MAT 2322 April 21st, 2005

Professor: Dr. Arian Novruzi

FINAL EXAM (winter 2005)

(with Solution)

LAST NAME, First name:

Student number:

Notes

1) No books or any other documents are allowed.
2) A simple calculator with no programming and graphical capabilities can be used.
3) Solve each problem using the space following it; if more space is needed use the back of any page or additional white pages after the last problem and indicate when doing so.

Problem	Points	Your score
1	4	
2	5	
3	3	
4	4	
5	2	
6	4	
7	5	
8	5	
9	5	
Total	37	

Problem 1 *Find the critical points of $f(x, y) = x^3 + y^3 - 3xy + 1$ and classify them.*

Solution

- $\nabla f = 0$, $\begin{cases} f_x = 3x^2 - 3y, \\ f_y = 3y^2 - 3x \end{cases}$ $\begin{cases} x^2 = y \\ x = y^2 \end{cases}$ $\Rightarrow$ $x^4 = x$ $\Rightarrow$ $\boxed{\begin{matrix} x = 1 \\ y = 1 \end{matrix}}$, $\boxed{\begin{matrix} x = 0 \\ y = 0 \end{matrix}}$ (thus, $x, y \geq 0$). 2

- $\left.\begin{matrix} f_{xx} = 6x \\ f_{yy} = 6y \\ f_{xy} = -3 \end{matrix}\right\}$ $D = 36xy - 9$. 1

-

(x,y)	$(0,0)$	$(1,1)$
D	-9	27.
f_{xx}	-9	6.
	S. P.	min

1

Problem 2 *Find the max/min values of* $f(x,y) = xy^2$ *subject to the constraint* $g(x,y) := 2x^2 + y^2 - 6 = 0$.

Solution

- $\nabla f = \lambda \nabla g$

$$\begin{cases} y^2 = \lambda 4x \\ 2xy = \lambda 2y \\ 2x^2 + y^2 = 6 \end{cases}$$ 1

- $y \neq 0 \Rightarrow \begin{cases} y^2 = \lambda 4x \\ x = \lambda \\ 2x^2 + y^2 = 6 \end{cases} \Rightarrow \begin{cases} y^2 = 4x^2 \\ 2x^2 + y^2 = 6 \end{cases} \Rightarrow x^2 = 1$

Thus $x = \pm 1 \Rightarrow \lambda = \pm 1, \quad y^2 = 4, \quad y = \pm 2$ 2

$\boxed{(-1, \pm 2), \quad (1, \pm 2)}$

- $y = 0 \Rightarrow x^2 = 3, \quad \begin{cases} x = \pm\sqrt{3} \\ y = 0 \end{cases}, \quad \lambda = 0$ 1

$\boxed{(\pm\sqrt{3}, 0)}$

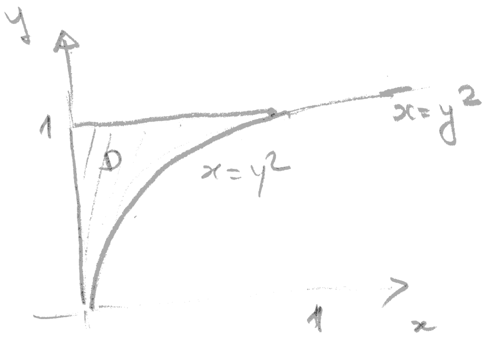
- min/max

(x,y)	$(\sqrt{3}, 0)$	$(-\sqrt{3}, 0)$	$(1, -2)$	$(1, 2)$	$(-1, -2)$	$(-1, 2)$
f	0	0	4	4	-4	-4
			Max	Max	min	min

1

Problem 3 *Find the mass of the plate with density $\rho(x, y) = 3\cos(y^3)$ occuppying the area enclosed by the curve $x = y^2$, y axis and $y = 1$.*

Solution



$$m = \iint_D \rho \, dA$$

$$D = \{0 \le y \le 1,\ 0 \le x \le y^2\}$$

1

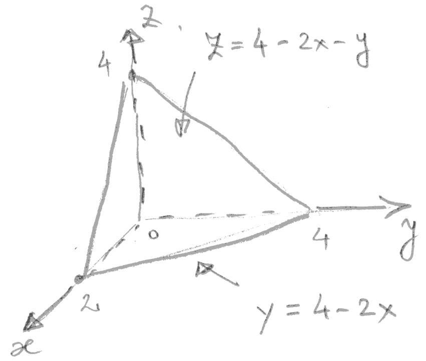
$$= 3\int_0^1 dy \int_0^{y^2} dx \cos(y^3) = 3\int_0^1 dy\, y^2 \cos(y^3)$$

$$= \left[\sin(y^3)\right]_0^1 = \sin 1.$$

2.

