MAT 2322 ℓ
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April 18th, 2006

FINAL EXAM (winter 2006)

+ Solution

LAST NAME, First name:
Student number:

Notes
1) No books or any other documents are allowed.
2) A simple calculator with no programming and graphical capabilities can be used.
3) Solve each problem using the space following it; if more space is needed use the back of any page or additional white pages after the last problem and indicate when doing so.

Problem	Points	Your score
1		
2		
3		
4		
5		
6		
7		
8		
9		
Total	53	

Problem 1 (7 points) *Find the critical points of* $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$ *and classify them.*

Solution

1) C.P. $\begin{cases} f_x = 6xy - 6x = 0 \\ f_y = 3x^2 + 3y^2 - 6y = 0, \end{cases}$ $\begin{cases} x(y-1) = 0 \\ x^2 + y^2 - 2y = 0 \end{cases}$ $\Rightarrow$ $x = 0$ or $y = 1$ $\Downarrow$.

$\left.\begin{array}{l} \underline{x=0} \Rightarrow y^2 - 2y = 0, \quad y = 0, \; y = 2 \\ y = 1 \Rightarrow x^2 - 1 = 0, \quad x = \pm 1 \end{array}\right\} \Rightarrow \{(0,0), (0,2), (-1,1), (1,1)\}$

2) classification

$\begin{cases} f_{xx} = 6y - 6 = 6(y-1) \\ f_{yy} = 6y - 6 = 6(y-1) \\ f_{xy} = 6x, \end{cases}$ $D = 36(y-1)^2 - 36x^2 = 36\left((y-1)^2 - x^2\right)$

(x,y)	$(0,0)$	$(0,2)$	$(-1,1)$	$(1,1)$
D	36	36	-36	-36
f_{xx}	-6	6	0	0
Conclusion	loc. max	loc. min	S.P	S.P.

Problem 2 (6 points) *Find the extremal values of the function* $f(x,y) = x^2y^2$ *subject to the constraint* $g(x,y) := x^2 + 4y^2 - 24 = 0$.

Solution

I) inside D.

$\rightarrow$ C.P. $\begin{cases} f_x = 2xy^2 = 0 \\ f_y = 2x^2y = 0 \end{cases} \Rightarrow x = 0$ or $y = 0 \Rightarrow \{(0,y), (x,0)\}$

So all the point of x and y axes are C.P.

$\rightarrow$ classification.

$f_{xx} = 2y^2, \quad f_{yy} = 2x^2, \quad f_{xy} = 4xy$

$D = 4x^2y^2 - 16x^2y^2 = -12x^2y^2$

It follows $D = 0$ and so the test is not conclusive

II) on ∂D:

$\rightarrow$ C.P. $\begin{cases} 2xy^2 = \lambda \cdot 2x \\ 2x^2y = \lambda \cdot 8y. \\ x^2 + 4y^2 = 24 \end{cases}$

$\underline{x=0}$: $4y^2 = 24, \quad y = \pm\sqrt{6}, \quad \lambda = 0$

$\underline{y=0}$: $x^2 = 24, \quad x = \pm 2\sqrt{6}, \quad \lambda = 0$

$\underline{x,y \neq 0}$: $\begin{cases} y^2 = \lambda \\ x^2 = 4\lambda \\ x^2 + 4y^2 = 24 \end{cases}, \quad \begin{cases} 4y^2 = x^2 \\ x^2 + 4y^2 = 24 \end{cases}, \quad 2x^2 = 24, \quad \begin{cases} x = \pm 2\sqrt{3} \\ y = \pm\sqrt{\frac{1}{4}x^2} = \pm\sqrt{3} \end{cases}$

So, C.P.: $\{(0, \pm\sqrt{6}), (\pm 2\sqrt{6}, 0), (-2\sqrt{3}, \pm\sqrt{3}), (2\sqrt{3}, \pm\sqrt{3})\}$

