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Mathematical methods I
MAT1300B
Fall 2018
Final exam

+ Solution.

LAST name____________________ First name:____________________

Student number:____________________

Instructions:

- The duration of this exam is 3 hours.
- The use of books, notes or calculators is not allowed.
- This exam consists of 16 questions: 12 are multiple choice and 4 are long answer.
  - For the 12 multiple choice questions, only the chosen answer (letter 'A' to 'E') entered in the grid on the second page will be marked.
  - For the 4 long answer questions, the correct answer requires justification written legibly and logically. Answer these questions in the space provided. Use the backs of pages if necessary.
- Don't detach this exam.
- NB
  Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

  By signing below, you acknowledge that you have ensured that you are complying with the above statement.

  Signature:________________

Write your answers for the multiple choice questions in the next page.

Write your answers for the multiple choice questions in this grid

Q1	C	Q2	A	Q3	D	Q4	A.
Q5	E	Q6	D	Q7	B	Q8	D.
Q9	A.	Q10	C	Q11	C	Q12	D.

Grid for the marker:

	Multiple Choice	Q13	Q14	Q15	Q16	Total
Points						

Multiple Choice Questions (1-12)

Question 1 *(2 points) Consider the function*

$$f(x) = \frac{2x+4}{x^2 - x - 6}.$$

Find which of the following is correct (there is only one).

A) $x = -2$ is the only vertical assymptote of f
B) $x = -2$ and $x = 3$ are the vertical assymptotes of f
C) $x = 3$ is the only vertical assymptote of f
D) $y = -\frac{4}{6}$ is the horizontal assymptote of f
E) f has not vertical assymptote

Sol

$\rightarrow \quad f(x) = \dfrac{2\cancel{(x+2)}}{\cancel{(x+2)}(x-3)} = \dfrac{2}{x-3}$

$\left[\begin{array}{l} x^2 - x - 6 = 0 \\ x = \frac{1}{2}\left(1 \pm \sqrt{1+24}\right) = \frac{1}{2}(1 \pm 5) \\ \quad = -2, 3 \end{array} \right]$

$\rightarrow \quad \lim_{x \to 3} f(x) = +\infty$, so $x = 3$ is AV.

Question 2 *(2 points) Over what interval is the function $f(x) = xe^x$ concave up?*

A) $(-2, \infty)$ **B)** $(-\infty, 0)$ **C)** $(-2, 0)$ **D)** $(-\infty, \infty)$ **E)** $(0, \infty)$

Sol

$\rightarrow \quad f'(x) = e^x + xe^x = e^x (1+x)$

$f''(x) = e^x(1+x) + e^x = e^x(1+x+1)$

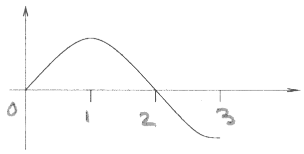
$= e^x(2+x)$

$\rightarrow \quad f''(x) = 0, \quad 2 + x = 0, \quad x = -2$

$\rightarrow$

x	$-\infty$		-2		$+\infty$
$f''(x)$		$-$	0	$+$	
$f(x)$		$\frown$		$\smile$	

Question 3 *(2 points) The picture below represents the graph of a function f. Find which of graphs in the pictures A)-E) represents the graph of f'.*

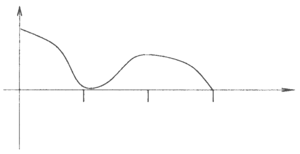


The graph of the function $y = f(x)$

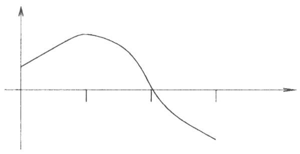
→ $f'(1) = 0, \quad f'(3) = 0$

→ $f'(x) > 0$ in $(0, 1)$

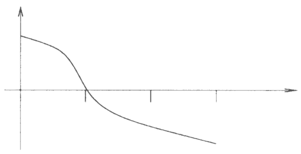
→ $f'(x) < 0$ in $(1, 3)$.



