LECTURES ON EXTENDED AFFINE LIE ALGEBRAS

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ABSTRACT. We give an introduction to the structure theory of extended affine Lie algebras, which provide a common framework for finite-dimensional semisimple, affine and toroidal Lie algebras. The notes are based on a lecture series given during the Fields Institute summer school at the University of Ottawa in June 2009.

CONTENTS

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Introduction

Extended affine Lie algebras form a category of Lie algebras containing finite-dimensional semisimple, affine, toroidal and some other interesting classes of Lie algebras.

Like finite-dimensional simple Lie algebras, extended affine Lie algebras are defined by a set of axioms prescribing their internal structure, rather than a potentially elusive presentation. The structure of extended affine Lie algebras is now well understood, and is quite similar to the construction of affine Lie algebras: They are obtained from a generalized loop algebra, a so-called invariant Lie torus, by taking a central extension and adding some derivations:

$$\begin{tikzcd} \begin{array}{c}\text{central extension of } L\\ \text{(another Lie torus)}\end{array} \arrow[r, squiggly, "\begin{array}{c}\text{add}\\ \text{derivations}\end{array}"] & \text{extended affine Lie algebra} \\ \text{invariant Lie torus } L \arrow[u, squiggly] & \end{tikzcd}$$

Invariant Lie tori have been classified. Although there are some rather sophisticated examples, many of them have a concrete matrix realization or can be described in terms of familiar objects like finite-dimensional simple Lie algebras and Laurent polynomial rings. This makes extended affine Lie algebras easily accessible. Since they are an emerging new area, there are many open questions, opportunities for research and applications, for example in physics. A short history of extended affine Lie algebras is given in section 2.1, in particular it describes the role physicists have played.

The goal of these notes is to provide a survey of the structure theory of extended affine Lie algebras, accessible to graduate students. The emphasis is on examples, and not on an exposition containing all proofs. Such an exposition will appear elsewhere. Thus, while we have endeavored to present a complete picture of the theory by giving precise definitions and theorems, most of the proofs have been left out. But references to proofs are provided, as far as possible.

Outline. Section 1 reviews the construction of affine Kac-Moody algebras and discusses some natural generalizations, like toroidal algebras. It also contains an exposition of central extensions of Lie algebras, which are crucial for the theory. The following section 2 starts with the definition of an extended affine Lie algebra and then presents some easily proven properties. We also give examples of extended affine Lie algebras: finite-dimensional split simple, affine Kac-Moody and untwisted multi-loop algebras. Part of the axioms for an extended affine Lie algebra is the existence of a root space decomposition. Section 3 describes the structure of the roots occurring in an extended affine Lie algebra, naturally called extended affine root systems. They turn out to be special types of so-called affine reflection systems. In section 4 we reverse the picture above: We start with an extended affine Lie algebra and, using the structure of affine reflection systems, we associate to it a graded ideal, the so-called core, and its central quotient, the centreless core. Both are Lie tori. This section also presents properties of Lie tori and examples. Finally, in section 5 we survey the general construction of extended affine Lie algebras, as summarized in the picture above.

Prerequisites. We assume that the reader is familiar with the basic structure theory of complex finite-dimensional semisimple Lie algebras, as for example developed in [Hu]. Some familiarity with affine Kac-Moody algebras, e.g. chapters 7 and 8 of [Kac], is helpful but not essential, since section 1.1 will give a short review of the necessary background. Similarly, knowing split simple Lie algebras will facilitate reading the notes, but is not required. A short summary of the facts used here is presented in section 2.3.

Notation and setting. With some rare exceptions (in 1.1, 5.1 and 5.2), all vector spaces and algebras are defined over a field F of characteristic 0. *We will not assume that F is algebraically closed*, since this is not needed and would not do proper justice to the theory to be explained here. Thus, F could be, but need not be the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers or the field of rational numbers $\mathbb{Q}$ or ... Unless specified otherwise, linear maps will always be F-linear. All unadorned tensor products will be over F.

