Solution to Sample Final #2

Question 1.

Change the order of integration and obtain:

$$\int_1^{e^2}\int_0^{\ln x}\frac{x}{\ln x}\,dy\,dx=\int_1^{e^2}x\,dx=\frac{1}{2}(e^4-1).$$

Question 2.

Intersection between the sphere and the plane is given by

$$x^2+y^2+3^2=25\quad\Rightarrow\quad x^2+y^2=25-9=16.$$

So the region can be described in cylindrical coordinates:

$$0\le r\le 4,\quad 0\le\theta\le 2\pi,\quad 3\le z\le\sqrt{25-r^2}.$$

Thus,

$$\begin{aligned}\text{volume}&=\int_0^4\int_0^{2\pi}\int_3^{\sqrt{25-r^2}}dV\\&=\int_0^4\int_0^{2\pi}\int_3^{\sqrt{25-r^2}}r\,dz\,d\theta\,dr\\&=\\&=2\pi\left(\int_0^4 r\sqrt{25-r^2}\,dr-\int_0^4 3r\,dr\right)\\&=2\pi\left(\frac{98}{3}-24\right)=\frac{52}{3}\pi.\end{aligned}$$

Question 3.

$$\begin{aligned}\vec{F}(\vec{r}(t))&=t^2\,\vec{i}+4t^3\,\vec{j}+6t^2\,\vec{k},\\\vec{r}\,'(t)&=\vec{i}+\vec{j}+4t\vec{k},\\\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)&=t^2+28t^3,\\\int_C\vec{F}\cdot d\vec{r}&=\int_0^1(t^2+28t^3)dt=\frac{22}{3}.\end{aligned}$$

Question 4.

a)

(0,0) to (2,0)

Call this line segment C_1.

$$\begin{aligned}\vec{r}(t) &= 2t\vec{i}, \quad 0 \le t \le 1,\\ \vec{F}(\vec{r}(t)) &= 14t\vec{j},\\ \vec{r}\,'(t) &= 2\vec{i},\\ \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= 0,\\ \int_{C_1} \vec{F} \cdot d\vec{r} &= 0.\end{aligned}$$

(2,0) to (0,1)

Call this line segment C_2.

$$\begin{aligned}\vec{r}(t) &= 2\vec{i} + t(\vec{j} - 2\vec{i}), \quad 0 \le t \le 1,\\ \vec{F}(\vec{r}(t)) &= 2t\vec{i} + 7(2-2t)\vec{j},\\ \vec{r}\,'(t) &= \vec{j} - 2\vec{i},\\ \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= -4t + 7(2-2t) = 14 - 18t,\\ \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 (14-18t)dt = 14 - 9 = 5.\end{aligned}$$

(0,1) to (0,0)

Call this line segment C_3.

$$\begin{aligned}\vec{r}(t) &= (1-t)\vec{i}, \quad 0 \le t \le 1,\\ \vec{F}(\vec{r}(t)) &= 2(1-t)\vec{j},\\ \vec{r}\,'(t) &= -\vec{j},\\ \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= 0,\\ \int_{C_3} \vec{F} \cdot d\vec{r} &= 0.\end{aligned}$$

Thus, $\int_C \vec{F} \cdot d\vec{r} = 5$.

b) Scalar curl of $\vec{F}$ is $\frac{\partial}{\partial x}(7x) - \frac{\partial}{\partial y}(2y) = 5$.

If S is the region bounded by C, then, by Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S 5\, dA = 5 \cdot (\text{area of } S) = 5.$$

Question 5.

$$\begin{aligned}\operatorname{curl}\vec{F} &= (3x^2z^2 - 3x^2z^2)\vec{i} + (6xyz^2 - 6xyz^2)\vec{j} + [(2xz^3 + 2y) - (2xz^3 + 2y)]\vec{k} \\ &= \vec{0}.\end{aligned}$$

Thus, by the Curl Test, there must be a potential function $f(x, y, z)$.

$$\begin{aligned} f(x, y, z) &= \int (2xyz^3 + y^2)dx = x^2yz^3 + xy^2 + C(y, z) \\ f_y(x, y, z) &= x^2z^3 + 2xy + C_y(y, z) = x^2z^3 + 2xy. \end{aligned}$$

Thus, $C_y(y, z) = 0$. That is, C does not depend on y. So we can write:

$$f(x, y, z) = x^2yz^3 + xy^2 + C(z).$$

Now,

$$f_z(x, y, z) = 3x^2yz^2 + C'(z) = 3x^2yz^2.$$

Thus, $C'(z) = 0$. Hence, $C(z)$ is a constant.

So $f(x, y, z) = x^2yz^3 + xy^2 + C$ is a potential function for any constant C.

Question 6.

Use the formula for flux through the graph of a function.

Along S,

$$\vec{F} = x\vec{i} + y\vec{j} + (x^2 - y^2)\vec{k}.$$

Now, let $f(x, y, z) = x^2 - y^2$. Then $f_x = 2x$ and $f_y = 2y$.

Thus,

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \int_{-1}^{1}\int_{-1}^{1} \vec{F} \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\, dx\, dy \\ &= \int_{-1}^{1}\int_{-1}^{1} (-x^2 - 3y^2)\, dx\, dy \\ &= \int_{-1}^{1} (-\frac{2}{3} - 6y^2)dy \\ &= -\frac{4}{3} - 4 = -\frac{16}{3} \end{aligned}$$

Question 7.

Use the formula for flux through cylindrical surfaces.

Let S be the cylinder. It can be parametrized in cylindrical coordinates as follows.

$$r = 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 5.$$

With this parametrization,

$$\vec{F} = 2z\cos\theta\vec{i} + 2z\sin\theta\vec{j} + 4z^2\cos\theta\sin\theta\vec{k},$$

on S.

Then

$$\begin{aligned}\int_S \vec{F}\cdot d\vec{A} &= \int_0^{2\pi}\int_0^5 \vec{F}\cdot(\cos\theta\vec{i}+\sin\theta\vec{j})2\,dz\,d\theta\\ &= \int_0^{2\pi}\int_0^5 4z(\cos^2\theta+\sin^2\theta)\,dz\,\theta\\ &= \int_0^{2\pi}\int_0^5 4z\,dz\,d\theta = 100\pi.\end{aligned}$$

Question 8.

a)

$$\operatorname{div}\vec{F} = 4xy^2 + 2y^2z$$

b)

$$\operatorname{curl}\vec{F} = (2yz^2 - 3x^2)\vec{i} + (6xz - 4x^2y)\vec{k}$$

c) Divergence of a curl of any vector field is zero.

Question 9.

See answer to Question 4 in sample final exam #1.