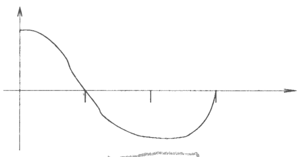
A)



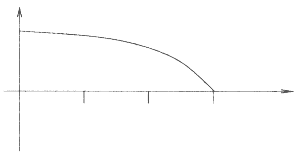
B)



C)



$\boxed{D)}$



E)

Question 4 *(2 points) Consider $f(x) = x^3 - 3x^2 + 1$. Which of the following statements is correct (there is only one)?*

$\boxed{\textbf{A)}}$ f has local minimum at $x = 2$
B) f has local maximum at $x = 1$
C) f has an inflection point at $x = 0$
D) f has a global minimum at $x = -4$
E) f has a global maximum at $x = 4$

<u>Sol</u>

→ $f'(x) = 3x^2 - 6x = 3x \cdot (x - 2)$.

→ $f'(x) = 0, \quad x = 0, \quad x = 2$

→

x		0		2	
$f'(x)$	+	0	−	0	+
$f(x)$	↗		↘		↗
		loc. max		loc. min	

Question 5 *(2 points) The function $y = y(x)$ is defined implicitly by*

$$x^3y - 3xy^2 + 4x + 3y = 5.$$

Knowing that $y(1) = 1$, find $y'(1)$.

A) -2 B) $-1/2$ C) 0 D) $1/2$ $\boxed{\text{E) } 2}$

Sol

→ $3x^2y + x^3y' - 3y^2 - 6xy\cdot y' + 4 + 3y' = 0$

→ $x=1,\ y=1$:

$\cancel{3} + y' - \cancel{3} - 6y' + 4 + 3y' = 0$

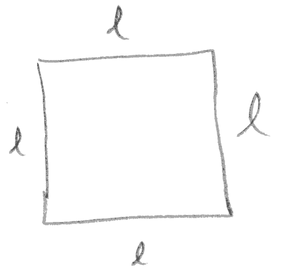
$-2y' + 4 = 0$

$y' = 2$

Question 6 *(2 points) A city grows in the form of a square. It is observed that its sides grow at a rate of 2 km/year. Find the rate change of its area when the sides has length 5 km.*

A) 2   B) 5   C) 10 $\boxed{\text{D) } 20}$ E) 25

Sol



→ $A = l^2$

→ $A' = 2\cdot l \cdot l'$

→ $l' = 2,\quad l = 5$:

$A' = 2\cdot 5 \cdot 2 = 20$

Question 7 *(2 points) The rate change of the balance of a bank account is given in the figure below for a period of five years. Find how much money is accumulated in the account during last four years?*

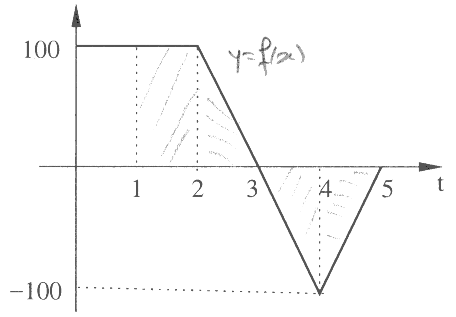
A) 25

$\boxed{\text{B) } 50}$

C) 75

D) 100

E) 150



Sol

$$M = \int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx + \int_4^5 f(x)dx$$

$$= 1 \cdot 100 + \frac{1}{2} \cdot 1 \cdot 100 - \frac{1}{2} \cdot 1 \cdot 100 - \frac{1}{2} \cdot 1 \cdot 100$$

$$= 100 + 50 - 50 - 50 = 50$$

Question 8 *(2 points) Suppose the marginal cost of a product is given by*

$$C'(x) = x^{1/3} - x^3 + 3.$$

Knowing that $C(0) = -\frac{1}{2}$ find the cost $C(1)$.

