Université d'Ottawa · University of Ottawa

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Mathematics and Statistics

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MAT2322, Calcul III
Midterm exam #1
Fall, 2017

Write clearly your
LASTNAME, firstname
and
Student number

+ sol

Instructions:

- The length of exam is 80 minutes
- The exam contains 7 pages; the last one is white and you can use it for different notes
- The exam has 5 problems, each worth 4 points
- Write the solution clearly, with detailed explanations, in the space following the problem; if more space is needed, continue the solution on the back of any other page, and indicate clearly when doing so
- Use of any textbook, course notes or any other document is not allowed
- You can use a calculator not featuring formal calculus and graphing capabilities

Results:

Problem	1	2	3	4	5	Total
Your result						(over 20)

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Problem 1 *Find and classify the critical point(s) of the function* $f(x,y) = x^2+y^3+x^2y-3y$.

Sol

→ C.P. of f:

$$\begin{cases} f_x(x,y) = 2x + 2xy = 0 & (1) \\ f_y(x,y) = 3y^2 + x^2 - 3 = 0 & (2) \end{cases}$$

(1) $\Rightarrow$ $2x(1+y)=0$; so; we get two cases:

→ $x=0$; then (2) $\Rightarrow$ $3y^2-3=0$, $y^2=1$, $y=\pm 1$;

so $(0,\pm 1)$ are C.P.

→ $1+y=0$, $y=-1$; then (2) $\Rightarrow$ $3\cdot(-1)^2+x^2-3=0$, $x^2=0$, $x=0$,

so, $(0,-1)$ is. C.P, which we fund also in previous case.

→ classification:

$f_{xx}(x,y) = 2+2y = 2(1+y)$,

$f_{yy}(x,y) = 6y$,

$f_{xy}(x,y) = 2x$; so $D(x,y) = 12y(1+y) - 4x$.

→ $D(0,-1) = 0$; the test is inconclusive

→ $D(0,1) = 24 > 0$

$f_{xx}(0,1) = 4 > 0$; so $(0,1)$ is C.P. of local. min.

Problem 2 *Use the method of Lagrange multipliers to find the global extrema of $f(x,y) = xy$ under the constraint $4x^2 + y^2 = 8$.*

Solution

→ $f(x,y) = xy, \quad g(x,y) = 4x^2 + y^2 - 8$

→ $\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \iff \begin{cases} y = \lambda \cdot 8x & (1) \\ x = \lambda \cdot 2y & (2) \\ 4x^2 + y^2 = 8 & (3). \end{cases}$

→ $\lambda \neq 0$; if not, (1) and (2) imply $x = y = 0$, so (3) cannot be solved

→ $x \neq 0, y \neq 0$; if not, say $x = 0$, from (1) we get $y = 0$, so (3) cannot be solved

→ then, it is legitimate to take $\frac{(1)}{(2)}$:

$$\frac{y}{x} = \frac{\lambda \cdot 8x}{\lambda \cdot 2y}, \quad \frac{y}{x} = 4\frac{x}{y}, \quad y^2 = 4x^2;$$

Sub in (3) gives:

$$8x^2 = 8, \quad x^2 = 1, \quad x = \pm 1;$$

From $y^2 = 4x^2$, for $x = \pm 1$ we get $y^2 = 4$ so $y = \pm 2$

Therefore:

$(-1,-2), \ (-1,2), \ (1,-2), \ (1,2)$ are solutions of (1),(2),(3)

→

(x,y)	$(-1,-2)$	$(-1,2)$	$(1,-2)$	$(1,2)$
$f(x,y)$	2	-2	-2	2

min

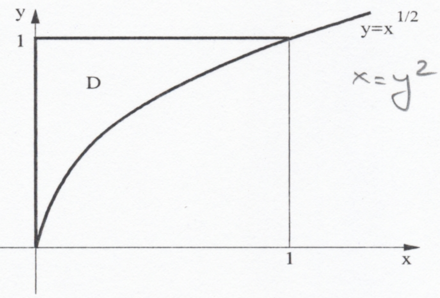
max

$\min = f(-1,2) = f(1,-2) = -2,$

$\max = f(-1,-2) = f(1,2) = 2.$

Problem 3 *Let D be as in figure, bounded by the bold lines. Evaluate the integral*

