MAT 2322 **February 28th, 2006**
Professor: Dr. Arian Novruzi

Midterm exam (A) (winter 2006)

+ Solution

LAST NAME, First name:
Student number:

Notes
1) No books or any other document are allowed
2) A simple calculator with no programming and graphical capabilities can be used
3) Solve each problem using the space following it; if more space is needed use the back of any page or additional white pages after the last problem and indicate when doing so

Problem	Points	You
1	9	
2	7	
3	4	
4	6	
5	6	
6	6	
Total	38	

Problem 1 *Find the critical points of* $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ *and classify them.*

Solution

a) C.P.

$$\begin{cases} f_x = 6x^2 + y^2 + 10x = 0 \\ f_y = 2xy + 2y = 0 \end{cases}, \qquad \begin{cases} 6x^2 + 10x + y^2 = 0 \\ y(x+1) = 0 \end{cases}$$

2 $\underline{y=0} \Rightarrow 6x^2 + 10x = 0 \Rightarrow x = 0, \; x = -\frac{5}{3}$ $\Big\} \Rightarrow$

2 $\underline{x+1=0} \Rightarrow x = -1 \Rightarrow y^2 = -6 + 10 = 4, \; y = \pm 2$

C.P: $(0,0)$, $(-\frac{5}{3}, 0)$, $(-1,-2)$, $(-1,2)$

b) classification.

$f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$.

$D = (12x+10)(2x+2) - 4y^2$

	$(0,0)$	$(-\frac{5}{3}, 0)$	$(-1,-2)$	$(-1,2)$
f_{xx}	10	-10	-2	-2
D	20	$\frac{40}{3}$	-16	-16.
$f(x,y)$	loc. min	loc. max	S.P.	S.P.
	0	$\left(\frac{5}{3}\right)^3$		

c) Conclusion.

at $(0,0)$: local min $= 0$

at $(-\frac{5}{3}, 0)$: local max $= \left(\frac{5}{3}\right)^3$.

Problem 2 *Find the max/min values of the function* $f(x,y) = e^{-xy}$ *in the region* $R = \{(x,y),\ g(x,y) := x^2 + 4y^2 - 1 \le 0\}$.

Solution

1) Inside R:

a) $\begin{cases} f_x = -y e^{-xy} = 0 \\ f_y = -x e^{-xy} = 0 \end{cases} \Rightarrow x = 0,\ y = 0$

b) classification: $f_{xx} = y^2 e^{-xy}$, $f_{yy} = x^2 e^{-xy}$, $f_{xy} = (xy-1)e^{-xy}$

$D = x^2y^2 e^{-2xy} - (xy-1)^2 e^{-2xy}$

at $(0,0)$: $f_{xx} = 0$, $D = 0 - 1 = -1 < 0 \Rightarrow$ S.P.

2) On ∂R.

a) C.P. $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases}$, $\begin{cases} -y e^{-xy} = 2x\lambda \\ -x e^{-xy} = 8y \cdot \lambda \\ x^2 + 4y^2 = 1 \end{cases}$; $\Rightarrow x, y \neq 0 \Rightarrow \begin{cases} \dfrac{y}{x} = \dfrac{x}{4y} \\ x^2 + 4y^2 = 1 \end{cases} \Rightarrow$

$\begin{cases} x^2 = 4y^2 \\ x^2 + 4y^2 = 1 \end{cases} \Rightarrow 2x^2 = 1$, $x = \pm \dfrac{1}{\sqrt{2}}$, $y = \pm \dfrac{x}{2} = \pm \dfrac{1}{2\sqrt{2}} \Rightarrow$

CP: $\left(\pm \dfrac{1}{\sqrt{2}}, \pm \dfrac{1}{2\sqrt{2}}\right)$

b) classification

	$(0,0)$	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$
$f(x,y)$	S.P.	$e^{-\frac{1}{4}}$	$e^{\frac{1}{4}}$	$e^{\frac{1}{4}}$	$e^{-\frac{1}{4}}$
		min	max	max	min

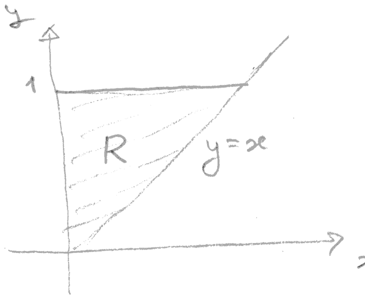
3) Conclusion

at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$: min $= e^{-\frac{1}{4}}$

at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$: max $= e^{\frac{1}{4}}$.

