Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Professor Arian Novruzi
Department of Mathematics and Statistics
University of Ottawa
email:novruzi@uottawa.ca
tel: x3530 or 613 8899 535

+ Sol

Calculus III for Engineers
MAT2322B
Fall 2017
Duration: 3 hours
CLOSED BOOK

FINAL EXAM

NAME: ______________________________ **Student number:** ______________

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and electronic devices must be turned off and put away, for example in your bag. You should not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediateley the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature____

Instructions:

- This exam has 14 questions and 17 pages. The last two pages are for your preliminary calculations.
- **Questions 1-8 are multiple choice questions (MCQ)**, each worth **5 points** (no partial points). Write your answer (a letter A-F) in the table below.
- **Questions 9-14 are long answer questions (LAQ)**, each worth **10 points**.
- A correct solution (for LAQ) requires a **clearly written and fully detailed solution**. Use the space following the question, or if necessary, the back of any page.
- **The only calculators which are allowed are TI-30, TI-34, Casio FX-260, FX-300, scientific and non prgrammable**

Multiple choice questions	1	2	3	4	5	6	7	8	Total (40)
Put your answer	B	B.	D	D	A	D	A	B	

Long answer questions	9	10	11	12	13	14	Total (60)
Your result							

Problem 1 *(MCQ) Let* $f(x,y) = 2x^2 + xy + y^3$ *with* $(x,y) \in \mathbb{R}^2$. *Find and classify the critical points of* f. *Which of the following statements is true?*

A. f has 1 local maximum and 1 saddle point

$\boxed{\text{B.}}$ f has 1 local minimum and 1 saddle point

C. f has 1 local maximum and 1 local minimum

D. f has 2 saddle points

E. none of above

Solution

→ $\begin{cases} f_x = 4x + y = 0 & (1); \\ f_y = x + 3y^2 = 0 & (2) \end{cases}$

(1) ⇒ $y = -4x$; (3)

(2) ⇒ $x + 3(-4x)^2 = 0$

$x \cdot (1 + 12x) = 0 \Rightarrow x = 0$ or $x = -\frac{1}{12}$

(3) ⇒ $x = 0, \ y = 0$;

$x = -\frac{1}{12}, \ y = \frac{1}{3}$.

So, two C.P. $(0,0), \left(-\frac{1}{12}, \frac{1}{3}\right)$

→ $f_{xx} = 4, \ f_{yy} = 6y, \ f_{xy} = 1$.

$D = 24y - 1$.

→ $(0,0)$: $D = -1 < 0$; so $(0,0)$ is S.P.

$\left(-\frac{1}{12}, \frac{1}{3}\right)$: $\left.\begin{array}{l} D = 24 \cdot \frac{1}{3} - 1 = 23 > 0 \\ f_{xx} = 4 > 0 \end{array}\right\}$; so $\left(-\frac{1}{12}, \frac{1}{3}\right)$ is loc. min.

Problem 2 *(MCQ) The double integral $\int_0^1 \int_{x^2}^1 f(x,y)dydx$ gives the integral over a region $D \subset \mathbb{R}^2$ which is described as a type I region. Which of the integrals below describes the same integral, but with the integral expressed using D as a type II region?*

A. $\int_0^1 \int_0^1 f(x,y)dxdy$

$\boxed{\text{B.}}$ $\int_0^1 \int_0^{\sqrt{y}} f(x,y)dxdy$

C. $\int_0^1 \int_{\sqrt{y}}^1 f(x,y)dxdy$

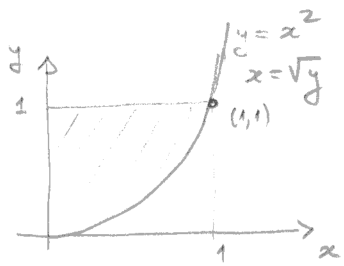
D. $\int_0^1 \int_0^{y^2} dxdy$

E. none of above

Solution

→ $D = \{ (x,y),\ 0 \le x \le 1,\ x^2 \le y \le 1 \}$

$= \{ (x,y),\ 0 \le y \le 1,\ 0 \le x \le \sqrt{y} \}$



→ $\int_0^1 \int_{x^2}^1 f(x,y)dydx = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$.

