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MAT2322, 2013
Final exam
(Fall)

LAST NAME, First name: + solution.
Student number:

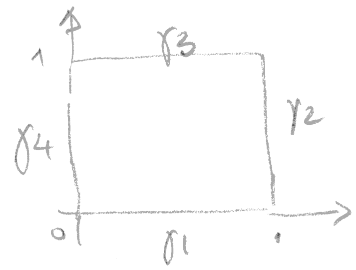
Instructions:

0) Write clearly your name and stuent number on this page
1) This exam has of 9 problems.
2) The duration of this exam is 3 hours.
3) No books or any other documents are allowed.
4) A simple calculator with no programming and graphical capabilities can be used.
5) Solve each problem using the space following it; if more space is needed use the back of any page or additional white pages after the last problem and indicate when doing so.

Problem	1	2	3	4	5	6	7	8	9	Total (60)
Your result	7	4	6	6	6	7	8	8	8	

Problem 1 (~~6~~ 7 *points*)
Find the global maximum and the global minimum values of $f(x,y) = x^2 + y^2 + xy^2 + 2$ *in the region* $D = \{(x,y) \in \mathbb{R}^2,\ 0 \le x \le 1,\ 0 \le y \le 1\}$.

<u>Sol</u>



$\to$ C.P. $\begin{cases} f_x = 2x + y^2 = 0 & (1) \\ f_y = 2y + 2xy = 0 & (2) \end{cases}$

(2) $\Rightarrow$ $y(x+1) = 0$:

$x = -1$ or $y = 0$

(1) $\Rightarrow$ $y = \pm\sqrt{2}$ or $x = 0$

C.P: $(-1, \pm\sqrt{2})$, $(0,0)$.

$\to$ $g(x) = f(x,0) = x^2 + 2$, $\min\limits_{\gamma_1} f = 2$, $\max\limits_{\gamma_1} f = 3$

$g(y) = f(y,0) = y^2 + 2$, $\min\limits_{\gamma_4} f = 2$, $\max\limits_{\gamma_4} f = 3$

$g(x) = f(x,1) = x^2 + x + 3$, $\min\limits_{\gamma_3} f = 3$, $\max\limits_{\gamma_3} f = 5$

$g(y) = f(y,1) = 2y^2 + 3$, $\min\limits_{\gamma_2} f = 3$, $\max\limits_{\gamma_2} f = 5$

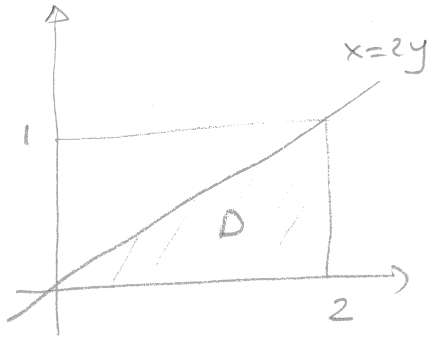
$\to$ $f(0,0) = 2$

$\to$ $\max\limits_{D} f = 5$, $\min\limits_{D} f = 2$.

Problem 2 *(4 points)*

Calculate the integral $\displaystyle\int_0^1 \int_{2y}^2 e^{x^2}\,dx\,dy.$

<u>Sol</u>



$$\int_0^1 \int_{2y}^2 e^{x^2}\,dx\,dy$$

$$= \iint_D e^{x^2}\,dA$$

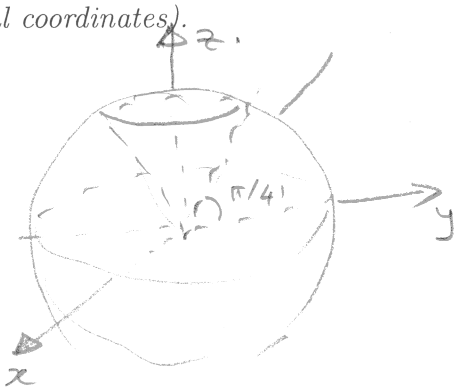
$$= \int_0^2 \int_0^{\frac{x}{2}} e^{x^2}\,dy\,dx$$

$$= \int_0^2 \frac{1}{2} x e^{x^2}\,dx$$

$$= \frac{1}{2}\,\frac{1}{2}\left[e^{x^2}\right]_0^2$$

$$= \frac{1}{4}\left(e^4 - 1\right).$$

Problem 3 *(6 points)*
Let $E \subset \mathbb{R}^3$ be the region bounded by the hemisphere $z = \sqrt{1-x^2-y^2}$ and the cone $z = \sqrt{x^2+y^2}$. Compute $\iiint_E (x^2+y^2+z^2)dV$ (you may use spherical coordinates).



