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MAT2322, Calculus III
Midterm #1
(Fall)

Write CLEARLY (in uppercase letters) your

LAST NAME, Firstname: + Sol
Student number:

Instructions:

- The length of the exam is de 80 minutes.
- The exam has 5 problems.
- Write the solution clearly in the space following it. If necessary, you can continue the solution in the back of any page - in this case, you must clearly indicate that the solution continues in the back of the page "n".
- **Use of manuals, courses notes, calculators or any other electronic devices is not allowed..**

Results:

Problem	1	2	3	4	5	Total
Your result						(over 20)

Problem 1 *(4 points) Find and classify the critical points of the function*

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Sol

$\rightarrow \begin{cases} f_x(x,y) = 6x^2 + y^2 + 10x = 0 & (1) \\ f_y(x,y) = 2xy + 2y = 0 & (2) \end{cases} \Rightarrow (x+1)y = 0 \Rightarrow$

$x + 1 = 0$ or $y = 0$; sub in (1):

$\Downarrow$

$x = -1$; sub in (1):

$6 + y^2 - 10 = 0$

$y^2 = 4$, $y = \pm 2$

$6x^2 + 10x = 0$

$2x(3x+5) = 0$

$x = 0$, $x = -\frac{5}{3}$

Hence, C.P. are: $(-1, \pm 2)$, $(0,0)$, $(-\frac{5}{3}, 0)$

$\rightarrow$ $f_{xx}(x,y) = 12x + 10$

$f_{yy}(x,y) = 2x + 2$

$f_{xy}(x,y) = 2y$,

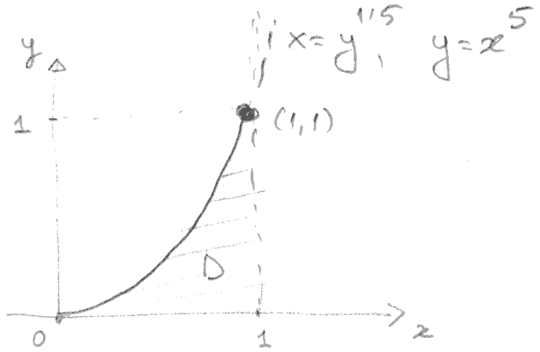
$D = (12x+10)(2x+2) - (2y)^2$

$\rightarrow$

	$(-1,-2)$	$(-1,+2)$	$(0,0)$	$(-5/3, 0)$
D	-16	-16	20	13.3
f_{xx}			10	-10
$f(x,y)$	S.P.	S.P.	loc. min	loc. max.

Problem 2 *(3 points) Evaluate the integral*

$$I = \int_0^1 \int_{y^{1/5}}^1 \frac{1}{1+x^6} dx dy.$$



Sol

$$I = \iint_D \frac{1}{1+x^6} dA,$$ where

$$D = \{(x,y),\ 0 \le y \le 1,\ y^{1/5} \le x \le 1\}$$

Note that $x = y^{1/5}$ gives $y = x^5$, and that $x = y^{1/5}$ and $x = 1$ intersect at $(1,1)$. Then

$$\iint_D \frac{1}{1+x^6} dA = \int_0^1 \int_0^{x^5} \frac{1}{1+x^6} dy\, dx$$

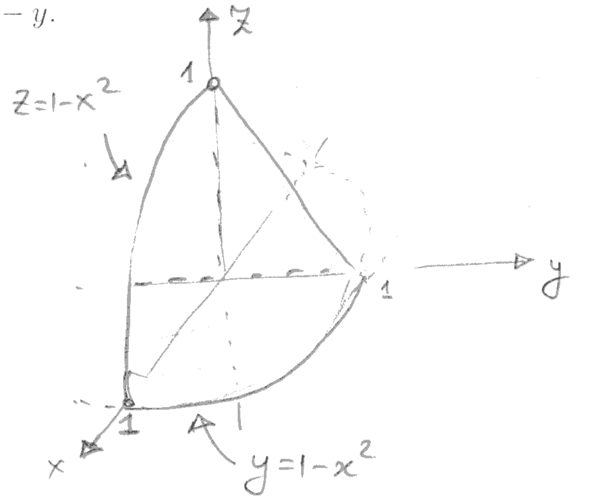
use D of type I

$$= \int_0^1 \frac{1}{1+x^6} [y]_{y=0}^{y=x^5} dx = \int_0^1 \frac{x^5}{1+x^6} dx$$

$$= \frac{1}{6} [\ln(1+x^6)]_0^1$$

$$= \frac{\ln 6}{2}$$

Problem 3 *(5 points) Find the volume of the solid E in the first octant $(x, y, z \geq 0)$ delimited by the coordinate planes and the surface $z = 1 - x^2 - y$.*



Sol

→ sketch the domain;

taking $x=0$ or $y=0$ or $z=0$

in $z=1-x^2-y$ we get:

$z=1-y$ or $z=1-x^2$ or $1-x^2-y=0$

It follows that the surface

$z=1-x^2-y$ intersect : → the plane xy along the parabola $y=1-x^2$

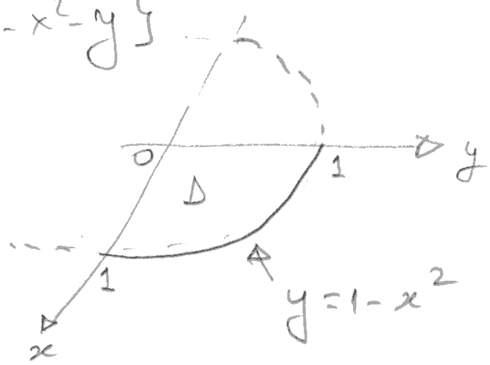
→ the plane xz along the parabola $z=1-x^2$

→ the plane yz along the line $z=1-y$.

→ then $E=\{(x,y,z),\ (x,y)\in D,\ 0\leq z\leq 1-x^2-y\}$

$D=\{(x,y),\ 0\leq x\leq 1,$

$0\leq y\leq 1-x^2\}.$



→ $V = \iiint\limits_E dV$

$$= \iint\limits_D \int_0^{1-x^2-y} dz\, dA = \iint\limits_D (1-x^2-y)\, dA$$

$$= \int_0^1 \int_0^{1-x^2} (1-x^2-y)\, dy\, dx$$

$$= \int_0^1 \frac{1}{2}(1-2x^2+x^4)\, dx = \frac{4}{15}$$

Problem 4 *(4 points) Use the Lagrange multipliers method to find the minimum and maximum of $f(x,y) = x^2y$ under the constraint $x^2 + 2y^2 = 6$.*

Sol

→ Solve $\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), \\ g(x,y) = 0, \end{cases}$ where $g(x,y) = x^2 + 2y^2 - 6$.

So:

$$\begin{cases} 2xy = \lambda \cdot 2x & (1) \\ x^2 = \lambda \cdot 4y & (2) \\ x^2 + 2y^2 = 6 & (3). \end{cases}$$

Note that (1) $\Rightarrow$ $2x \cdot (y - \lambda) = 0$; hence

$x = 0$ or $y - \lambda = 0$, $y = \lambda$.

Replace in (3):

$2y^2 = 6$, $y = \pm\sqrt{3}$

Hence

$(0, -\sqrt{3})$, $(0, +\sqrt{3})$

are solutions of (1)-(3)

(for an appropriate λ)

Replace in (2):

$x^2 = 4y^2$;

Replace in (3):

$6y^2 = 6$, $y = \pm 1$.

Replacing $y = \lambda = \pm 1$ in (2) gives

$x^2 = 4$, $x = \pm 2$.

Hence, $(-2,-1)$, $(-2,+1)$, $(2,-1)$, $(2,+1)$

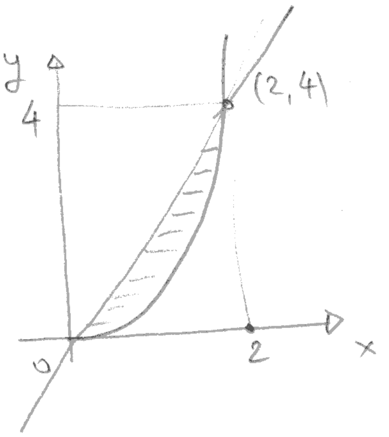
are solutions to (1) - (3).

(for appropriate λ).

→ comparison.

(x,y)	$(0, \pm\sqrt{3})$	$(\pm 2, -1)$	$(\pm 2, +1)$
$f(x,y)$	0	$\boxed{-4}$	$\boxed{4}$
		minimum. value	maximum value

Problem 5 *(4 points) Find the mass and the center of mass of the plate with density $\rho(x, y) = 3$ and delimited by the graphs of $y = x^2$ and $y = 2x$.*



Sol

$\to$ $D = ?$ $\quad x^2 = 2x, \quad (x=0,\ y=0)$

$\quad\quad\quad\quad\quad\quad\quad\quad (x=2,\ y=4)$

$$D = \{(x,y),\ 0 \le x \le 2,\ x^2 \le y \le 2x\}.$$

$\to$ $m = \iint_D 3\, dA$

$$= \int_0^2 \int_{x^2}^{2x} 3\, dy\, dx = \int_0^2 3(2x - x^2)\, dx = 4$$

$\to$ $m_y = \iint_D 3 \cdot x\, dA$

$$= \int_0^2 \int_{x^2}^{2x} 3x\, dy\, dx = \int_0^2 3x\,(2x - x^2)\, dx = 4$$

$\to$ $m_x = \iint_D 3 \cdot y\, dA$

$$= \int_0^2 \int_{x^2}^{2x} 3 \cdot y\, dy\, dx = \int_0^2 \frac{3}{2}\left((2x)^2 - (x^2)^2\right) dx = \frac{32}{5}$$

$\to$ $Q = \left(\frac{m_y}{m}, \frac{m_x}{m}\right) = \left(1, \frac{8}{5}\right)$