Problem 3 *(MCQ) Let E be the solid in the first octant, i.e. $x,\ y,\ z \geq 0$, and bounded by the coordinate planes and the plane $z = 1 - x - y$. Which of the following values gives the volume of E?*

A. 1

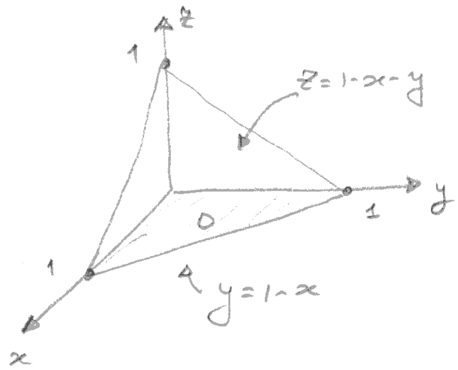
B. $\frac{1}{2}$

C. $\frac{1}{3}$

$\boxed{\text{D.}}$ $\frac{1}{6}$

E. none of above

Solution



→ $z = f(x,y) = 1 - x - y$

$z = 0 \Rightarrow 1 - x - y = 0, \quad y = 1 - x$

→ $E = \{ (x,y,z), \quad (x,y) \in D, \quad 0 \leq z \leq 1 - x - y \}$,

$D = \{ (x,y), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x \}$.

→ $V(E) = \iiint_E 1 \cdot dA = \iint_D \int_0^{1-x-y} dz\, dA$

$= \iint_D (1 - x - y)\, dA$

$= \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_{y=0}^{1-x} dx = \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$

$= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \int_0^1 (1 - 2x + x^2)\, dx$

$= \frac{1}{2} \left[x - x^2 + \frac{1}{3}x^3 \right]_0^1 = \frac{1}{2} \left(1 - 1 + \frac{1}{3} \right)$

$= \frac{1}{6}$.

Problem 4 *(MCQ) A wire has the shape of the parametrized curve* $C = \{\vec{r}(t) = (t^2, 3t, 4t),\ 0 \le t \le 6\}$*, and has a mass-density given by* $\rho(x, y, z) = y$*. Which of the values below gives the total mass* $\int_C \rho ds$ *of the wire?*

A. 49

B. 128

C. 254

$\boxed{\text{D.}}$ 518

E. none of above

Solution

→ $m = \int_C \rho\, ds$, $\vec{r}(t) = (x(t), y(t), z(t)) = (t^2, 3t, 4t)$

$\vec{r}\,'(t) = (2t, 3, 4)$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 9 + 16} = \sqrt{4t^2 + 25}$.

→ $m = \int_0^6 3t \cdot \sqrt{4t^2 + 25}\, dt$

$= 3 \cdot \left[\frac{(4t^2+25)^{3/2}}{3/2} \cdot \frac{1}{8} \right]_0^6$

$= \frac{1}{4} \cdot \left((4 \cdot 36 + 25)^{3/2} - (25)^{3/2} \right)$

$= \frac{1}{4} \cdot \left((13^2)^{3/2} - (5^2)^{3/2} \right)$

$= \frac{1}{4} \left((13)^3 - 5^3 \right)$

$= \frac{1}{4} (2197 - 125)$

$= \frac{1}{4} \cdot 2072 = 518$

Problem 5 *(MCQ) Let C be the portion of the circle $x^2 + y^2 = 1$ which lies in the first quadrant ($x \geq 0,\ y \geq 0$), oriented counterclockwise. Let $\vec{F}(x, y) = (xy, 0)$. Which of the following values equals the line integral $\int_C \vec{F} \cdot d\vec{r}$?*

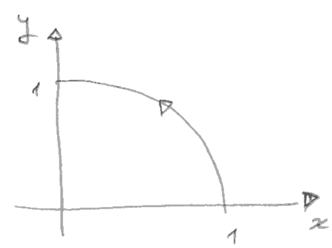
$\boxed{\text{A.}}$ $-\frac{1}{3}$

B. 1

C. $\frac{1}{3}$

D. 3

E. none of above



Solution

$\rightarrow$ $W = \int_C \vec{F} \cdot d\vec{r} = ?$

$\rightarrow$ $\vec{r}(t) = (\cos t, \sin t), \quad t \in [0, \frac{\pi}{2}]$.