Sol

$\to$ $E = \{(\rho, \theta, \varphi),\quad 0 \le \rho \le 1,\quad 0 \le \theta \le 2\pi,\quad 0 \le \varphi \le \pi/4\}$

$$\to \iiint_E (x^2+y^2+z^2)\,dV =$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\varphi \cdot \rho^2 \,d\rho\, d\varphi\, d\theta =$$

$$\int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \sin\varphi\, d\varphi \cdot \int_0^1 \rho^4\, d\rho =$$

$$[\theta]_0^{2\pi} \cdot [-\cos\varphi]_0^{\pi/4} \cdot \frac{1}{5}[\rho^5]_0^1 =$$

$$2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{5} =$$

$$= \frac{1}{5}\pi\cdot(2-\sqrt{2}).$$

rmk One can use cylindrical coordinates; then $E = \{(r, \theta, z),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1/\sqrt{2},\ r \le z \le \sqrt{1-r^2}\}$ and

$$\iiint_E (x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{1/\sqrt{2}} r \int_r^{\sqrt{1-r^2}} (r^2+z^2)\,dz = \dots = \frac{1}{5}\pi(2-\sqrt{2}).$$

Problem 4 *(6 points)*
Check whether the vector field $\vec{F}(x, y, z) = (y^2 + z, 2xy + z, x + y)$ *is conservative. If yes, find its potential.*

Sol

$\to$ $\operatorname{curl} \vec{F} = \vec{0}$?

$$\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ y^2+z & 2xy+z & x+y & y^2+z & 2xy+z \end{array}$$

$\operatorname{curl} \vec{F} = (1-1, 1-1, 2y-2y) = \vec{0}$; so $\vec{F} = \nabla f$.

$\to$ $f_x = y^2 + z$; (1)

$f_y = 2xy + z$ (2)

$f_z = x + y$ (3)

(3) $\Rightarrow$: $f(x,y,z) = (x+y)z + g(x,y)$.

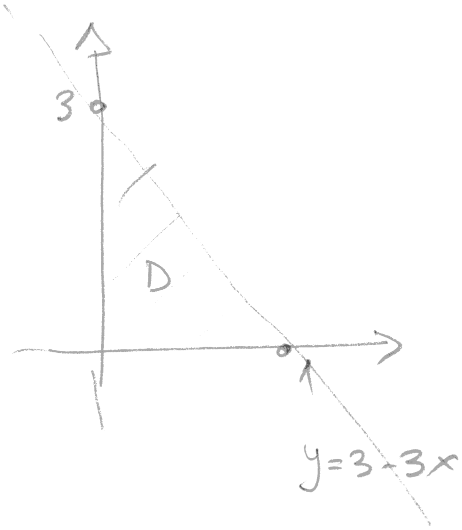
(2) $\Rightarrow$ $z + g_y = 2xy + z \Rightarrow g_y = 2xy$, $g = xy^2 + h(x)$

hence, $f(x,y,z) = (x+y)z + xy^2 + h(x)$.

(1) $\Rightarrow$ $z + y^2 + h' = y^2 + z \Rightarrow h' = 0$, $h = C$.

Thus $f(x,y,z) = (x+y)z + xy^2 + C$.

Problem 5 *(6 points)*
Let $\vec{F}(x,y,z) = (4y, 9x)$*,* $D \subset \mathbb{R}^2$ *be triangle with nodes* $(0,0)$*,* $(1,0)$*,* $(0,3)$ *and* Γ *the boundary of* D *oriented counterclockwise. Use Green's theorem to calculate* $\int_\Gamma \vec{F} \cdot d\vec{r}$*.*



Sol

$$\int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma 4y \cdot dx + 9x\, dy$$

$$= \iint_D \left((9x)_x - (4y)_y \right) dA$$

$$= \iint_D (9-4)\, dA$$

$$= \int_0^1 \int_0^{3-3x} 5\, dy\, dx$$

$$= \int_0^1 5\,(3-3x)\, dx$$

$$= \left[15 \cdot x - 15 \tfrac{1}{2} x^2 \right]_0^1$$

$$= 15 - \frac{15}{2} = \frac{15}{2}.$$

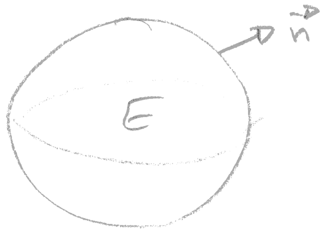
Problem 6 *(7 points)*

Find the flux of $\vec{F}(x,y,z) = (-y, x, z)$ through the sphere $S = \{(x,y,z),\ x^2+y^2+z^2 = 2^2\}$, oriented positively (i.e. the unit normal vector is oriented outward), with two methods:

a) Directly, and

b) using Gauss (divergence) theorem.