Problem 3 *Let R be the region defined by y axis and the lines $y = x$, $y = 1$. Evaluate the double integral* $I = \int\int(\cos(y^2) + e^y)dA$

Solution



$$I = \int_0^1 \int_0^y (\cos y^2 + e^y)\, dx\, dy$$

$$= \int_0^1 \left[y \cos y^2 + y e^y\right]_0^y dy$$

$$= \int_0^1 y \cos y^2\, dy + \int_0^1 y e^y\, dy$$

$$\begin{cases} t = y^2 \\ dt = 2y\, dy. \end{cases} \qquad \begin{cases} u = y \\ v' = e^y \end{cases}, \quad \begin{matrix} u' = 1 \\ v = e^y \end{matrix}$$

$$= \int_0^1 \frac{1}{2} \cos t\, dt + \left[y e^y\right]_0^1 - \int_0^1 e^y\, dy$$

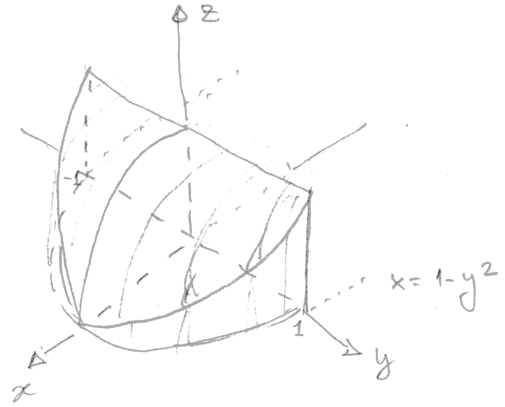
$$= \frac{1}{2}\left[\sin t\right]_0^1 + e - (e - 1)$$

$$= \frac{1}{2} \sin 1 + 1$$

Problem 4 *Let E be the solid under the surface $z = 1 - x^2$ and above the region D in xy plane. The region D is defined by the curves $x = 1 - y^2$ and $x = 0$.*
a) Sketch the solid E.
b) Evaluate the volume of E.

Solution

a)



b) $I = \int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{1-x^2} dz\, dx\, dy$

$= \int_{-1}^{1} \int_{0}^{1-y^2} (1 - x^2)\, dx\, dy$

$= \int_{-1}^{1} \left[x - \frac{1}{3} x^3 \right]_0^{1-y^2} dy = \int_{-1}^{1} \left((1-y^2) - \frac{1}{3}(1-y^2)^3 \right) dy$

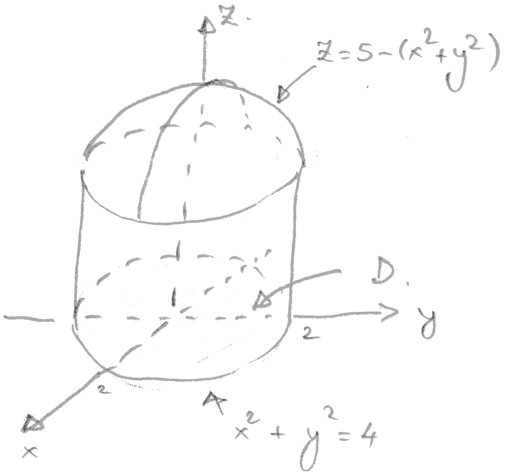
$= \int_{-1}^{1} \left(1 - y^2 - \frac{1}{3}\left(1 - 3y^2 + 3y^4 - y^6\right) \right) dy$

$= \left[y - \frac{1}{3}y^3 - \frac{1}{3}y + \frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{21}y^7 \right]_{-1}^{1}$

$= 2 - \frac{2}{3} - \frac{2}{3} + \frac{2}{3} - \frac{2}{5} + \frac{2}{21} = \frac{36}{35}$

Problem 5 *Using cylindrical coordinates evaluate* $I = \iiint_E 3x^2 dV$, *where* E *is the solid that lies within the cylinder* $x^2 + y^2 = 4$, *above the plane* $z = 0$ *and below the paraboloid* $z = 5 - (x^2 + y^2)$.