$\rightarrow$ $W = \int_0^{\pi/2} (\cos t \cdot \sin t, 0) \cdot (-\sin t, \cos t) \cdot dt$

$= \int_0^{\pi/2} - \sin^2 t \cdot \cos t \, dt$

$= -\frac{1}{3} \left[\sin^3 t\right]_0^{\pi/2}$

$= -\frac{1}{3}$

Problem 6 *(MCQ) A thin sheet S has the form of the portion of the cone $K = \{(x, y, z),\ z = \sqrt{x^2+y^2}\}$ which lies in the first octant, i.e. $x \geq 0,\ y \geq 0,\ z \geq 0$, and between $z = 0$ and $z = 2$. The mass-density of this sheet is given by $\rho(x, y, z) = x$. Which of the values below equals the total mass $\iint_S \rho dS$ of this sheet?*

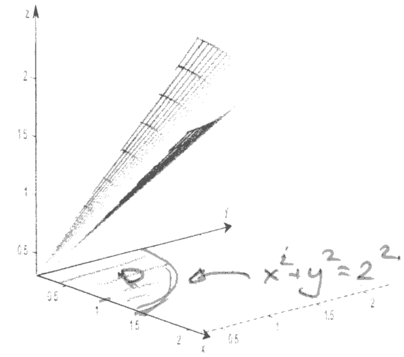
A. $\frac{1}{3}\sqrt{2}$

B. $\frac{8}{3}$

C. $8\sqrt{2}$

D. $\boxed{\frac{8}{3}\sqrt{2}}$

E. none of above



Solution

$\to$ $S = \{ (x,y,z) = (x,y,\sqrt{x^2+y^2}),\ 0 \leq \sqrt{x^2+y^2} \leq 2 \}$.

$= \{ (x,y,g(x,y)),\ x^2+y^2 \leq 2^2 \}$

$= \{ (x,y,g(x,y)),\ (x,y) \in D \}$, $\quad g(x,y) = \sqrt{x^2+y^2}$;

$D = \{ (x,y),\ x,y \geq 0,\ x^2+y^2 \leq 2^2 \}$

$= \{ (r,\theta),\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2 \}$.

$\to$ $g_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $g_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$\sqrt{1+g_x^2+g_y^2} = \left(1 + \dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2}\right)^{1/2} = \sqrt{2}$;

$\to$ $m = \iint_D x \cdot \sqrt{1+g_x^2+g_y^2}\, dA = \iint_D x \cdot \sqrt{2}\, dA$

$= \sqrt{2} \cdot \int_0^2 \int_0^{\pi/2} r \cdot r\cos\theta\, d\theta\, dr$

$= \sqrt{2} \cdot \left[\frac{1}{3} r^3\right]_0^2 \left[\sin\theta\right]_0^{\pi/2} = \sqrt{2} \cdot \frac{8}{3}\ 1$.

$= \frac{8}{3}\sqrt{2}$

Problem 7 *(MCQ) Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = (-y, x, z)$, and let S be the cylinder $x^2 + y^2 = 4$, between $z = 0$ and $z = 4$, oriented away from the z-axis. Which of the values below equals the flux integral $\iint_S \vec{F} \cdot d\vec{S}$?*

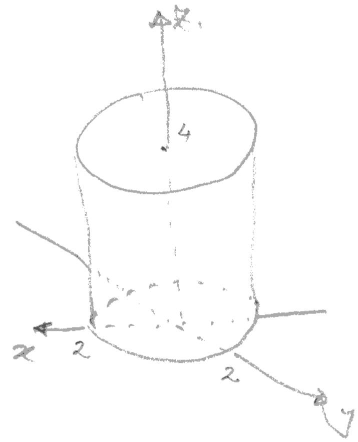
$\boxed{\text{A.}}$ 0

B. 1

C. 2

D. 4

E. none of above



Solution

→ $S = \{ (x,y,z) = (2\cos\theta, 2\sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 4 \}$

$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$.

→ $\vec{r}_\theta = (-2\sin\theta, 2\cos\theta, 0) \quad -2\sin\theta, \; 2\cos\theta$

$\vec{r}_z = (0, 0, 1) \quad 0, \; 0 \quad \Rightarrow \quad \vec{r}_\theta \times \vec{r}_z = (2\cos\theta, 2\sin\theta, 0)$.