The symbol $\mathfrak{g}$ will always denote a split simple finite-dimensional Lie algebra. We let $Z(L) = \{z \in L : [z, L] = 0\}$ denote the centre of a Lie algebra L. We will say that L is *centreless* if $Z(L) = 0$. If K is a subspace of a Lie algebra E, the *centralizer* of K in E is $C_E(K) = \{c \in E : [c, K] = 0\}$.

With the exception of some remarks, all algebras will be associative or Lie algebras. For an F-algebra A we denote by $\mathrm{Der}_F(L)$ the Lie algebra of all derivations of L (recall that an F-linear map $d : L \to L$ is a *derivation* if $d([l_1, l_2]) = [d(l_1), l_2] + [l_1, d(l_2)]$ holds for all $l_1, l_2 \in L$).

The algebras considered here will often be graded by some abelian group, usually denoted Λ and always written additively. A Λ-*grading* of a vector space V by the abelian group Λ is a decomposition $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$ into subspaces V^λ. Suppose V is such a Λ-graded vector space. Then the Λ-*support* of V is defined as $\mathrm{supp}_\Lambda V = \{\lambda \in \Lambda : V^\lambda \neq 0\}$. A *graded subspace of* V is a subspace U of V satisfying $U = \bigoplus_\lambda (U \cap V^\lambda)$. We will say that V has *finite bounded dimension* if there exists a constant M such that for all $\lambda \in \Lambda$ we have $\dim V^\lambda \leq M$. Note that this is a stronger condition than requiring that V has *finite homogeneous dimension*, which by definition just means that every V^λ, $\lambda \in \Lambda$, is finite-dimensional.

Given two Λ-graded vector spaces $V = \bigoplus_\lambda V^\lambda$ and $W = \bigoplus_\lambda W^\lambda$, we say an F-linear map $f : V \to W$ has *degree* λ if $f(V^\mu) \subset W^{\lambda+\mu}$ holds for all $\mu \in \Lambda$. We denote by $\mathrm{Hom}_F(V, W)^\lambda$ the linear maps of degree λ and put

$$\mathrm{grHom}_F(V, W) = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F(V, W)^\lambda \quad \text{and} \quad \mathrm{grEnd}_F(V) = \mathrm{grHom}_F(V, V).$$

We note that $\mathrm{grEnd}_F(V)$ is a Λ-graded associative algebra with respect to composition of maps. We give F the trivial grading $F = F^0$ and define the *graded dual space of* V as

$$V^{\mathrm{gr}*} = \mathrm{grHom}_F(V, F) = \bigoplus_{\lambda\in\Lambda} (V^{\mathrm{gr}*})^\lambda.$$

Observe that $(V^{\mathrm{gr}*})^\lambda$ consists of those linear forms $\varphi : V \to F$ which satisfy $\varphi(V^\mu) = 0$ whenever $\lambda + \mu \neq 0$ and can therefore be identified with the usual dual space $(V^{-\lambda})^*$.

Given a symmetric bilinear form on a vector space V, an endomorphism d of V is called *skew-symmetric* if $(d(v) \mid v) = 0$ for all $v \in V$. Since we assume that our base field has characteristic 0, this is equivalent to the condition $(d(v_1) \mid v_2) + (v_1 \mid d(v_2)) = 0$ for all $v_1, v_2 \in V$. A bilinear form is *nondegenerate* if $(v \mid u) = 0$ for all $u \in V$ implies $v = 0$.

If A is an algebra, a Λ-*grading of the algebra* A is a Λ-grading of the underlying vector space A, say $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$, for which in addition $A^\lambda A^\mu \subset A^{\lambda+\mu}$ holds for all $\lambda, \mu \in \Lambda$. Since we will often deal with algebras with two gradings, it is convenient to use superscripts and subscripts to distinguish them.

These notes grew out of my notes for a lecture series during the Fields Institute summer school on Geometric Representation Theory and Extended Affine Lie Algebras, held at the University of Ottawa in June 2009. I would like to thank all the participants of the summer school for their interest and questions. I also thank Bruce Allison and Juana Sánchez Ortega for their careful reading of an earlier version of these notes.