Problem 4 *Find the mass of the solid with density $\rho(x, y, z) = 2x$, which occupies the region in the 1st octant enclosed by the plane $2x + y + z = 4$.*

Solution



$m = \iiint_E \rho \, dV.$

E of 1^{st} type: $D = \{0 \le x \le 2, \quad 0 \le y \le 4-2x\}$.

$z_1 = 0, \; z_2 = 4 - 2x - y$ 1

$$m = \int_0^2 dx \int_0^{4-2x} dy \int_0^{4-2x-y} dz \; 2x$$ 1

$$= \int_0^2 dx \int_0^{4-2x} dy \cdot 2x(4-2x-y)$$

$$= \int_0^2 dx \int_0^{4-2x} dy \,(8x - 4x^2 - 2xy) = \int_0^1 dx \left[(8x-4x^2)y - xy^2\right]_{y=0}^{y=4-2x}$$

$$= \int_0^2 dx \left((8x-4x^2)(4-2x) - x\cdot(4-2x)^2\right)$$

$$= \int_0^2 dx \cdot \left(32x - 16x^2 - 16x^2 + 8x^3 - 16x + 16x^2 - 4x^3\right)$$ 2

$$= \int_0^2 dx \,(16x - 16x^2 + 4x^3) =$$

$$= \left[8x^2 - \frac{16}{3}x^3 + x^4\right]_0^2 = 32 - \frac{16}{3}\cdot 8 + 16$$

$$= 48 - \frac{128}{3} = \frac{144-128}{3} = \frac{16}{3}$$

Problem 5 *A point moves along a curve* $\vec{r} = \vec{r}(t)$. *It is known that the acceleration is* $\vec{a}(t) = \sqrt{2}\vec{j} + 2t\vec{k}$ *and also the initial velocity and position* $\vec{v}(0) = \vec{i}$, $\vec{r}(0) = \vec{0}$.
a) Find the equations of the curve $\vec{r} = \vec{r}(t)$.
b) What distance travels the point in 3 seconds?

Solution

a) $\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(\tau)\,d\tau = \vec{i} + \sqrt{2}\,t\cdot\vec{j} + t^2\vec{k}$

$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau)\,d\tau = t\vec{i} + \frac{t^2}{\sqrt{2}}\vec{j} + \frac{t^3}{3}\vec{k}$

1

b) $S(3) = \int_0^3 |\vec{v}(t)|\,dt = \int_0^3 (1 + 2t^2 + t^4)^{1/2}\,dt$

$= \int_0^3 (1+t^2)^{2 \cdot 1/2}\,dt = \int_0^3 (1+t^2)\,dt =$

$= \left[t + \frac{1}{3}t^3\right]_0^3 = 3 + \frac{1}{3}\cdot 3\cdot 3^2$

$= 3 + 9 = 12.$

1.

Problem 6 *Let* $\vec{F} = yz\vec{i} + xz\vec{j} + (xy+2z)\vec{k}$ *and* Γ *be the curve joining the points* $(1,0,-2)$, $(4,6,3)$.
a) Find $f(x,y,z)$ *such that* $\vec{F} = \nabla f$.
b) Compute $\int_\Gamma \vec{F}\cdot d\vec{r}$, *where* $\vec{r}$ *gives the parametric equations of* Γ.