→ $\phi = \iint_S \vec{F} \cdot d\vec{S}$

$= \iint_{[0,2\pi]\times[0,4]} \vec{F} \cdot (\vec{r}_\theta \times \vec{r}_z) \, dA$

$= \int_0^{2\pi} \int_0^{4} (-2\sin\theta, 2\cos\theta, z) \cdot (2\cos\theta, 2\sin\theta, 0) \, dz \, d\theta$

$= \int_0^{2\pi} \int_0^{4} (-2^2\sin\theta\cos\theta + 2\sin\theta \cdot \cos\theta) \, dz \, d\theta$

$= 0$.

Problem 8 *(MCQ) Of the vector fields below, only one is such that there exists a scalar function $f(x, y, z)$ with $\vec{F} = \nabla f(x, y, z)$ in $\mathbb{R}^3$. Which one?*

A. $\vec{F} = (z, 0, -x)$

$\boxed{B.}$ $\vec{F} = (x^2, y^3, z^4)$

C. $\vec{F} = (y, -x, 0)$

D. $\vec{F} = (0, z, -y)$

E. $\vec{F} = (xyz, 0, 0)$

Solution

$\rightarrow$ $\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ z & 0 & -x & z & 0 \end{array}$ $\Big|$ $\nabla \times \vec{F} = (0, 2, 0)$ $\times$

$\rightarrow$ $\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ x^2 & y^3 & z^4 & x^2 & y^3 \end{array}$ $\Big|$ $\nabla \times \vec{F} = (0, 0, 0)$ $\checkmark$

$\rightarrow$ $\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ y & -x & 0 & y & -x \end{array}$ $\Big|$ $\nabla \times \vec{F} = (0, 0, -2)$ $\times$

$\rightarrow$ $\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ 0 & z & -y & 0 & z \end{array}$ $\Big|$ $\nabla \times \vec{F} = (-2, 0, 0)$ $\times$

$\rightarrow$ $\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ xyz & 0 & 0 & xyz & 0 \end{array}$ $\Big|$ $\nabla \cdot \vec{F} = (0, xy, -xz)$ $\times$.

Problem 9 *(LAQ) Let $f(x,y) = x^2 - y^2 + 2$ and $D = \{(x,y) \mid x^2+y^2 \leq 1\}$. Find the maximum and minimum values of f in D.*

Sol

→ C.P. of f in D: $\begin{cases} f_x = 2x = 0 \\ f_y = -2y = 0 \end{cases} \Rightarrow (0,0)$.

→ C.P. of f on ∂D:

- $\partial D = \{(\cos\theta, \sin\theta),\ 0 \leq \theta \leq 2\pi\}$.
- $g(\theta) = f(\cos\theta, \sin\theta) = \cos^2\theta - \sin^2\theta + 2 = \cos(2\theta) + 2$.
- $g'(\theta) = -2\sin(2\theta) + 2 = 0$, $\sin(2\theta) = 1$, $2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

  $\theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi$

→ Comparison:

C.P.	$(0,0)$	$(\cos 0, \sin 0)$	$(\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$	$(\cos\pi, \sin\pi)$	$(\cos\frac{3}{2}\pi, \sin\frac{3}{2}\pi)$.
f (C.P)	2	3	1	3	1.

→ max = 3 @ $(1,0)$ and $(-1,0)$

→ min = 1 @ $(0,1)$ and $(0,-1)$.

rmk C.P. on ∂D can be fund with Lagrange multipliers:

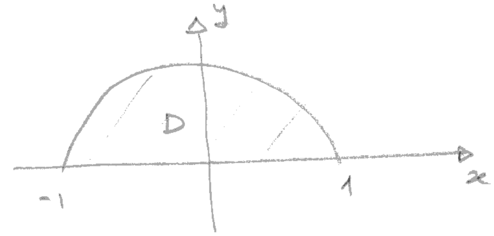
$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2+y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 2x = \lambda 2x \\ 2y = \lambda.(-2y) \\ x^2+y^2 = 1. \end{cases} \Leftrightarrow \begin{cases} x(1-\lambda) = 0 & (1) \\ y(1+\lambda) = 0 & (2) \\ x^2+y^2 = 1. & (3) \end{cases}$$

(1) $\Rightarrow$ $x = 0$; then (3) $\Rightarrow$ $y = \pm 1$, so $(0, \pm 1)$

(1) $\Rightarrow$ $\lambda = 1$; then (2) $\Rightarrow$ $y = 0$; so (3) $\Rightarrow$ $x = \pm 1$, $(\pm 1, 0)$.