Sol



$\vec{n} = \frac{1}{2}(x, y, z)$

a) $\iint_S \vec{F} \cdot d\vec{S} =$

$= \iint_S (\vec{F} \cdot \vec{n})\, dS = \iint_S (-y, x, z) \cdot \frac{1}{2}(x, y, z)$

$= \iint_S \frac{1}{2}(-xy + xy + z^2)\, dS$

$= \iint_S \frac{1}{2} z^2\, dS$

$= \int_0^{2\pi} \int_0^{\pi} 2^2 \sin\varphi \ \frac{1}{2}\, 2^2 \cos^2\varphi \, d\varphi\, d\theta$

$= 2^3 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \cdot \cos^2\varphi \, d\varphi$

$= 2^3 \cdot 2\pi \cdot \left[-\frac{1}{3}\cos^3\varphi\right]_0^{\pi}$

$= \frac{2^5}{3}\pi$

b). $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$

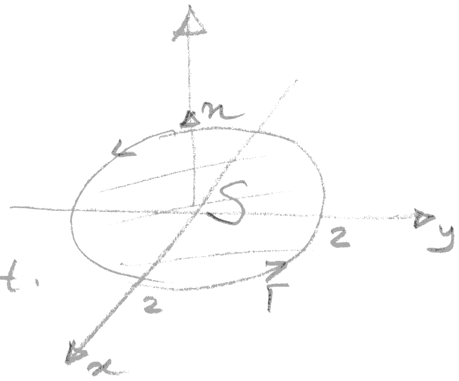
$= \iiint_E (\partial_x(-y) + \partial_y(x) + \partial_z z)\, dV =$

$= \iiint_E dV =$ volume of E $= \frac{4}{3}\pi \cdot 2^3$

$= \frac{2^5}{3}\pi.$

Problem 7 *(8 points)*
Let $\vec{F}(x,y,z) = (yz^4 + y,\ xz^4 + 3x,\ 4xyz^3)$ and Γ be the circle $\{(x,y,0)\ x^2 + y^2 = 4\}$, oriented counterclockwise when seen from $z > 0$. Calculate $\int_\Gamma \vec{F} \cdot d\vec{r}$, with two methods:

a) Directly, and
b) using Stokes' theorem.



Sol

a) $\int_\Gamma \vec{F}\cdot d\vec{r}$ (use $z=0$)

$$= \int_0^{2\pi} (2\sin t,\ 3\cdot 2\cos t,\ 0)\cdot(-2\sin t,\ 2\cos t,\ 0)\,dt.$$

$\Gamma = \{\vec{r}(t),\ t \in [0, 2\pi]\}$
$\vec{r}(t) = (2\cos t,\ 2\sin t,\ 0).$

$$= \int_0^{2\pi} (-4\sin^2 t + 12\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left(-4\cdot\frac{1-\cos(2t)}{2} + 12\cdot\frac{1+\cos(2t)}{2}\right)dt$$

$$= -2\cdot 2\pi + 6\cdot 2\pi = 8\pi$$

b) $\int_\Gamma \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$

But $\vec{n} = (0,0,1)$ and $\operatorname{curl}\vec{F} = (*,\ *,\ 2)$

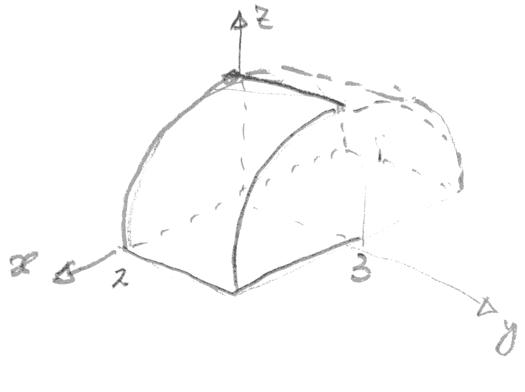
$$\begin{array}{ccccc} \partial_x & \partial_y & \partial_z & \partial_x & \partial_y \\ yz^4+y & xz^4+3x & 4xyz^3 & yz^4+y & xz^4+3x \end{array}$$

Then

$$\int_\Gamma \vec{F}\cdot d\vec{r} = \iint_S (*,*,2)\cdot(0,0,1)\,dS = \iint_S 2\,dS$$

$$= 2\iint_S dS = 2\cdot\pi\cdot 2^2 = 8\pi.$$

Problem 8 *(8 points)*
Let S be the part of the cylinder $z = \sqrt{4-x^2}$ over the first quadrant and for $y \in [0,3]$, with the unit normal vector oriented upward, and $\vec{F} = (y,\ x^2y^2z^4,\ xyz)$. Compute $\iint_S \vec{F}\cdot d\vec{S}$ (you may use cartesian or cylindrical coordinates).