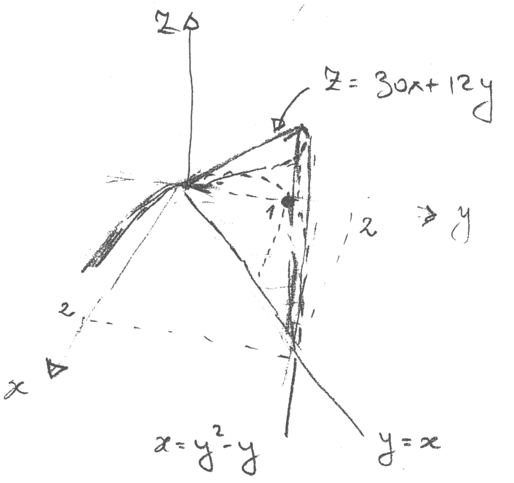
$\rightarrow$ classification

(x,y)	$(0, \pm\sqrt{6})$	$(\pm 2\sqrt{6}, 0)$	$(\pm 2\sqrt{3}, \pm\sqrt{3})$
$f(x,y)$	0	0	36
	min	min	max.

Problem 3 (7 points) *Let E be the solid under the surface $z = 30x + 12y$ and above the domain D, in the first quadrant, enclosed by the curves $y = x$ and $x = y^2 - y$.*
a) Sketch the domain D and the solid E.
b) Find the volume of E.

Solution

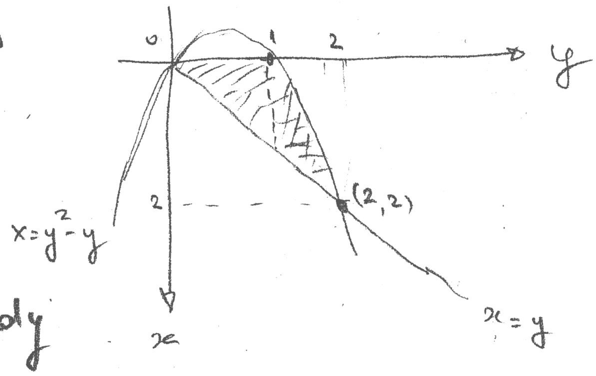
1) Sketch D and E



2 Intersection $\begin{cases} y = x \\ x = y^2 - y \end{cases}$, $y^2 - 2y = 0$, $\begin{cases} y = 0, & x = 0 \\ y = 2, & x = 2 \end{cases}$

1 b) $E = \{ (x, y, z),\ (x,y) \in D,\ 0 \le z \le 30x + 12y \}$

1 $D = D_1 \cup D_2$, $D_1 = \{ 0 \le y \le 1,\ 0 \le x \le y \}$
$D_2 = \{ 1 \le y \le 2,\ y^2 - y \le x \le y \}$



1 $$V = \iint_{D_1} \int_0^{30x+12y} dz \cdot dA + \iint_{D_2} \int_0^{30x+12y} dz\, dA$$

$$= \int_0^1 \int_0^y (30x + 12y)\, dx\, dy + \int_1^2 \int_{y^2-y}^{y} (30x + 12y)\, dx\, dy$$

$$= \int_0^1 (15y^2 + 12y^2)\, dy + \int_1^2 \left(15\left(y^2 - (y^2 - y)^2\right) + 12y\left(y - (y^2 - y)\right)\right) dy$$

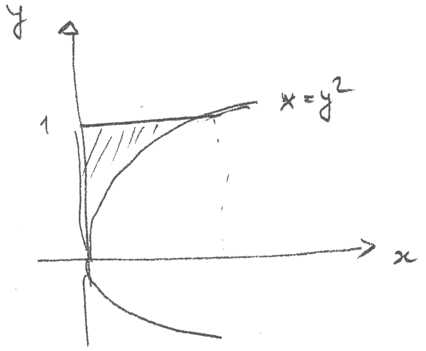
2. $$= \int_0^1 27y^2\, dy \quad + \int_0^1 (-15y^4 + 18y^3 + 24y^2)\, dy$$

$$= 9 \quad + \frac{61}{2}$$

$$= \frac{79}{2}.$$

Problem 4 (4 points) *A thin plate occupying the region R, enclosed by the lines $x = y^2$, $y = 1$ and $x = 0$, has density $\rho(x, y) = 3\cos(y^3) + \frac{2}{y}$. Find the mass of the plate.*