Problem 10 *(LAQ) Compute* $\iint_D \sin(x^2+y^2)dA$*, where* $D \subset \mathbb{R}^2$ *is the region above* x*-axis and bounded by the circle* $x^2+y^2=1$ *and the* x*-axis (upper half disk with radius 1 and center at the origin).*

Solution



→ $D = \{(x,y),\quad x^2+y^2 \le 1,\quad y \geq 0\}$

$= \{(r\cos\theta, r\sin\theta),\quad 0 \le r \le 1,\quad 0 \le \theta \le \pi\}$

→ $\iint_D \sin(x^2+y^2)\, dA = \int_0^{\pi} \int_0^1 r \sin(r^2)\, dr\, d\theta$

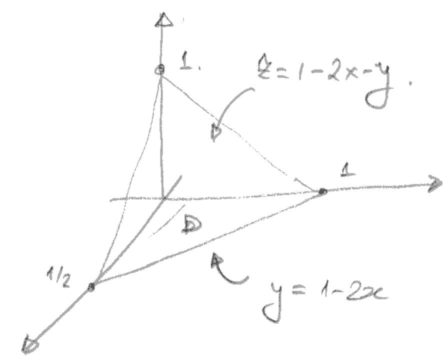
$= \int_0^{\pi} \left[-\frac{\cos(r^2)}{2} \right]_{r=0}^{1} d\theta$

$= \int_0^{\pi} \frac{1}{2} \cdot (1-\cos(1))\, d\theta$

$= (1-\cos(1))\frac{\pi}{2}.$

Problem 11 *(LAQ) A solid object occuppies the region E in the first octant (i.e. $x, y, z \geq 0$) and is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and the plane with equation $2x+y+z = 1$. This object has mass-density given by $\rho(x, y, z) = 48y$. Find the total mass of this object.*

Solution



- $E = \{(x,y,z),\ (x,y) \in D,\quad 0 \le z \le 1-2x-y\}$,

  $D = \{(x,y),\ 0 \le x \le \frac{1}{2},\ 0 \le y \le 1-2x\}$.

- $$m = \iiint_E 48y \, dV$$

$$= \iint_D \int_0^{1-2x-y} 48y \, dz = \iint_D 48y(1-2x-y)\, dA$$

$$= \int_0^{1/2} \int_0^{1-2x} 48\left((1-2x)y - y^2\right) dy\, dx$$

$$= \int_0^{1/2} 48 \left[(1-2x)\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{y=0}^{1-2x} dx$$

$$= \int_0^{1/2} 48 \cdot \left((1-2x)^3 \frac{1}{2} - \frac{1}{3}(1-2x)^3\right) dx$$

$$= \int_0^{1/2} 48 \cdot \frac{1}{6}(1-2x)^3 \, dx$$

$t = 1-2x$

$dt = -2dx$

$$= \int_1^0 8 \cdot t^3 \cdot \left(-\frac{1}{2}dt\right).$$

$$= 4\int_0^1 t^3 \, dt = 4 \cdot \left[\frac{1}{4}t^4\right]_0^1$$

$$= 1.$$

Problem 12 *(LAQ) Compute*[1] *the triple integral* $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^3 dzdydx$.

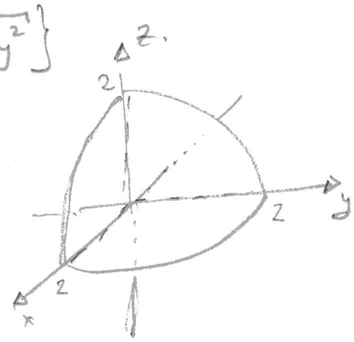
Sol

$\rightarrow$ $I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^3 \, dz \, dy \, dx$

$= \iiint_E (x^2+y^2+z^2)^3 dV$, where.

