MAT 2322 **March 1st, 2005**
Professor: Dr. Arian Novruzi

Midterm exam (winter 2005)
SOLUTION

LAST NAME, First name:
Student number:

Notes
1) No books or any other document are allowed
2) A simple calculator with no programming and graphical capabilities can be used
3) Solve each problem using the space following it; if more space is needed use the back of any page or additional white pages after the last problem and indicate when doing so

Problem	Points	You
1	5	
2	5	
3	3	
4	4	
5	5	
6	5	
Total	27	

Problem 1 *[5] Find the critical points of* $f(x, y) = x^2 + 2y^2 - x^2y + 4$ *and classify them.*

Solution.
a) CP:
$\nabla f(x, y) = (2x - 2xy, 4y - x^2)$. Then solve

$$\begin{cases} 2x(1-y) &= 0, \\ x^2 &= 4y, \end{cases} \quad \begin{cases} (x, y) &= (0, 0), \\ (x, y) &= (-2, 1), \\ (x, y) &= (2, 1). \end{cases}$$

b) Classification

$$f_{xx}(x, y) = 2(1-y), \quad f_{yy}(x, y) = 4, \quad f_{xy}(x, y) = -2x, \quad D = 8(1-y) - 4x^2.$$

It follows

(x, y)	$(0, 0)$	$(-2, 1)$	$(2, 1)$
D	8	-16	-16
f_{xx}	2	0	0
	loc. min	*SP*	*SP*

Problem 2 *[5] Find the max/min values of* $f(x, y) = x^2y$ *subject to the constraint* $g(x, y) := x^2 + 2y^2 - 6 = 0$.

Solution
a) $\nabla f(x, y) = (2xy, x^2)$, $\nabla g(x, y) = (2x, 4y)$. Then

$$\begin{cases} 2xy &= \lambda 2x, \\ x^2 &= \lambda 4y, \\ x^2 + 2y^2 &= 6, \end{cases}$$

If $x = 0$, then $y = \pm\sqrt{3}$, $\lambda = 0$.
If $x \neq 0$, dividing by $2x$ the first equation gives $\lambda = y$; the second equation will give $x^2 = 4y^2$ and the third one yields $y^2 = 1$, or $y = \pm 1$. It follows that $x^2 = 4$, so $x = \pm 2$ (and $\lambda = \pm 1$).

Thus the solutions are

(x, y)	$(0, \pm\sqrt{3})$	$(2, \pm 1)$	$(-2. \pm 1)$
λ	0	± 1	± 1

b) The circle $x^2 + 2y^2 = 6$ is colsed, bounded. So, we need just to compute $f(x, y)$ on the found points and find the max/min.

(x, y)	$(0, -\sqrt{3}$	$(0, \sqrt{3})$	$(2, -1)$	$(2, 1)$	$(-2, -1)$	$(-2, 1)$
$f(x, y)$	0	0	-4	4	-4	4
			min	*max*	*min*	*max*

Problem 3 *[3] Let* $R = [-1, 3] \times [0, 2]$*. Find an approximation of* $I = \iint_R (y^2 - 2x^2) dA$ *using a Riemann sum with* $m = 4$ *(number of divisions on* x *direction) and* $n = 2$ *(number of divisions on* y *direction) and by taking for* (x_i^*, y_j^*) *the upper-right corner.*

Solution
As $m = 4$, $n = 2$ we get

$$x_i = i, \quad i = -1, 0, 1, 2, 3, \quad y_j = j, \quad j = 0, 1, 2; \qquad \Delta x = 1, \quad \Delta y = 1, \quad \Delta A = 1.$$

Also $(x_i^*, y_j^*) = (i, j)$; then, the approximation is as follows

$$I \approx \sum_{i=0}^{i=3} \sum_{j=1}^{j=2} (j^2 - 2i^2) = -36$$

Problem 4 *[4] Evaluate the double integral* $I = \iint_D e^{y^2}\, dA$, *where* D *is defined by*

$$D = \{(x, y),\ \ 0 \le x \le 1,\ x \le y \le 1\}.$$

Solution.
D is of type I and II. Using type I does not allow easy integration.
Consider D as of type II:

$$D = \{(x, y),\ \ 0 \le y \le 1,\ 0 \le x \le y\}.$$

Then

$$\begin{aligned} I &= \int_0^1 \int_0^y e^{y^2}\, dx dy = \int_0^1 y e^{y^2}\, dy \\ &= \int_0^1 \frac{1}{2} d(e^{y^2}) = \frac{1}{2}(e-1). \end{aligned}$$

Problem 5 *[5] Using a triple integral find the volume of the solid bounded by the (inside of) cylinder* $x = y^2$ *and the planes* $z = 0$ *and* $x + z = 1$.

Solution.
a) The plane $x + z = 1$ intersects the xy plane along the line $x = 1$ (take $z = 0$). Then, the line $x = 1$ intersects the curve $x = y^2$ on points $(1, -1)$, $(1, 1)$.
It follows that the solid is above the domain D defined by $x = y^2$ and $x = 1$, above $z = 0$ and below the plane $z = 1 - x$.
b) We can compute the integral by considering D as of type II:

$$D = \{(x, y),\ -1 \leq y \leq 1,\ y^2 \leq x \leq 1\}.$$

It follows

$$\begin{aligned} I &= \int_{-1}^{1}\int_{y^2}^{1}(1-x)dxdy = \int_{-1}^{1}\left[x - \frac{1}{2}x^2\right]_{x=y^2}^{x=1} dy = \int_{-1}^{1}(\frac{1}{2} - y^2 + \frac{1}{2}y^4)dy \\ &= \frac{1}{2}(1-(-1)) - \frac{1}{3}(1-(-1)) + \frac{1}{10}(1-(-1)) \\ &= \frac{8}{15}. \end{aligned}$$

Problem 6 *[5] Using cylindrical coordinates evaluate* $I = \iiint_E x^2 dV$*, where* E *is the solid that lies within the cylinder* $x^2 + y^2 = 1$*, above the plane* $z = 0$ *and below the cone* $z^2 = 4(x^2 + y^2)$*.*

Solution.
Let D denote the base (in xy plane) of the cylinder. So, the domain E is of type I and

$$(x, y) \in D; \quad u_1(x, y) = 0, \quad u_2(x, y) = 2\sqrt{x^2 + y^2}.$$

Of course, in polar coordinates

$$D = \{(r, \theta), \quad 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}.$$

Then, we have

$$\begin{aligned}
I &= \iint_D \int_0^{2\sqrt{x^2+y^2}} x^2 dz dA = \iint_D x^2 \ 2\sqrt{x^2 + y^2} dA \\
&= \int_0^{2\pi} \int_0^1 r \ (r\cos\theta)^2 \ 2r \ dr d\theta \\
&= \int_0^{2\pi} \int_0^1 2r^4 \ (\cos\theta)^2 \ dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(\cos(2\theta) + 1) d\theta \int_0^1 2r^4 dr \\
&= \frac{1}{2} \left[\frac{1}{2}\sin(2\theta) + \theta\right]_0^{2\pi} \frac{2}{5} \left[r^5\right]_0^1 \\
&= \frac{2}{5}\pi.
\end{aligned}$$