Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Arian Novruzi
Department of Mathematics and Statistics
University of Ottawa
email:novruzi@uottawa.ca

Calculus III for Engineers
MAT2322B
Fall 2017
Duration: 80 mins
Closed book

+ Sol.

Midterm exam #2

NAME:_______________________________ Student number:______________

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and electronic devices must be turned off and put away, for example in your bag. You should not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediateley the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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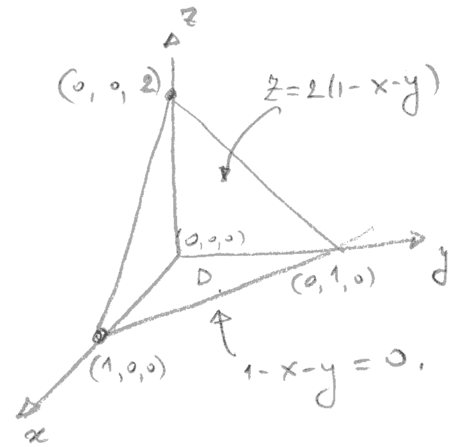
Instructions:

- The exam has 5 problems and 8 pages. The last two pages are for your preliminary calculations.
- Write the solution clearly in the space following it. If necessary, you can continue the solution in the back of any page - in this case, you must clearly indicate that the solution continues in the back of the page "n".
- **Use of manuals, courses notes or any other document is not allowed.**
- **The only calculators which are allowed are TI-30, TI-34, Casio FX-260, FX-300, and any other scientific and any non prgrammable calculator**

Problem	1	2	3	4	5	Total
Your result						

Question 1 *(10 points)*
Use the triple integral to find the volume of the solid E, which is the tetrahedron with nodes $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,2)$.



Sol

→ Let D the triangle with nodes $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$; then.

$$E = \{(x,y,z),\ (x,y)\in D,\quad 0 \le z \le ax+by+c\},$$

where $z = ax+by+c$ is the equation of the plane passing through the nodes $(1,0,0)$, $(0,1,0)$, $(0,0,2)$.

→ $z = ax+by+c$:

$(1,0,0)$: $\quad 0 = a\cdot 1 + b\cdot 0 + c, \qquad 0 = a + c$

$(0,1,0)$: $\quad 0 = a\cdot 0 + b\cdot 1 + c, \qquad 0 = b + c$

$(0,0,2)$: $\quad 2 = a\cdot 0 + b\cdot 0 + c, \qquad 2 = c$

So, $a = b = -2$, $c = 2$ and $z = 2(1-x-y)$.

→ $$V(E) = \iiint_E 1\cdot dV = \iint_{D} \int_0^{2(1-x-y)} 1\cdot dz\cdot dA = \iint_D 2(1-x-y)\, dA.$$

← type I domain

→ But $D = \{(x,y),\ 0 \le x \le 1,\ 0 \le y \le 1-x\}$; therefore

$$V(E) = \int_0^1 \int_0^{1-x} 2(1-x-y)\, dy\, dx$$

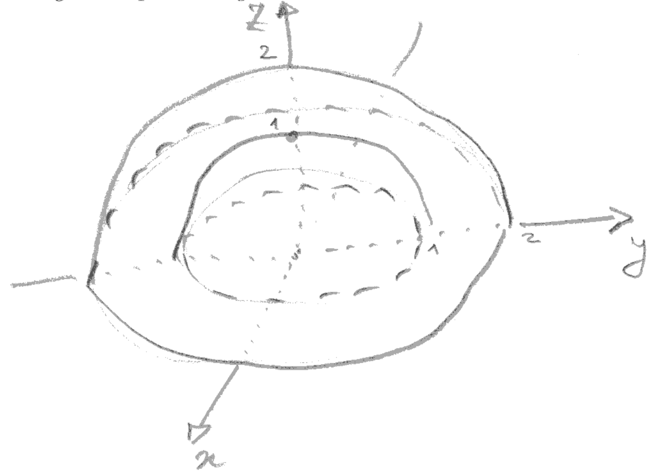
$$= \int_0^1 2\cdot\left[y - xy - \tfrac{1}{2}y^2\right]_0^{1-x} dx = \int_0^1 \left(2(1-x) - 2x(1-x) - (1-x)^2\right) dx$$