$E = \{ (x,y,z), \ 0 \le x \le 2, \ 0 \le y \le \sqrt{4-x^2}, \ 0 \le z \le \sqrt{4-x^2-y^2} \}$

$= \{ (x,y,z), \ x,y,z \geq 0, \ x^2+y^2+z^2 \le 2^2 \}$



$\rightarrow$ use spherical coordinates:

$E = \{ (\rho, \theta, \varphi) \ 0 \le \rho \le 2, \ 0 \le \theta \le \frac{\pi}{2}, \ 0 \le \varphi \le \frac{\pi}{2} \}$

$I = \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \sin\varphi \cdot \rho^6 \, d\varphi \, d\theta \, d\rho$

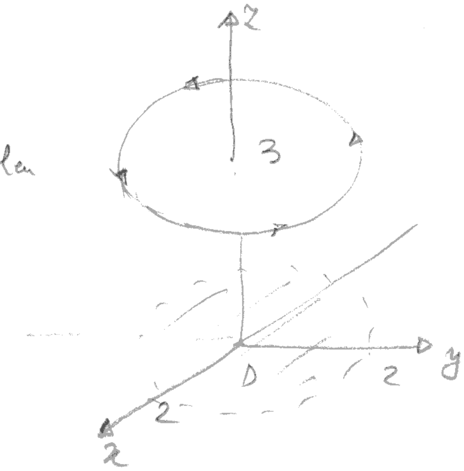
$= \left(\int_0^2 \rho^8 d\rho \right) \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\pi/2} \sin\varphi \, d\varphi \right)$

$= \left[\frac{1}{9} \rho^9 \right]_0^2 \left[\theta \right]_0^{\pi/2} \left[-\cos\varphi \right]_0^{\pi/2}$

$= \frac{2^8}{9} \pi .$

[1]Hint: cartesian coordinates may not be the most appropriate choice of coordinates to do this computation

Problem 13 *(LAQ) A vector field $\vec{F}(x,y,z)$ is such that $\nabla \times \vec{F} = (0,0,2z)$. Using Stokes theorem, compute the line integral $I = \int_C \vec{F} \cdot d\vec{r}$ where C is the oriented closed curve given by $C = \{\vec{r}(t) = (2\cos(t), 2\sin(t), 3),\ 0 \le t \le 2\pi\}$.*



Solution

→ Let S the surface (planar) that C encloses; then

$$S = \{(x,y,3),\quad x^2+y^2 \le 2^2\}$$
$$= \{(x,y,g(x,y)),\quad (x,y) \in D\},$$
$$g(x,y) = 3, \qquad D = \{(x,y),\quad x^2+y^2 \le 2^2\}$$

→ $\operatorname{curl} \vec{F} = (0,0,2z)$

$$I = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D (-0 \cdot \cancel{g_x} - 0 \cdot \cancel{g_y} + 2z)\, dA$$

$$= \iint_D 2 \cdot 3 \, dA$$

$$= 6 \iint_D dA$$

$$= 6 \cdot \pi \cdot 2^2$$

$$= 24\pi.$$

Problem 14 *(LAQ) Consider the vector field* $\vec{F}(x,y,z) = (yz + x, xz^2 - y, z^2)$.
(a) Compute the divergence of $\vec{F}$, *i.e. compute* $\nabla \cdot \vec{F}$.
(b) Let E *denote the three-dimensional parallelepiped (rectangular box) region defined by* $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$, *and let* S *denote the surface which is the boundary of* E, *oriented outwards. Compute the flux integral* $\iint_S \vec{F} \cdot d\vec{S}$ *by using Gauss Divergence theorem.*

Solution

$\rightarrow \nabla \cdot \vec{F} = 1 - 1 + 2z = 2z$

$$\rightarrow \phi = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_E \operatorname{div} \vec{F} \cdot dV = \iiint_E 2z \, dV$$

$$= \int_0^1 \int_0^2 \int_0^3 2z \, dz \, dy \, dx$$

$$= \left(\int_0^1 dx\right)\left(\int_0^2 dy\right)\left(\int_0^3 2z \, dz\right)$$

$$= [x]_0^1 \, [y]_0^2 \cdot [z^2]_0^3$$

$$= 1 \cdot 2 \cdot 3^2$$

$$= 18.$$