1. Affine Lie algebras and some generalizations

We will always assume that F is a field of characteristic 0. Occasionally we will need some roots of unity in F, so certainly an algebraically closed field like $\mathbb{C}$ will do.

We denote by $\mathfrak{g}$ a split simple finite-dimensional Lie algebra over F. For example, if F is algebraically closed then this just means that $\mathfrak{g}$ is a simple and finite-dimensional. Their structure theory is explained in most standard textbooks, for example in [Hu]. For more general fields, an example of a split simple $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_n(F)$ of $n \times n$-matrices over F which have trace 0. These types of Lie algebra are investigated in [Bou3, Ch. VII], [D, Ch. 1] or [J, Ch. IV].

1.1. **Realization (construction of affine Kac-Moody Lie algebras).** Let $\zeta \in F$ be a primitive mth root of 1. In other words, the multiplicative subgroup of F generated by ζ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. For example, in $F = \mathbb{C}$ we can take $\zeta = \exp(2\pi i/m)$.

Let σ be an automorphism of $\mathfrak{g}$ of finite order $m \in \mathbb{N}$. Thus, the subgroup $\langle \sigma \rangle$ of the automorphism group of $\mathfrak{g}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. For example, if $\mathfrak{g} = \mathfrak{sl}_n(F)$ an example of such an automorphism is $\sigma(x) = axa^{-1}$, where a is an $n \times n$-matrix of order m, and an example of such a matrix is $a = \zeta E_n$ where E_n is the $n \times n$ identity matrix.

Observe that σ is diagonalizable. Indeed, its minimal polynomial divides the polynomial $t^m = 1$ and therefore has no multiple roots in F. For a general field F this would of course only say that σ is a semisimple endomorphism. But since as we assumed that F contains all roots of unity which we need, σ is diagonalizable over F. To describe its eigenspaces we need some notation. In anticipation of the later developments we put

$$\Lambda = \mathbb{Z} \quad \text{and} \quad \bar{\Lambda} = \mathbb{Z}/m\mathbb{Z},$$

and denote the canonical map $\Lambda \to \bar{\Lambda}$ by $\lambda \mapsto \bar{\lambda}$. That σ is diagonalizable, means

$$\mathfrak{g} = \textstyle\bigoplus_{\bar{\lambda} \in \bar{\Lambda}} \mathfrak{g}_{\bar{\lambda}} \quad \text{for } \mathfrak{g}_{\bar{\lambda}} = \{x \in \mathfrak{g} : \sigma(x) = \zeta^\lambda x\} \tag{1.1}$$

Of course, some of the $\mathfrak{g}_{\bar{\lambda}}$ could be zero. The eigenspaces of σ are precisely the non-zero among the subspaces $\mathfrak{g}_{\bar{\lambda}}$. It is also appropriate to note that $\mathfrak{g}_{\bar{\lambda}}$ is well-defined: if $\bar{\lambda} = \bar{\mu}$ then $\zeta^\lambda = \zeta^\mu$. Finally we point out that the decomposition (1.1) is a $\bar{\Lambda}$-grading, which means that it satisfies

$$[\mathfrak{g}_{\bar{\lambda}}, \mathfrak{g}_{\bar{\mu}}] \subset \mathfrak{g}_{\bar{\lambda}+\bar{\mu}} \quad \text{for all } \bar{\lambda}, \bar{\mu} \in \bar{\Lambda}. \tag{1.2}$$

Let $F[t^{\pm 1}]$ be the ring of Laurent polynomials. This is a unital associative commutative F-algebra with F-basis $\{t^\lambda : \lambda \in \mathbb{Z}\}$ and multiplication rule $t^\lambda t^\mu = t^{\lambda+\mu}$.