$$\begin{cases} t = 1-x, & x = 1-t \\ dt = -dx \end{cases}$$

$$= \int_0^1 \left(2t - 2(1-t)t - t^2\right) dt$$

$$= \left[\cancel{t^2} - \cancel{t^2} + \tfrac{2}{3}t^3 - \tfrac{1}{3}t^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Question 2 *(10 points)*
The solid E occuppies the space between the spheres $x^2+y^2+z^2=1^2$, $x^2+y^2+z^2=2^2$ for $z \geq 0$. Assuming E has density $2z$, find its mass by using a triple integral.



Sol

→ Here, we use spherical coordinates.

(ρ, θ, ϕ),

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

→ Note that

$$E = \{(\rho, \theta, \phi), \quad 1 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \tfrac{\pi}{2}\};$$ then.

→ $$m(E) = \iiint_E 2z \cdot dV$$

$$= \int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^2 \sin\phi \cdot 2\rho \cos\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_1^2 \int_0^{2\pi} \rho^3 2 \cdot \left[\tfrac{1}{2} \sin^2\phi\right]_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta \, d\rho$$

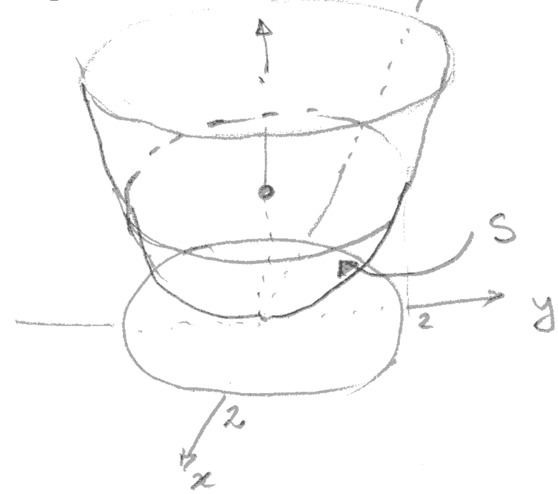
$$= \int_1^2 \int_0^{2\pi} \rho^3 2 \cdot \tfrac{1}{2} \, d\theta \, d\rho$$

$$= \tfrac{1}{2} \int_1^2 \rho^3 2 \cdot 2\pi \, d\rho$$

$$= \tfrac{1}{2} \tfrac{1}{4} \left[\rho^4\right]_1^2 \cdot 2\pi \cdot 2$$

$$= \frac{\pi}{4}\left(2^4 - 1^4\right) = \frac{15}{4} \pi \cdot 2 = \frac{15}{2} \pi$$

Question 3 *(10 points)*
The surface S is the part of the graph of the function $f(x,y) = \frac{1}{2}(x^2+y^2)$ above the domain $D = \{(x,y),\ x^2+y^2 \le 2^2\}$. Find the area of S.



Sol

$\to$ $A(S) = \iint_D \sqrt{1 + |f_x(x,y)|^2 + |f_y(x,y)|^2}\, dA.$

$\to$ $f_x(x,y) = x, \quad f_y(x,y) = y$; so

$A(S) = \iint_D \sqrt{1 + x^2 + y^2}\, dA$

$\to$ $D = \{(x,y),\ x^2+y^2 \le 2^2\} = \{(r,\theta),\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$; then

$$A(S) = \int_0^2 \int_0^{2\pi} r \cdot \sqrt{1+r^2}\, d\theta . dr$$

$$= \int_0^2 r\sqrt{1+r^2} \cdot 2\pi\, dr$$

$$= \left[\cancel{2}\pi \cdot \frac{(1+r^2)^{3/2}}{\frac{3}{2}} \frac{1}{\cancel{2}} \right]_0^2$$

$$= \frac{2}{3} \left[(1+r^2)^{3/2} \right]_0^2$$

$$= \frac{2}{3} \left(5^{3/2} - 1 \right) \pi.$$

Question 4 *(5 points)*
Let C be the straight segment from $(2,0)$ to $(5,4)$. Compute the line integral $I = \int_C xe^y ds$.