Solution

i) use cartesian coordinates.

$S = \{ (x,y,z),\ 0 \le x \le 2,\ 0 \le y \le 3,\ z = \sqrt{4-x^2} \}.$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^2 \int_0^3 (-P\, g_x - Q\, g_y + R)\, dy\, dx$$

$$= \int_0^2 \int_0^3 \left(-y \cdot \frac{-x}{\sqrt{4-x^2}} - x^2y^2z^4 \cdot 0 + x\cdot y \sqrt{4-x^2} \right) dy\, dx.$$

$$= \frac{9}{2} \int_0^2 \left(\frac{x}{\sqrt{4-x^2}} + x\sqrt{4-x^2} \right) dx$$

$$= \frac{9}{2} \left[-\sqrt{4-x^2} - \frac{1}{3}(4-x^2)^{3/2} \right]_0^2 = 21.$$

ii) use cylindrical coordinates:

$S = \{ (2\cos\theta,\ y,\ 2\sin\theta),\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le y \le 3 \}, \quad \vec{n} = \left(\frac{x}{2}, 0, \frac{z}{2}\right):$

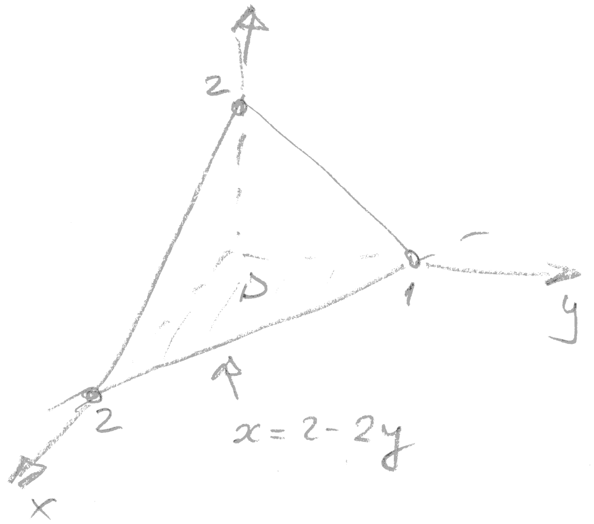
$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{\pi/2} \int_0^3 2\cdot (y,\ x^2y^2z^4,\ xyz)\cdot\left(\frac{x}{2}, 0, \frac{z}{2}\right) dy\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 (2\cos\theta\cdot y + 8\cos\theta\cdot\sin^2\theta\cdot y)\, dy\, d\theta$$

$$= 9\int_0^{\pi/2} (\cos\theta + 4\sin^2\theta\cdot\cos\theta)\, d\theta = 9\left[\sin\theta + \frac{4}{3}\sin^3\theta\right]_0^{\pi/2} = 21.$$

Problem 9 *(8 points)*
Let $\vec{F} = (x^2y, xy^2, 2xyz)$ *and* S *be the boundary (surface) of the tetrahedron with faces the coordinate planes* ($x = y = z = 0$) *and the plane* $x + 2y + z = 2$. *Assume* S *is oriented positively, i.e. the unit normal vector points to the exterior.*

Find the flux $\iint_S \vec{F} \cdot d\vec{S}$ *by using Gauss (divergence) theorem.*



Sol

- $z=0: \quad x+2y=2, \quad x=2-y$

- $E = \{(x,y,z), \quad (x,y) \in D, \quad 0 \le z \le 2-x-2y.\}$

  $D = \{(x,y), \quad 0 \le y \le 1, \quad 0 \le x \le 2-y\}$

- $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \cdot dV$$

$$= \iiint_E (2xy + 2xy + 2xy)\, dV = \iiint_E 6xy\, dV$$

$$= \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy \; dz \cdot dx\, dy$$

$$= \int_0^1 \int_0^{2-2y} 6xy \cdot (2-x-2y)\, dx\, dy$$

$$= \dots =$$

$$= \frac{2}{5}$$