection with the xy-plane, *i.e.*, when $z = 0$. This gives us

$$0 = 1 - x - y.$$

The domain of f is the triangular region (we'll call it T) bounded by $0 = 1 - x - y$, the x-axis, and the y-axis.

We will use this and Stokes' Theorem to determine the line integral.

Now,

$$\operatorname{curl} \vec{F} = -2(z\vec{i} + x\vec{j} + y\vec{k}),$$

so

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl} \vec{F} \cdot d\vec{A}.$$

Now use the formula for flux through a graph.

$$\begin{aligned}
\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} &= \int_T \operatorname{curl} \vec{F} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dx\,dy \\
&= \int_0^1 \int_0^{1-x} \operatorname{curl} \vec{F} \cdot (\vec{i} + \vec{j} + \vec{k})\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} -2(x + y + (1 - x - y))\,dy\,dx \\
&= -2 \int_0^1 \int_0^{1-x} dy\,dx \\
&= -2 \cdot \frac{1}{2} = -1.
\end{aligned}$$