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MAT2322, Calculus III
2013, Midterm #2

Write CLEARLY (in uppercase letters) your

LAST NAME, Firstname:
Student number:

+ Sol

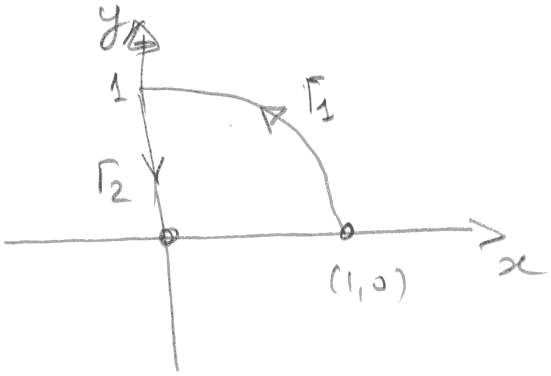
Instructions:

- The length of the exam is de 80 minutes.
- The exam has 5 problems.
- Write the solution clearly in the space following it. If necessary, you can continue the solution in the back of any page - in this case, you must clearly indicate that the solution continues in the back of the page "n".
- **Use of manuals, courses notes, calculators or any other electronic devices is not allowed.**.

Results

Problem	1	2	3	4	5	Total
Your result						

Problem 1 *(5 points)*
Calculate the line integral of $\vec{F}(x,y) = (x-y, x+y)$ along the curve Γ, where Γ consists of the arc of the circle $x^2+y^2=1$ from $(1,0)$ to $(0,1)$ oriented counterclockwise, and of the segment from $(0,1)$ to $(0,0)$.



$$\Gamma = \Gamma_1 \cup \Gamma_2$$

$$\Gamma_1 = \{ (\cos t, \sin t), \quad t \in [0, \tfrac{\pi}{2}] \}$$

$$\Gamma_2 = \{ (0, 1-t), \quad t \in [0,1] \}.$$

$$\int_\Gamma (x-y)\,dx + (x+y)\,dy =$$

$$\int_{\Gamma_1} (x-y)\,dx + (x+y)\,dy + \int_{\Gamma_2} (x-y)\,dx + (x+y)\,dy =$$

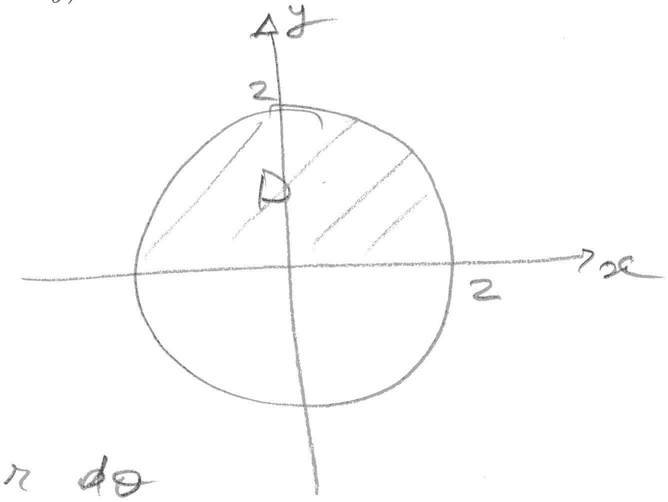
$$= \int_0^{\pi/2} \Big((\cos t - \sin t)(-\sin t) + (\cos t + \sin t).\cos t \Big) dt + \int_0^1 \Big((0-(1-t))\,0 + (0+1-t)(-1) \Big) dt$$

$$= \int_0^{\pi/2} (\sin^2 t + \cos^2 t)\,dt + \int_0^1 (t-1)\,dt$$

$$= \frac{\pi}{2} + \left(\frac{1}{2} - 1\right) =$$

$$= \frac{\pi - 1}{2}$$

Problem 2 *(5 points)*
Let $D = \{(x, y),\ x^2 + y^2 \leq 4,\ y \geq 0\}$. *Evaluate* $\iint_D (x + y)dA$.



$D = \{(r, \theta),\ 0 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$.

$$\iint_D (x+y)\, dA$$

$$= \int_0^\pi \int_0^2 r \cdot (r\cos\theta + r\sin\theta)\, dr\, d\theta$$

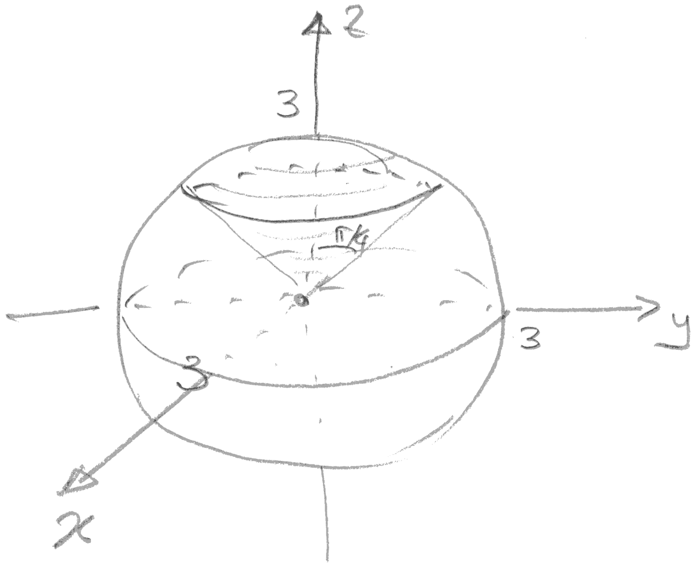
$$= \int_0^\pi (\cos\theta + \sin\theta) \left[\frac{r^3}{3}\right]_0^2 d\theta$$