Solution



a) Sketch the region R.

b) Consider R as of type II

$R = \{(x,y),$ with $0 \le y \le 1, \ 0 \le x \le y^2\}$

$$m = \int_0^1 \int_0^{y^2} \left(3\cos(y^3) + \frac{2}{y}\right) dx\, dy$$

$$= \int_0^1 \left[3\cos(y^3) + \frac{2}{y}\right]_{x=0}^{x=y^2} dy = \int_0^1 \left(3y^2\cos(y^3) + 2y\right) dy$$

$$= \left[\sin(y^3) + y^2\right]_0^1$$

$$= \sin 1 + 1 - \sin 0 - 0 = 1 + \sin 1$$

Problem 5 (4 points) *A point moves along a space curve $\vec{r} = \vec{r}(t)$.*
a) Knowing that $\vec{a}(t) = \vec{k}$, $\vec{v}(0) = \vec{i} - \vec{j}$ and $\vec{r}(0) = \vec{0}$ find $\vec{r}(t)$.
b) Find the distance d that the point travels in 1 second.
c) Use one step of midpoint rule for the integral to give an approximated value of the distance d.

Solution

1 a) $\vec{v}'(t) = \vec{a}(t) = \vec{k}$,

$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(\tau)d\tau = \vec{i} - \vec{j} + \vec{k} \cdot t$

$\vec{r}'(t) = \vec{v}(t)$

1 $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau)d\tau = \int_0^t (\vec{i} - \vec{j} + \vec{k}\,\tau)\,d\tau$

$= t \cdot \vec{i} - t \cdot \vec{j} + \frac{t^2}{2}\vec{k}$

1 b). $d = \int_0^1 |\vec{v}(t)|dt = \int_0^1 \sqrt{1^2 + 1^2 + t^2}\,dt$

$d = \int_0^1 \sqrt{2 + t^2}\,dt.$

1 c) $d \simeq (1-0) \cdot \sqrt{2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$

Problem 6 (6 points) *Let $\vec{F} = e^y\vec{i} + xe^y\vec{j} + (z+1)e^z\vec{k}$ and Γ a curve starting at $(0,0,0)$ and ending at $(1,1,1)$ with parametric equations $\vec{r} = \vec{r}(t)$.*
a) Explain why $\vec{F}$ is a conservative vector field and find the potential f such that $\nabla f = \vec{F}$.
b) Compute $\int_\Gamma \vec{F} \cdot d\vec{r}$.

Solution

2

a) $\vec{F}$ is defined in $\mathbb{R}^3$, a simple, connected domain; $\vec{F}$ is C^∞; moreover

$\to$ curl $\vec{F}$ = det $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ e^y & xe^y & (z+1)e^z \end{bmatrix}$

$= 0 \cdot \vec{i} - 0 \cdot \vec{j} + (e^y - e^y)\vec{k} = \vec{0}$;

therefore, $\vec{F}$ is conservative.

3

$\to \nabla f = \vec{F} \Rightarrow \begin{cases} f_x = e^y, \\ f_y = xe^y, \\ f_z = (z+1)e^z, \end{cases}$ $\quad f = xe^y + g(y,z) \Rightarrow$

$\begin{cases} xe^y + g_y = xe^y, \\ f_z = (z+1)e^z \end{cases}$ $\quad g_y = 0 \Rightarrow g = h(z) \Rightarrow$

$f = xe^y + h(z)$; replace:

$h' = (z+1)e^z$,

$h = \int (ze^z + e^z)\, dz = ze^z. \Rightarrow$

$f(x,y,z) = xe^y + ze^z.$

1

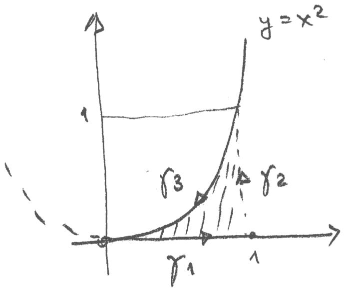
b) $\int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma \nabla f \cdot d\vec{r} =$

$= f(1,1,1) - f(0,0,0)$

$= (1e + 1 \cdot e) - (0 \cdot 1 + 0 \cdot 1) = 2e$

Problem 7 (5 points) *Let D be the domain enclosed by the parabola $y = x^2$ and the lines $x = 1$, $y = 0$. Let Γ be the boundary of D oriented counterclockwise.*
a) Sketch the domain D and the boundary Γ.
b) Verify Green's theorem in D, i.e. prove that $\int_\Gamma Pdx + Qdy = \iint_D (Q_x - P_y)dA$, where $P = x^2 + y^2$, $Q = 2xy$.

Solution



a) Sketch D, Γ

b)

b.1) $\iint_D (Q_x - P_y)\, dA =$

$D = \{0 \le x \le 1,\ 0 \le y \le x^2\}$

1 $= \int_0^1 \int_0^{x^2} (2y - 2y)\, dy\, dx = 0$

b.2) $\int_\Gamma Pdx + Qdy = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$

1 $\to \int_{\gamma_1} Pdx + Qdy = \int_0^1 (x^2 + 0^2)\, dx + \cancel{2\cdot x \cdot 0 \cdot d0} = \int_0^1 x^2\, dx = \frac{1}{3}$

$\gamma_1 = \{(x,0),\ 0 \le x \le 1\}$

1 $\to \int_{\gamma_2} Pdx + Qdy = \int_0^1 \cancel{(1^2 + y^2)\, d1} + 2\cdot 1 \cdot y\, dy = \int_0^1 2y\, dy = 1$

$\gamma_2 = \{(1,y),\ 0 \le y \le 1\}$

$\to \int_{\gamma_3} Pdx + Qdy = -\int_{-\gamma_3} Pdx + Qdy = -\int_0^1 \left(x^2 + (x^2)^2\right) dx + 2\cdot x \cdot x^2\, dx^2$

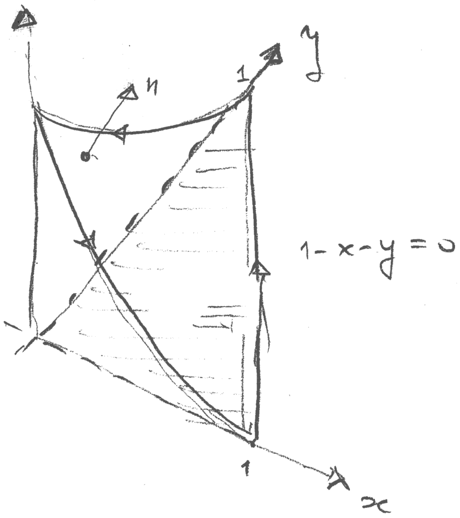
$-\gamma_3 = \{(x, x^2),\ 0 \le x \le 1\}$

2 $= -\int_0^1 (x^2 + x^4 + 4x^4)\, dx = -\left(\frac{1}{3} + \frac{1}{5} + \frac{4}{5}\right) = -\left(\frac{1}{3} + 1\right)$

Thus $\int_\gamma Pdx + Qdy = \frac{1}{3} + 1 - \left(\frac{1}{3} + 1\right) = 0$ ✓

Problem 8 (7 points) *Let S be the part of the surface $z = (1-x-y)^2$ in the first octant, with normal vector $\vec{n}$ oriented upward. Let Γ be the boundary of S and $\vec{r} = \vec{r}(t)$ be the parametric equations of Γ.*
a) Sketch the surface S and the boundary Γ.
b) Use Stokes theorem to compute $\int_\Gamma \vec{F}\cdot d\vec{r}$, where $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$.

