Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT1300: Mathematical methods I
Fall 2019
Midterm exam #1

Professor: Arian Novruzi
Department of Mathematics and Statistics
University of Ottawa
email: novruzi@uottawa.ca

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LAST name____________________ First name:________________

Student number:________________

Instructions:

- The duration of this exam is 80 minutes.
- **NO CALCULATORS. NO BOOKS. NO NOTES.**
- This midterm exam consists of 5 multiple choice questions and 3 long answer questions. The multiple choice questions are worth 5 points each, and the long answer questions are as indicated. The total value of the exam is 50 points.
- For the **multiple choice** questions: write your answer (letter 'A' to 'E') in the table below.
- For the **long answer questions**: write clearly the solution in the space following the problem. You may use the back of any other page if necessary, but you have to clearly indicate the page number where your solution continues.
- Don't detach this examem.
- **NB**
  Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.
  By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature:________________

Answers:

	1	2	3	4	5	6	7	8	Total
Problem	multiple choice (write a letter A-F)					long solution (don't write anything here)			
Votre résultat	F	B	B	E	C				

Multiple Choice Section Questions (1-5)

Question 1 *Solve the following logarithmic equation.*

$$\ln(x+1) - \ln(2x-3) = \ln 3.$$

A) 11 **B)** $\frac{5}{2}$ **C)** $\frac{15}{4}$ **D)** 5 **E)** $\frac{e}{e+5}$ $\boxed{\textbf{F)}\ 2}$

$\rightarrow x+1>0$ and $2x-3>0$, so $x > \frac{3}{2}$

$\rightarrow \ln(x+1) = \ln 3 + \ln(2x-3)$

$\ln(x+1) = \ln(3\cdot(2x-3))$

$x+1 = 3\cdot(2x-3)$

$x+1 = 6x-9$

$5x = 10$

$x = 2$

Question 2 *Let* $f(x) = \frac{8}{\sqrt{5x-6}}$. *Find the equation of the tangent line to the graph determined by the above equation at* $x = 2$.

A) $y = -3x + 2$

$\boxed{\textbf{B)}\ y = -\frac{5}{2}x + 9}$

C) $y = 2x - 8$

D) $y = -\frac{3}{2}x + \frac{1}{\sqrt{2}}$

E) $y = -\frac{3}{2}x + 7$

F) $y = \frac{3}{4}x + 4$

$\rightarrow x_0 = 2, \quad f(x_0) = \frac{8}{\sqrt{5\cdot2-6}} = \frac{8}{\sqrt{4}} = 4.$

$\rightarrow y = f'(x_0)(x-x_0) + f(x_0)$

$\rightarrow f'(x) = \left(8\cdot(5x-6)^{-\frac{1}{2}}\right)' = 8.\left(-\frac{1}{2}\right)(5x-6)^{-\frac{3}{2}}).5$

$\rightarrow f'(2) = -\cancel{8}\frac{1}{2}\cdot\frac{1}{\cancel{4^{3/2}}}\cdot 5 = -\frac{5}{2}$

$\rightarrow y = \left(-\frac{5}{2}\right)(x-2) + 4$

$y = -\frac{5}{2}x + 5 + 4$

$y = -\frac{5}{2}x + 9$

Question 3 *Find a and b so that the function f given below is continuous everywhere.*

$$f(x) = \begin{cases} -ax + 10 & x < -2 \\ 6 & x = -2 \\ bx^2 + x & x > -2 \end{cases}$$

A) $\begin{cases} a = 1 \\ b = 2 \end{cases}$ B) $\boxed{\begin{cases} a = -2 \\ b = 2 \end{cases}}$ C) $\begin{cases} a = -2 \\ b = -2 \end{cases}$ D) $\begin{cases} a = 2 \\ b = 2 \end{cases}$ E) $\begin{cases} a = -2 \\ b = 3 \end{cases}$ F) $\begin{cases} a = 2 \\ b = -2 \end{cases}$

$$\begin{cases} \lim_{x \to -2^-} f(x) = f(-2) \\ \lim_{x \to -2^+} f(x) = f(-2) \end{cases} \iff \begin{cases} \lim_{x \to -2^-} (ax + 10) = 6 \\ \lim_{x \to -2^+} (bx^2 + x) = 6 \end{cases} \iff$$

$$\begin{cases} 2a + 10 = 6 \\ 4b - 2 = 6 \end{cases} \iff \begin{cases} a = -2 \\ b = 2 \end{cases}$$

Question 4 *Find the limit (this is ONLY a one-sided limit)*

$$\lim_{x \to 3^+} \frac{|x - 3|}{x^2 - 9}.$$

A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 0 D) $\frac{1}{4}$ $\boxed{\text{E) } \frac{1}{6}}$ F) The limit does not exist.