Solution

a) $\begin{cases} f_x = yz, \\ f_y = xz, \\ f_z = xy+2z \end{cases}$ $\quad f = xyz + g(y,z).$

$xz = xz + \partial_y g \Rightarrow \partial_y g = 0 \Rightarrow g = h(z) \Rightarrow$

$xy + 2z = xy + h' \Rightarrow h' = 2z \Rightarrow h = z^2 + C \Rightarrow$

$f(x,y,z) = xyz + z^2 + C$

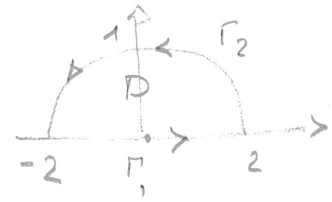
3

b) $\int_C \vec{F}\cdot d\vec{r} = f(4,6,3) - f(1,0,-2)$

$= 77$

1

Problem 7 *Let D be the upper half part of the disk $x^2+y^2=4$ and Γ be its boundary. Consider $P(x,y)=xy$, $Q(x,y)=-x^2+x$. Verify directly the Green's theorem (thus, compute $\oint_\Gamma Pdx+Qdy$ and $\int_D(Q_x-P_y)dA$, and compare them).*



Solution

a) $\iint_D (Q_x - P_y)\, dA = \iint_D (-2x+1-x)\, dA$

$= \iint_D (-3x+1)\, dA = \int_0^2 dr \int_0^\pi d\theta \; r(-3r\cos\theta+1)$

$= \int_0^2 dr \cdot r\left[-3r\sin\theta+\theta\right]_0^\pi = \left[\frac{r^2}{2}\right]_0^2 (-3r\cdot 0+\pi) = 2\cdot\pi$

1

2

b). $\oint_\Gamma P dx + Q dy = \oint_\Gamma \left(xy\, dx + (-x^2+x)\, dy\right) = \int_{\Gamma_1} + \int_{\Gamma_2}$.

- $\Gamma_1: \begin{cases} x = x \\ y = 0 \end{cases}$ $\quad -2 \le x \le 2.$

$\int_{\Gamma_1} = \int_{-2}^{2} \left(x\cdot 0\cdot dx + (-x^2+x)\cdot 0\right) = 0.$

0.5

- $\Gamma_2: \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta, \end{cases}$ $\quad 0 \le \theta \le \pi$

$\int_{\Gamma_1} = \int_0^\pi (2\cdot\cos\theta\cdot 2\sin\theta\cdot 2\cdot(-\sin\theta))\,d\theta + \left(-2^2\cos^2\theta + 2\cos\theta\right) 2\cos\theta\, d\theta$

$= \int_0^\pi \left(-8\sin^2\theta\cdot\cos\theta - 8\cos^3\theta + 4\cos^2\theta\right) d\theta$

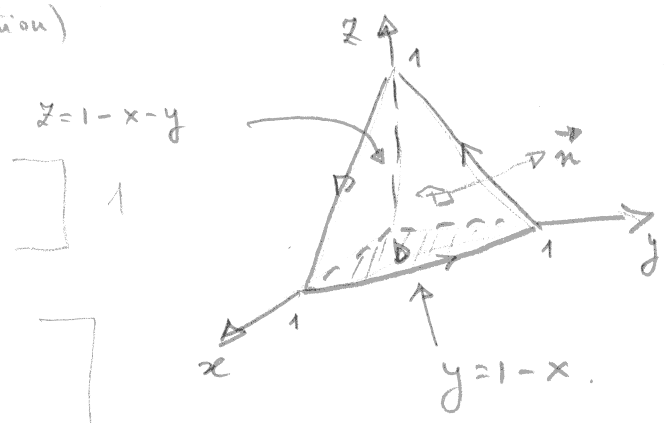
$= \int_0^\pi \left(-8\cos\theta + 4\cos^2\theta\right) d\theta = \int_0^\pi \left(-8\cos\theta + 4\cos 2\theta + 2\right) d\theta$

$= \left[-8\sin\theta + 2\sin 2\theta + 2\theta\right]_0^\pi = 2\pi$

1.5

Problem 8 *Consider the vector field $\vec{F} = (x + y^2)\vec{i} + (y - z^2)\vec{j} + (z - x^2)\vec{k}$ and let Γ be the closed space curve joining the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Using Stokes theorem compute $\oint_\Gamma \vec{F} \cdot d\vec{r}$, the work of the vector field $\vec{F}$ on Γ.* (on Γ, anticlockwise orientation)

Solution



Stokes:

$$W = \oint_\Gamma \vec{F} \cdot d\vec{r} = \iint_S (\text{rot}\,\vec{F} \cdot \vec{n})\, dS.$$ 1