$$\iint_D \frac{1}{1+y^3}\,dA.$$



Sol

→ $D = \{ (x,y),\ 0 \le x \le 1,\ \sqrt{x} \le y \le 1 \}$ ← type I

$= \{ (x,y),\ 0 \le y \le 1,\ 0 \le x \le y^2 \}$ ← type II

→ $I = \int_0^1 \int_{\sqrt{x}}^1 \frac{1}{1+y^3}\,dy \cdot dx \quad = ?$

↑ D as of type I

$= \int_0^1 \int_0^{y^2} \frac{1}{1+y^2}\,dx\,dy$

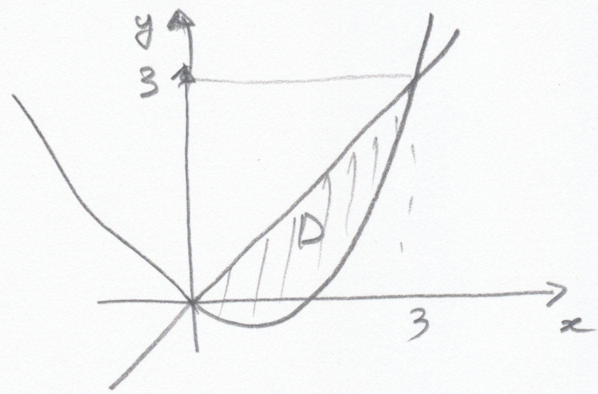
↑ D as of type II

$= \int_0^1 \frac{1}{1+y^3} [x]_0^{y^2}\,dy = \int_0^1 \frac{y^2}{1+y^3}\,dy$

$= \left[\frac{1}{3} \ln(1+y^3) \right]_0^1 = \frac{1}{3}(\ln 2 - \cancel{\ln 1})$

$= \frac{1}{3} \ln 2.$

Problem 4 *Find the mass of the plate with density $\rho(x,y) = x$, bounded by the graphs of functions $y = x^2 - 2x$ et $y = x$.*



Sol

→ $D = ?$

→ intersection of graphs

$x^2 - 2x = x, \quad x^2 - 3x = 0$

$x(x-3) = 0$

$\begin{cases} x=0 \\ y=0 \end{cases}, \begin{cases} x=3 \\ y=3 \end{cases}$

→ $D = \{(x,y), \ 0 \le x \le 3, \ x^2 - 2x \le y \le x\}$ ← type I

→ $m = \iint_D \rho(x,y)\, dA$

$= \int_0^3 \int_{x^2-2x}^{x} x \cdot dy\, dx$

$= \int_0^3 x \cdot [y]_{x^2-2x}^{x}\, dx = \int_0^3 x \cdot (x - (x^2 - 2x))\, dx$

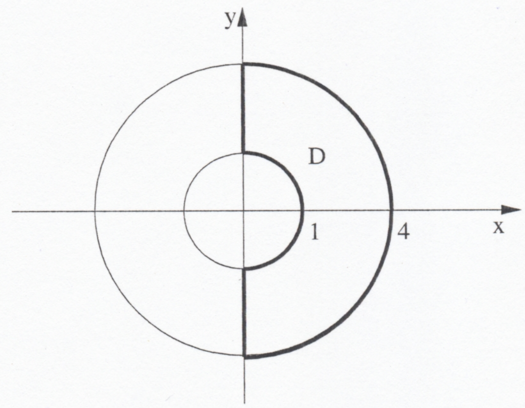
$= \int_0^3 (3x^2 - x^3)\, dx$

$= [x^3 - \frac{1}{4}x^4]_0^3$

$= 3^3 - \frac{1}{4}3^4 = 3^3(1 - \frac{3}{4}) = \frac{1}{4} \cdot 3^3$

$= \frac{27}{4}.$

Problem 5 *The plate D is shown in figure, bounded by the bold lines. Find its mass, knowing that the density of the plate is given by $\rho(x,y) = y^2$.*



Solution

$\to$ $D = \{ (r,\theta),\ 1 \le r \le 4,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \}$

$= [1,4] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ in polar coordinates (r,θ);

Note: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$ $\quad r = \sqrt{x^2+y^2}$.

$\to$ $m = \iint_D \rho(x,y)\, dA$

$= \iint_D y^2 \cdot dA$ [use polar coords]

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^4 r \cdot (r\sin\theta)^2\, dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cdot \left[\frac{1}{4} r^4\right]_1^4 d\theta = \frac{1}{4}(4^4-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta$

$= \frac{1}{4}(4^4-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1-\cos(2\theta))\, d\theta$

$= \frac{1}{4}(4^4-1) \cdot \frac{1}{2} \left[\theta - \cancel{\frac{1}{2}\sin(2\theta)}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\pi}{8}(4^4-1)$.

$= \frac{255}{8}\pi$.