A) $\frac{9}{2}$ B) $\frac{5}{2}$ C) $\frac{1}{2}$ $\boxed{\text{D) } 3}$ E) 4

Sol

$$\to \quad C(x) = \int C'(x)\,dx = \int x^{1/3} dx - \int x^3 dx + \int 3 dx$$

$$= \frac{x^{4/3}}{4/3} - \frac{x^4}{4} + 3x + K$$

$$\to \quad K = ? \quad C(0) = -\frac{1}{2} \Leftrightarrow 0 - 0 + 0 + K = -\frac{1}{2}, \quad K = -\frac{1}{2}$$

$$\to \quad C(1) = \frac{1^{4/3}}{4/3} - \frac{1^4}{4} + 3 \cdot 1 - \frac{1}{2}$$

$$= \frac{3}{4} - \frac{1}{4} + 3 - \frac{1}{2} = \frac{3 - 1 + 12 - 2}{4} = \frac{12}{4} = 3.$$

Question 9 *(2 points) Consider the function*

$$F(x) = \int_0^x \frac{1}{\sqrt{t^2+1}} dt.$$

Compute $F'(1)$.

$\boxed{\textbf{A)}\ \frac{1}{\sqrt{2}}}$ **B)** $-\sqrt{2}$ **C)** $-2\sqrt{2}$ **D)** $\sqrt{2}$ **E)** $2\sqrt{2}$

Sol

$\rightarrow F'(x) = \left(\int_0^x \frac{1}{\sqrt{t^2+1}} \cancel{dx}\right)' = \frac{1}{\sqrt{x^2+1}}.$

$\rightarrow F'(1) = \frac{1}{\sqrt{1^2+1}} = \frac{1}{\sqrt{2}}$

Question 10 *(2 points) Compute*

$$\int_0^4 \frac{1}{\sqrt{x}} dx.$$

A) 1 **B)** 2 $\boxed{\textbf{C)}\ 4}$ **D)** 8 **E)** 16

Sol

$\rightarrow \int_0^4 \frac{1}{\sqrt{x}} dx = \int_0^4 x^{-\frac{1}{2}} dx$

$= \left[\frac{x^{1/2}}{1/2}\right]_0^4 = \frac{4^{1/2}}{1/2} - \frac{0^{1/2}}{1/2}$

$= \frac{2}{1/2} = 4.$

Question 11 *(2 points) Find how many critical points has the function*

$$f(x,y) = -3x^2 - 2y + x^3 + y^2 + 5.$$

A) 0 **B)** 1 $\boxed{\textbf{C)}\ 2}$ **D)** 3 **E)** 4

<u>Sol</u>

$\rightarrow \begin{cases} f_x(x,y) = -6x + 3x^2 = 0, & (1) \\ f_y(x,y) = -2 + 2y = 0, & (2) \end{cases}$

$\rightarrow$ (1) : $3x \cdot (-2+x) = 0,$ $x=0,$ $x=2$

$\Rightarrow (0,0), (2,0)$

(2) : $2(-1+y) = 0,$ $y=0$

Question 12 *(2 points) Consider the function*

$$f(x,y) = x\sqrt{x^2+y^2+1}.$$

Find $f_y(2,2)$.

A) $\frac{8}{3}$ **B)** 4 **C)** $\frac{13}{3}$ $\boxed{\textbf{D)}\ \frac{4}{3}}$ **E)** 12

<u>Sol</u>

$\rightarrow f_y(x,y) = x \cdot \frac{1}{2}(x^2+y^2+1)^{-\frac{1}{2}} \cdot 2y$

$= x \cdot y \cdot \dfrac{1}{\sqrt{x^2+y^2+1}}$

$\rightarrow f_y(2,2) = 2 \cdot 2 \cdot \dfrac{1}{\sqrt{2^2+2^2+1}} = \dfrac{4}{\sqrt{9}} = \dfrac{4}{3}.$