The *loop algebra* associated to the data $(\mathfrak{g}, \sigma)$ is the Lie algebra

$$\mathcal{L} = L(\mathfrak{g}, \sigma) = \textstyle\bigoplus_{\lambda\in\Lambda} \mathfrak{g}_{\bar\lambda} \otimes Ft^\lambda \tag{1.3}$$

with product

$$[u_{\bar\lambda} \otimes t^\lambda,\, v_{\bar\mu} \otimes t^\mu] = [u_{\bar\lambda}, v_{\bar\mu}] \otimes t^{\lambda+\mu}. \tag{1.4}$$

We will sometimes use more precise terminology: If $\sigma = \mathrm{Id}$, i.e., $m = 1$, we will call $L(\mathfrak{g}, \mathrm{Id}) = \mathfrak{g} \otimes F[t^{\pm 1}]$ the *untwisted loop algebra*, and we will call $L(\mathfrak{g}, \sigma)$ a *twisted loop algebra* if it is clear that $\sigma \neq \mathrm{Id}$ and we want to emphasize this.

We point out that we consider $L(\mathfrak{g}, \sigma)$ as a Lie algebra over F. It is therefore infinite-dimensional. It is also important to note that $\mathcal{L}$ is a Λ-graded algebra, whose homogenous spaces are $\mathcal{L}^\lambda = \mathfrak{g}_{\bar\lambda} \otimes Ft^\lambda$ for $\lambda \in \Lambda$. For the reader with some background in algebraic geometry, a more geometric definition of $L(\mathfrak{g}, \sigma)$ is the following: It is (isomorphic to) the Lie algebra of equivariant maps $F^\times \to \mathfrak{g}$, where σ acts on $F^\times$ by $\sigma(x) = \zeta x$.)

Let κ be the Killing form of $\mathfrak{g}$, i.e., $\kappa(u, v) = \mathrm{tr}(\mathrm{ad}\, u \circ \mathrm{ad}\, v)$, and define

$$\psi : \mathcal{L} \times \mathcal{L} \to F, \quad \psi(u \otimes t^\lambda, v \otimes t^\mu) = \lambda\, \delta_{\lambda,-\mu}\, \kappa(u, v) \tag{1.5}$$

where $\delta_{\lambda,-\mu}$ is the Kronecker delta: It has the value 1 if $\lambda = -\mu$ and is zero otherwise.

Exercise 1.1. Check that the map ψ of (1.5) is a 2-*cocycle* of $\mathcal{L}$, i.e. an F-bilinear map satisfying

$$\psi(l, l) = 0 = \psi([l_1, l_2], l_3) + \psi([l_2, l_3], l_1) + \psi([l_3, l_1], l_2) \tag{1.6}$$

for $l, l_i \in \mathcal{L}$.

A consequence of Exercise 1.1 is that we can enlarge our Lie algebra $\mathcal{L}$ by adjoining a 1-dimensional space, denoted Fc here:

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\mathfrak{g}, \sigma) = L(\mathfrak{g}, \sigma) \oplus Fc \tag{1.7}$$

is a Lie algebra over F with respect to the product

$$[l_1 \oplus s_1 c,\, l_2 \oplus s_2 c]_{\tilde{\mathcal{L}}} = [l_1, l_2]_L \oplus \psi(l_1, l_2)c$$

for $l_i \in \mathcal{L}$ and $s_i \in F$. We have added subscripts on the products to emphasize where the product is calculated, in $\tilde{\mathcal{L}}$ or in $\mathcal{L}$. It is obvious from the product formula, that it is important to know in which Lie algebra the product is being calculated. But in the future we will leave out the subscripts, if it is clear in which algebra the product is calculated.

The equations (1.6) are exactly what is needed to make $\tilde{\mathcal{L}}$ a Lie algebra. The map

$$\mathfrak{u} : \tilde{\mathcal{L}} \to \mathcal{L}, \quad \mathfrak{u}(l \oplus sc) = l$$

is a surjective Lie algebra homomorphism with kernel $\mathrm{Ker}(\mathfrak{u}) = Fc = Z(\tilde{\mathcal{L}})$, the centre of $\tilde{\mathcal{L}}$. In other words, $\mathfrak{u}$ is a *central extension* (see 1.3 for a short review of central extensions). In fact, $\mathfrak{u}$ is the "biggest" central extension, the so-called *universal central extension*, see [G] and [Wi] for a proof.