$\to \frac{0}{0}$

$$\to \lim_{x \to 3^+} \frac{|x-3|}{x^2 - 9} = \lim_{x \to 3^+} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$$

$$= \lim_{x \to 3^+} \frac{1}{x+3}$$

$$= \frac{1}{6}$$

Question 5 *For which value(s) x, the graph of the following function has horizontal tangent lines? (i.e. $f'(x) = 0$)*

$$f(x) = xe^{-3x+2}$$

A) $\begin{cases} x = \frac{3}{2} \\ x = 0 \end{cases}$ B) $\begin{cases} x = \frac{2}{3} \\ x = 0 \end{cases}$ $\boxed{\text{C) } x = \frac{1}{3}}$ D) $\begin{cases} x = -\frac{1}{2} \\ x = 0 \end{cases}$ E) $x = \frac{1}{2}$ F) $\begin{cases} x = \frac{1}{3} \\ x = 0 \end{cases}$

$\to$ $f'(x) = 0$

$x' \cdot e^{-3x+2} + x \cdot e^{-3x+2} \cdot (-3) = 0$

$e^{-3x+2} \cdot (1 - 3x) = 0$

$1 - 3x = 0$

$x = \frac{1}{3}$

Long Answer Questions (6-8)

Question 6 (8 points) *Using* **only** *<u>the definition of derivative</u> as a limit, calculate $f'(x)$ where*

$$f(x) = \sqrt{x-2}.$$

$\rightarrow$ $x \geq 2$

$\rightarrow$ for $x > 2$:

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(f(x+h) - f(x)\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left(\sqrt{x+h-2} - \sqrt{x-2}\right) \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}}$$

[use $(a-b)\cdot(a+b) = a^2 - b^2$]

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(\cancel{x}+h\cancel{-2}) - (\cancel{x}\cancel{-2})}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \frac{1}{2\sqrt{x-2}}.$$

Question 7 (8 points) *A deposit A is invested at a annual rate of* 4 %.

i) *Assuming the deposit is \$2,000 and the interest is compounded every* 3 *months, how much money is in the account after* 10 *years? (you do not need to simplify your answers)*

ii) *Find the time needed for any initial deposit to double when the interest compounds continually.*

i) $A = 2\,000$, $r = 0.04$, $n = 4$. (every 3 months/year, so 4 times)

$$C(t) = A \cdot \left(1 + \frac{r}{n}\right)^{nt} = 2000 \cdot \left(1 + \frac{0.04}{4}\right)^{4 \cdot t}$$

$$C(t) = 2\,000 \cdot (1.01)^{4t}$$

$$C(10) = 2\,000 \cdot (1.01)^{40}$$

ii) Here $n = +\infty$, so $C(t) = A \cdot e^{r \cdot t}$, so

$$C(t) = A \cdot e^{0.04 \cdot t}$$

Look for t with

$$C(t) = 2A, \text{ or}$$

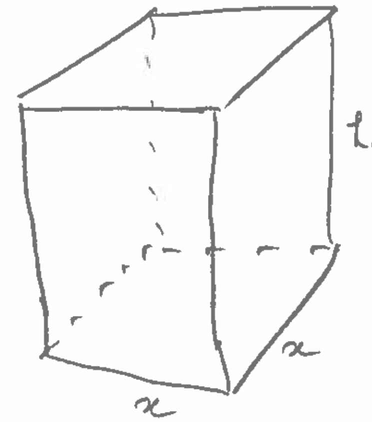
$$\cancel{A} \cdot e^{0.04 \cdot t} = 2 \cdot \cancel{A}$$

$$e^{0.04 \cdot t} = 2$$

$$0.04 \cdot t = \ln 2$$

$$t = \frac{\ln 2}{0.04}$$

Question 8 (9 points) *With 54 cm^2 of cardboard we want to make rectangular boxes with square bases. Find the dimensions of box which maximize the volume.*



→ x, h the dimensions

It is given

area $= 54 = 2\cdot x^2 + 4\cdot x h$, or

$x^2 + 2xh = 27$.

→ $V = x^2 h$ (volume).

Eliminate h:

$2xh = 27 - x^2$, $\quad h = \dfrac{27 - x^2}{2x}$; hence

$$V = x^2 \cdot \frac{1}{2\cdot x}(27 - x^2) = \frac{1}{2}\cdot x\cdot(27 - x^2) = \frac{1}{2}(27x - x^3),$$

Hence

maximize $V(x) = \frac{1}{2}(27x - x^3)$, for $x \in I = (0, +\infty)$

→ C.P. $V'(x) = \frac{1}{2}(27 - 3x^2) = 0$

$27 - 3x^2 = 0$, $x^2 = 9$, $x = 3$ $\cancel{(x=-3)}$

→ classification:

$V''(x) = \frac{1}{2}(-6x)$, $\quad V''(3) = -9 < 0$

so, $x = 3$ is local max; as it is the only C.P. in I, it is global max.

→ $x = 3$, $h = \dfrac{27 - 3^2}{2\cdot 3} = \dfrac{18}{6} = 3$ are dimensions which maximize the volume (the box is in fact a cube)