Long Answer Questions (13-16)

Question 13 *(6 points) Compute the following integrals:*

$$\int x \ln x dx =$$

$f(x) = \ln x$ | $f'(x) = \frac{1}{x}$

$g'(x) = x$ | $g(x) = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{x} \cdot \frac{1}{2}x^2 \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$$\int \frac{1}{x \ln x} dx =$$

$t = \ln x$

$dt = \frac{1}{x} dx.$

$= \int \frac{1}{\ln x} \boxed{\frac{1}{x} dx}$

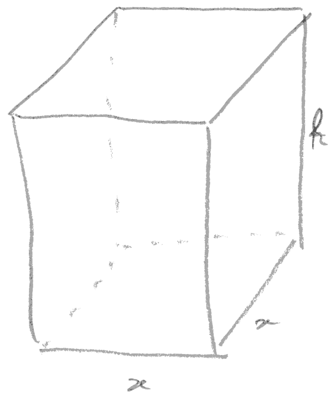
$= \int \frac{1}{t} \, dt$

$= \ln|t| + C$

$= \ln|\ln x| + C.$

Question 14 *(6 points) You have been asked to bid on the construction of square-bottomed box with no top which will hold 64 cubic centimeters of water. Suppose that the material of the bottom face costs \$20 per centimeter square and the material of the side faces costs \$10 per centimeter square. What are the dimensions of the least expensive box which holds 64 cubic centimeters of water?*

Sol



→ x, h the dimensions

→ $x^2 \cdot h = 64$;

→ $C =$ cost

$$= \underbrace{20 \cdot x^2}_{\text{cost of bottom face}} + \underbrace{10 \cdot 4\, x \cdot h}_{\text{cost of four side faces.}}$$

→ From $x^2 \cdot h = 64$, $h = \frac{64}{x^2}$; Then the cost is given by

$$C = 20 \cdot x^2 + 40 \cdot x \cdot \frac{64}{x^2},$$ or.

$$C(x) = 20x^2 + \frac{40 \cdot 64}{x}.$$

→ minimize $C(x)$, for $x \in I = (0, +\infty)$.

- $C'(x) = 40 \cdot x - \frac{40 \cdot 64}{x^2} = \frac{40 \cdot x^3 - 40 \cdot 64}{x^2} = 40 \frac{x^3 - 64}{x^2}$

$C'(x) = 0$, $x^3 - 64 = 0$, $x^3 = 64$,

$x = (64)^{1/3} = (8 \cdot 8)^{1/3} = 8^{1/3} \cdot 8^{1/3} = 2 \cdot 2 = 4.$

-

x	0		4		$+\infty$
$C'(x)$		$-$	ϕ	$+$	
$C(x)$		$\searrow$	loc. min.	$\nearrow$	

- $C(4)$ is loc. min; as $x = 4$ is the only C.P in I then $C(4)$ is global min;
- $x = 4$, $h = \frac{64}{4^2} = 4$ are dimensions that minimize the cost.

Question 15 *(6 points) Consider two functions*

$$f(x) = x^2 - 4, \quad g(x) = 4 - x^2.$$

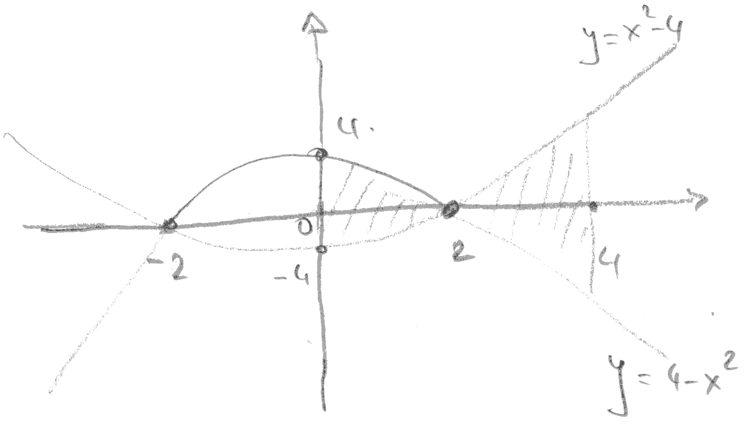
i) Find the values of x where the graphs of f and g intersect.
ii) Find the area of the region between the graphs of f and g and over the interval $[0, 4]$.