Sol

→ $C = \{ \vec{r}(t),\ t \in [0,1] \}$, with

$$\vec{r}(t) = (2,0) + t\left((5,4)-(2,0)\right) = (2,0) + t(3,4)$$
$$= (2+3t,\ 4t) = (x(t), y(t))$$

→ Then.
$$I = \int_C xe^y ds = \int_0^1 x(t).\ e^{y(t)} \cdot |\vec{r}\,'(t)|dt.$$

→ But: $\vec{r}\,'(t) = (3,4)$, so
$$|\vec{r}\,'(t)| = (3^2+4^2)^{1/2} = 5;$$ therefore

→ $$I = \int_0^1 (2+3t)\, e^{4t}.\ 5\, dt$$
$$= \int_0^1 10\, e^{4t} dt + 15 \int_0^1 t e^{4t} dt.$$
$$= \frac{10}{4}\left[e^{4t}\right]_0^1 + 15.\left[\frac{1}{4} te^{4t} - \frac{1}{4^2} e^{4t}\right]_0^1$$
$$= \frac{10}{4.}(e^4 - 1) + 15.\frac{1}{4.}(e^4 - 0) - \frac{15}{4^2.}(e^4 - 1)$$
$$= \frac{25}{16}(e^4 - 1) + \frac{15}{4.} e^4.$$
$$= \frac{85}{16} e^4 - \frac{25}{16.}$$

Question 4 *(5 points)*
Let C be the straight segment from $(1,0)$ to $(4,4)$. Compute[2] *the line integral $I = \int_C xe^y ds$.*

Sol

→ $C = \{ \vec{r}(t),\ t \in [0,1] \}$ with

$$\vec{r}(t) = (1,0) + t\big((4,4) - (1,0)\big) = (1,0) + t\cdot(3,4)$$
$$= (1+3t, 4t);$$

→ Then

$$I = \int_C x e^y ds = \int_0^1 x(t)\, e^{y(t)} |\vec{r}\,'(t)| dt.$$

→ But $\vec{r}\,'(t) = (3,4)$, so

$$|\vec{r}\,'(t)| = (3^2+4^2)^{1/2} = 5;$$ therefore

→ $$I = \int_0^1 (1+3t)\, e^{4t} \cdot 5 \cdot dt$$

$$= \int_0^1 5e^{4t} dt + 15 \int_0^1 te^{4t} dt$$

$$= \frac{5}{4}\left[e^{4t}\right]_0^1 + 15 \cdot \left[\frac{1}{4} te^{4t} - \frac{1}{16} e^{4t}\right]_0^1$$

$$= \frac{5}{4}(e^4 - 1) + \frac{15}{4}(e^4 - 0) - \frac{15}{16}(e^4 - 1)$$

$$= \frac{5}{16}(e^4-1) + \frac{15}{4} e^4$$

$$= \frac{65}{16} e^4 - \frac{5}{16}$$

[2] note that $\int te^{4t} dt = \frac{1}{4} te^{4t} - \frac{1}{16} e^{4t}$

Question 5 *(5 points)*
Let $\vec{F} = (xy^2, -x^2)$ *and* $C = \{\vec{r}(t),\ t \in [0,1]\}$, *where* $\vec{r}(t) = (t^3, t^2)$. *Compute the work* $W = \int_C \vec{F} d\vec{r}$.

Sol

$\rightarrow$ $W = \int_C \vec{F} \cdot d\vec{r}$

$= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \cdot dt.$

$\rightarrow$ But $\vec{F}(\vec{r}(t)) = \vec{F}(t^3, t^2) = (t^3 \cdot (t^2)^2, -(t^3)^2)$

$= (t^7, -t^6),$

$\vec{r}\,'(t) = (3t^2, 2t)$; hence

$\rightarrow$ $W = \int_0^1 (t^7, -t^6) \cdot (3t^2, 2t)\, dt$

$= \int_0^1 (3t^9 - 2t^7)\, dt.$

$= \left[3 \cdot \frac{t^{10}}{10} - 2 \frac{t^8}{8} \right]_0^1$

$= \frac{3}{10} - \frac{2}{8} = \frac{3.4 - 2.5}{40} = \frac{2}{40}$

$= \frac{1}{20}.$