Solution



1 a) use method of traces to sketch S

b) $$\int_\Gamma \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F}\cdot\vec{n})\, dS$$

1 S is above the region D in xy plane

$$D = \{0 \le x \le 1,\ 0 \le y \le 1-x\}.$$

1 S is given by $z = g(x,y) = (1-x-y)^2$, so

$$\begin{cases} g_x = -2(1-x-y) \\ g_y = -2(1-x-y) \end{cases}$$

1 Also, we have $$\text{curl } \vec{F} = U\vec{i} + V\vec{j} + W\vec{k} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{bmatrix}$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

Therefore

1 $$\int_\Gamma \vec{F}\cdot d\vec{r} = \iint_D (-Ug_x - Vg_y + W)\, dA$$

1 $$= \int_0^1\int_0^{1-x} (-2(1-x-y) - 2(1-x-y) - 1)\, dy\, dx.$$

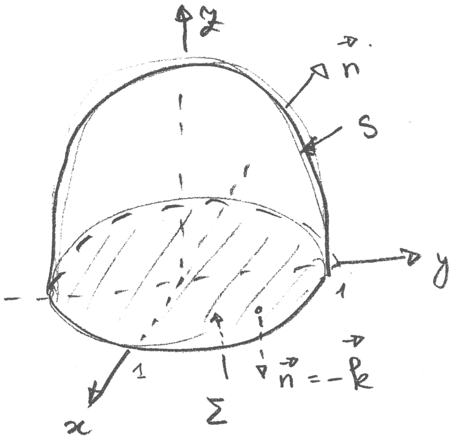
$$= -\int_0^1\int_0^{1-x} (5 - 4x - 4y)\, dy\, dx$$

1 $$= -\int_0^1 (3 - 5x + 2x^2)\, dy$$

$$= -\frac{7}{6}.$$

Problem 9 (7 points) *Let S the part of the surface $z = 1 - (x^2 + y^2)$ above the plane $z = 0$.*
a) Sketch the surface S
b) Use divergence theorem to compute $\iint_S (\vec{F} \cdot \vec{n}) dS$, where $\vec{F} = xz\vec{i} + yz\vec{j} + 3z^2\vec{k}$ and $\vec{n}$ is the upward normal vector to S.

Solution



a) use method of traces.
1 the intersection of the paraboloid with (x,y) plane is a circle with radius 1.

1 b) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \cdot dV - \iint_\Sigma \vec{F} \cdot d\vec{S}$.

1 $\rightarrow$ $E = \{(x,y,z),\ (x,y) \in D =$ circle with radius 1, $0 \le z \le 1 - (x^2+y^2)\}$

$\operatorname{div} \vec{F} = z + z + 6z = 8z$

1 $\iiint_E \operatorname{div}\vec{F} dV = \iint_D \int_0^{1-(x^2+y^2)} 8z \cdot dz\, dA = \iint_D 4 \left[z^2\right]_{z=0}^{z=1-(x^2+y^2)} dA = \iint_D 4\left(1 - (x^2+y^2)\right)^2 dA$

[pass in polar coordinates]

$= \int_0^{2\pi} \int_0^1 r \cdot 4(1-r^2)^2\, dr\, d\theta$

2 $= 2\pi \int_0^1 4r(1 - 2r^2 + r^4)\, dr = 2\pi \left[2r^2 - 2r^4 + \frac{4}{6} r^6\right]_0^1$

$= 2\pi \cdot \left(2 - 2 + \frac{2}{3}\right) = \frac{4}{3}\pi$

1 $\rightarrow$ $\iint_\Sigma \vec{F} \cdot d\vec{S} = \iint_\Sigma 3z^2(-1)\, dS = \iint_\Sigma 0 \cdot dS = 0$;

Conclusion:

$\iint_S \vec{F} \cdot d\vec{S} = \frac{4}{3}\pi$.