Sol

$\rightarrow$ $x^2 - 4 = 4 - x^2$

$2x^2 = 8, \quad x^2 = 4, \quad x = \pm 2.$

Intersection at $(\pm 2, 0)$.



$\rightarrow$ $A = \int_0^4 |f(x) - g(x)|\, dx$

$= \int_0^4 |2x^2 - 8|\, dx = \int_0^4 2 \cdot |x^2 - 4|\, dx$

$= 2 \int_0^2 |x^2 - 4|\, dx + 2 \int_2^4 |x^2 - 4|\, dx$

$= 2 \int_0^2 (4 - x^2)\, dx + 2 \int_2^4 (x^2 - 4)\, dx$

$= 2 \cdot \left[4x - \frac{1}{3}x^3\right]_0^2 + 2 \cdot \left[\frac{1}{3}x^3 - 4x\right]_2^4$

$= 2\left(\left(8 - \frac{8}{3}\right) - 0\right) + 2 \cdot \left(\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right)$

$= 16 - \frac{16}{3} + \frac{128}{3} - 32 - \frac{16}{3} + 16$

$= \frac{1}{3}(128 - 32) = \frac{96}{3}$

$= 32$

Question 16 *(8 points) Let (q_1, p_1) be the (demand, price) for the product 1 and (q_2, p_2) be the (demand, price) for the product 2, related by*

$$\begin{aligned} q_1 &= 200 - 3p_1 + p_2, \\ q_2 &= 150 + p_1 - 2p_2. \end{aligned}$$

i) Are these products complementary goods or substitute goods?
ii) Find the function $R(p_1, p_2)$ that expresses the total revenue from these two products.
iii) Find the price for each product that maximizes the revenue.

Sol

→ If p_1 increases then q_2 increases b/c $\frac{\partial q_2}{\partial p_1} = 1 > 0$; similarly if p_2 increases then q_1 increases b/c $\frac{\partial q_1}{\partial p_2} = 1 > 0$; so the goods are substitute.

→ $R(p_1,p_2) = p_1 \cdot q_1 + p_2 \cdot q_2 = 200 \cdot p_1 - 3p_1^2 + p_1 p_2 + 150 p_2 + p_1 p_2 - 2p_2^2.$

$R(p_1,p_2) = 200 p_1 + 150 p_2 - 3p_1^2 + 2p_1 p_2 - 2p_2^2.$

→ $\begin{cases} R_{p_1} = 200 - 6p_1 + 2p_2 \\ R_{p_2} = 150 + 2p_1 - 4p_2. \end{cases}$

C.P. $\begin{cases} 200 - 6p_1 + 2p_2 = 0; \\ 150 + 2p_1 - 4p_2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3p_1 - p_2 = 100 & (1) \\ -p_1 + 2p_2 = 75. & (2) \end{cases}$

$2*(1) + (2)$: $5p_1 = 275$, $p_1 = 55$

$(1) + 3*(2)$: $5p_2 = 325$, $p_2 = 65$

$(55, 65)$ is the only C.P.

second derivatives and D:

$R_{p_1p_1} = -6$, $R_{p_1p_2} = 2$, $R_{p_2p_2} = -4$,

$$D = (-6)\cdot(-4) - 2^2 = 20$$

Classification

$(55, 65)$: $D = 20 > 0$, $R_{p_1 p_1} = -6 < 0$

$(55, 65)$ is a C.P. of local maximum.

$\rightarrow$ $p_1 = 55$, $p_2 = 65$ maximize the revenue.