$$= \frac{8}{3} \left[\sin\theta - \cos\theta\right]_0^\pi$$

$$= \frac{8}{3} \left((0 - (-1)) - (0 - 1) \right) =$$

$$= \frac{16}{3}.$$

Problem 3 *(5 points)*
Find the volume of the bounded domain (solid) enclosed by the surfaces $x^2 + y^2 + z^2 = 9$ *(sphere) and* $z = \sqrt{x^2 + y^2}$ *(cone).*



Use spherical coordinates

$$\begin{cases} \text{sphere: } \rho^2 = 3^2 \\ \text{cone: } \rho\cos\varphi = \rho\sin\theta. \end{cases}$$

Intersection:

$$\begin{cases} \rho^2 = 3^2 \\ \rho\cos\varphi = \rho\sin\varphi \end{cases} \Rightarrow \begin{cases} \rho = 3 \\ \cos\varphi = \sin\varphi \end{cases} \Rightarrow \begin{matrix} \rho = 3 \\ \varphi = \frac{\pi}{4}. \end{matrix}$$

Then

$$E = \left\{ (\rho, \theta, \varphi), \quad \begin{matrix} 0 \le \rho \le 3, \\ 0 \le \theta \le 2\pi, \\ 0 \le \varphi \le \frac{\pi}{4} \end{matrix} \right\}$$

$$V(E) = \iiint_E dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \cdot \left[-\cos\varphi\right]_0^{\pi/4} \left[\frac{1}{3}\rho^3\right]_0^3 = 9\pi(2 - \sqrt{2})$$

Problem 4 *(5 points)*
Find which of the following vector fields is conservative in $\mathbb{R}^2$. For the conservative field(s) find the corresponding potential, for not conservative field(s) explain why they are not conservative.

a) $\vec{F}(x,y) = (y, -x)$
b) $\vec{G}(x,y) = (x^2y^2, x^2y)$
c) $\vec{H}(x,y) = (ye^x + \sin y, e^x + x\cos y + 2y)$

a) $Q_x = -1 \neq 1 = P_y$; $\vec{F}$ not conservative

b) $Q_x = 2xy \neq 2x^2y = P_y$; —//—

c) $\left.\begin{array}{l} Q_x = e^x + \cos y = P_y \\ D = \mathbb{R}^2 \text{ simply connected} \end{array}\right\} \Rightarrow \vec{F} = \nabla f$ in $\mathbb{R}^2$.

Hence:

$$\begin{cases} f_x = ye^x + \sin y & (1) \\ f_y = e^x + x\cos y + 2y & (2). \end{cases}$$

$(1) \Rightarrow f(x,y) = ye^x + x \cdot \sin y + h(y)$

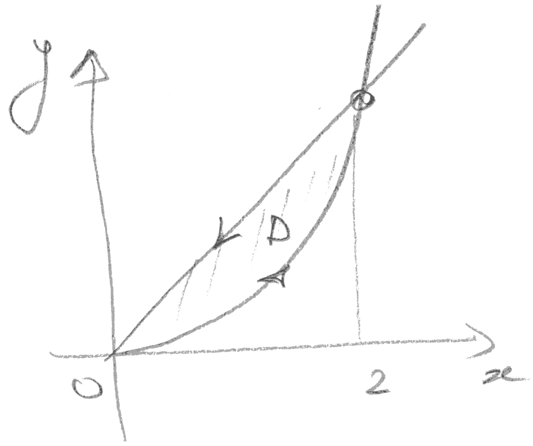
$(2) \Rightarrow e^x + x\cos y + h'(y) = e^x + x\cos y + 2y$

$h'(y) = 2y \Rightarrow h(y) = y^2 + C$

Thus

$$f(x,y) = ye^x + x\sin y + y^2 + C.$$

Problem 5 *(5 points)*
Let $D \subset \mathbb{R}^2$ be the bounded domain delimited by the graphs of functions $g(h) = x^2$ and $h(x) = 2x$. Evaluate $\int_\Gamma (x^2 - y^2)dx + 2xydy$, where Γ is the boundary of D, oriented positively (counterclockwise).



intersection:

$x^2 = 2x, \quad x = 0, \quad x = 2$

$D = \{ (x,y), \quad 0 \le x \le 2, \quad x^2 \le y \le 2x \}$

$$\int_\Gamma (x^2 - y^2)dx + 2xy\,dy =$$

$$= \iint_D \left((2xy)_x - (x^2-y^2)_y \right) dA$$

$$= \iint_D 4y \, dA$$

$$= \int_0^2 \int_{x^2}^{2x} 4y \, dy \, dx$$

$$= \int_0^2 2[y^2]_{x^2}^{2x} \, dx$$

$$= \int_0^2 2(4x^2 - x^4)dx = \left[8\frac{x^3}{3} - 2\frac{x^5}{5} \right]_0^2$$

$$= \frac{128}{15}.$$