But:

$$S: \{(x, y, g(x,y)),\ g(x,y) = z = 1 - x - y,\ (x,y) \in D\}$$

$$D: \{0 \le x \le 1,\ 0 \le y \le 1 - x\}$$

$$g_x = -1, \quad g_y = -1.$$ 1.5

$$\text{rot}\,\vec{F} = \underbrace{2z}_{U}\,\vec{i} + \underbrace{2x}_{V}\,\vec{j} \underbrace{- 2y}_{W}\,\vec{k}$$ 1

Then

$$W = \iint_D (-U g_x - U g_y + W)\, dA.$$

$$= \iint_D (2z + 2x - 2y)\, dA = 2\iint_D (\underbrace{1 - x - y}_{z} + x - y)\, dA$$

$$= 2\int_0^1 dx \int_0^{1-x} dy\, (1 - 2y) = 2\int_0^1 dx\, \{y - y^2\}_{y=0}^{y=1-x}$$

$$= 2\int_0^1 dx \left((1-x) - (1-x)^2\right)$$

$$t = 1 - x, \quad dt = -dx$$

$$= 2\int_1^0 (t - t^2)(-dt) = 2\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1$$

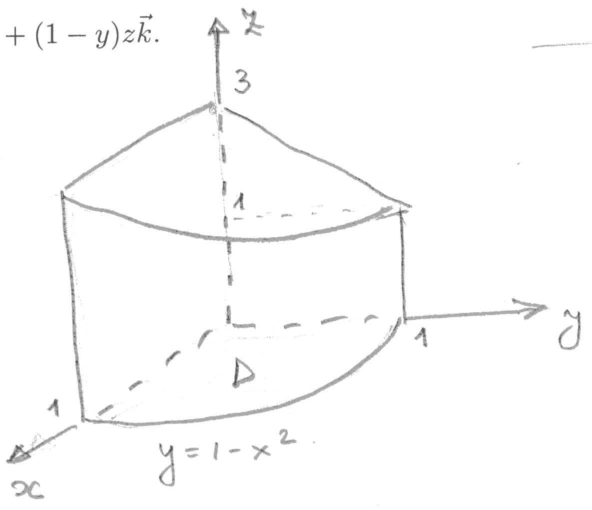
$$= \frac{1}{3}$$ 1.5

Problem 9 *Let E be the region in the 1st octant enclosed by the surfaces $y = 1 - x^2$, $z = 3 - 2y$. Moreover, let Γ be the boundary of E and $\vec{n}$ the normal vector to Γ pointing outside E.*
a) Sketch the solid E.
b) Using divergence theorem compute $\iint_\Gamma (\vec{F} \cdot \vec{n}) dS$, the flux of the vector field $\vec{F}$ over Γ, where

$$\vec{F} = (\ln(yz) + xy)\vec{i} + (e^{xz} + (x-1)y)\vec{j} + (1-y)z\vec{k}.$$

Solution

a)



1

b) $\phi = \iint_\Gamma (\vec{F} \cdot \vec{n})\, ds = \iiint_E \operatorname{div} \vec{F}\, dV$

$E : (x,y,z) : (x,y) \in D = \{ 0 \le x \le 1,\ 0 \le y \le 1 - x^2 \}$
$0 \le z \le 3 - 2y$ 1

$\operatorname{div} \vec{F} = \cancel{y} + x \cancel{- 1} \cancel{+ 1} \cancel{- y} = x$ 1

$\phi = \iint_D dA \int_0^{3-2y} dz\, x = \iint_D x(3-2y)\, dA$

$= \int_0^1 dx \int_0^{1-x^2} dy\, (3x - 2xy) = \int_0^1 dx\, [3xy - xy^2]_0^{1-x^2}$

$= \int_0^1 dx \cdot \left(3x(1-x^2) - x \cdot (1-x^2)^2\right)$ 2

$= \int_0^1 (3x - 3x^3 - x + 2x^3 - x^5)\, dx = \int_0^1 (2x - x^3 - x^5)$

$= \left[x^2 - \frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 1 - \frac{1}{4} - \frac{1}{6} = \frac{12 - 3 - 2}{12} = \frac{7}{12}$