Solution



$$I = \iint_D \int_0^{5-(x^2+y^2)} 3x^2 \, dz \, dA$$

$$= \iint_D 3x^2 \cdot \left(5 - (x^2+y^2)\right) dA .$$

$$D = \{0 \le \rho \le 2,\ 0 \le \theta \le 2\pi\}$$

$$= \int_0^{2\pi} \int_0^2 \rho \cdot 3\rho^2 \cos^2\theta \, (5-\rho^2) \, d\rho \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^2 3\rho^3 (5-\rho^2) \, d\rho$$

$$= \int_0^{2\pi} \frac{1}{2}(\cos 2\theta + 1) \, d\theta \cdot 3\left[5 \cdot \frac{\rho^4}{4} - \frac{\rho^6}{6}\right]_0^2$$

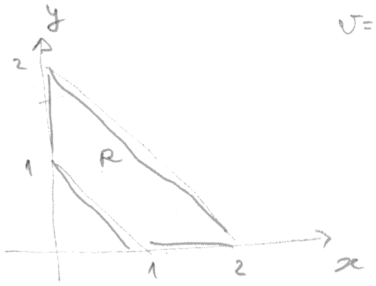
$$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{2\pi} \quad 3 \cdot \left(5 \cdot \frac{2^4}{4} - \frac{2^6}{6}\right)$$

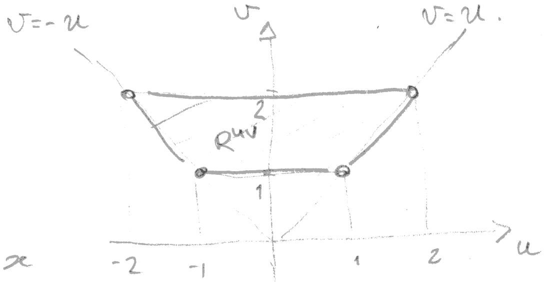
$$= \frac{1}{2}(0 + 2\pi) \cdot 3 \cdot \left(5 \cdot 4 - \frac{1}{3} \cdot 2^5\right)$$

$$= \pi \cdot 3 \cdot \left(20 - \frac{32}{3}\right) = 28 \cdot \pi .$$

Problem 6 *Let R be the quadrilateral region in xy plane defined by the points $(1,0)$, $(2,0)$, $(0,1)$, $(0,2)$. Using a transformation of variables (x,y) compute the integral $I = \iint_R \cos\frac{y-x}{y+x}dA$.*

Solution





Set $\begin{cases} u = y - x \\ v = y + x. \end{cases}$

$$\left.\begin{array}{l} (1,0) \to (-1,1) \\ (2,0) \to (-2,2) \\ (0,1) \to (1,1) \\ (0,2) \to (2,2) \end{array}\right\} \Rightarrow R^{uv} \text{ is a trapezoid.}$$

Solve (x,y): $\begin{cases} x = \frac{1}{2}(v-u) \\ y = \frac{1}{2}(u+v) \end{cases}$

Jacobian: $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

$$I = \iint_{R^{uv}} \cos\frac{u}{v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R^{uv}} \cos\frac{u}{v}\,\frac{1}{2}\,du\,dv$$

$R^{uv}: \{1 \le v \le 2,\quad -v \le u \le v\}$: type II

$$= \int_1^2 \int_{-v}^{v} \frac{1}{2}\cos\frac{u}{v}\,du\,dv = \frac{1}{2}\int_1^2 v\left[\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv =$$

$$= \frac{1}{2}\int_1^2 v\cdot\left(\sin 1 - \sin(-1)\right)dv = \frac{1}{2}\,2\sin 1 \cdot \int_1^2 v\,dv = \sin 1 \cdot \left[\frac{v^2}{2}\right]_1^2$$

$$= \frac{3}{2}\sin 1.$$