The Lie algebra $\tilde{\mathcal{L}}$ has a canonical derivation d, the so-called *degree derivation*

$$d\big((u \otimes t^\lambda) \oplus sc\big) = \lambda u \otimes t^\lambda, \quad (\lambda \in \mathbb{Z}, u \in \mathfrak{g}_{\bar\lambda}, s \in F). \tag{1.8}$$

Hence we can form the semidirect product $\hat{\mathcal{L}} = L(\mathfrak{g}, \sigma)\hat{} = \tilde{\mathcal{L}} \rtimes Fd$ with product

$$[\tilde{l}_1 \oplus s_1 d, \tilde{l}_2 \oplus s_2 d]_{\hat{\mathcal{L}}} = [\tilde{l}_1, \tilde{l}_2]_{\tilde{\mathcal{L}}} + s_1 d(\tilde{l}_2) - s_2 d(\tilde{l}_1)$$

for $\tilde{l}_i \in \tilde{\mathcal{L}}$ and $s_i \in F$. In untangled form,

$$\hat{\mathcal{L}} = \big(\textstyle\bigoplus_{\lambda \in \mathbb{Z}} (\mathfrak{g}_{\bar\lambda} \otimes F t^\lambda)\big) \,\oplus\, Fc \,\oplus\, Fd \tag{1.9}$$

is the Lie algebra with product

$$\begin{aligned} &[u_{\bar\lambda} \otimes t^\lambda \oplus s_1 c \oplus s_1' d,\, v_{\bar\mu} \otimes t^\mu \oplus s_2 c \oplus s_2' d] \\ &\quad = \big([u_{\bar\lambda}, v_{\bar\mu}] \otimes t^{\lambda+\mu} + \mu s_1' v_{\bar\mu} \otimes t^\mu - \lambda s_2' u_{\bar\lambda} \otimes t^\lambda\big) \oplus \lambda\, \delta_{\lambda,-\mu}\, \kappa(u_{\bar\lambda}, v_{\bar\mu})\, c. \end{aligned} \tag{1.10}$$

Exercise 1.2. Show $[\hat{\mathcal{L}}, \hat{\mathcal{L}}] = \tilde{\mathcal{L}}$ and $Z(\tilde{\mathcal{L}}) = Fc = Z(\hat{\mathcal{L}})$.

The importance of the Lie algebras $L(\mathfrak{g}, \sigma)\hat{}$ stems from the following.

Theorem 1.3. (**Realization Theorem** [Kac, Th. 7.4, Th. 8.3, Th. 8.5]) *Suppose F is algebraically closed.*

(a) *The Lie algebra $L(\mathfrak{g}, \sigma)\hat{}$ is an affine Kac-Moody Lie algebra, and every affine Kac-Moody Lie algebra is isomorphic (as F-algebra) to some $L(\mathfrak{g}, \sigma)\hat{}$.*

(b) *$L(\mathfrak{g}, \sigma)\hat{} \cong L(\mathfrak{g}, \sigma')\hat{}$ where σ' is a diagram automorphism with respect to some Cartan subalgebra of $\mathfrak{g}$.*

We note that diagram automorphisms have order $1, 2$ or 3, with the latter case only occurring for $\mathfrak{g}$ of type D_4.

1.2. **Multiloop and toroidal Lie algebras.** We will discuss some (straightforward) generalizations of $\mathcal{L} = L(\mathfrak{g}, \sigma)$, the central extension $\tilde{\mathcal{L}}$ and the big Lie algebra $\hat{\mathcal{L}}$.

The first idea is to replace the Laurent polynomial ring $F[t^{\pm 1}]$ by a ring with similar properties. Instead of one variable we will use the Laurent polynomial ring $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ in n variables. This ring has indeed very similar properties to the ring $F[t^{\pm 1}]$. We put $\Lambda = \mathbb{Z}^n$ and define

$$t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n} \quad \text{for } \lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$$

Then $\{t^\lambda : \lambda \in \Lambda\}$ is an F-basis of $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and the multiplication rule in $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is $t^\lambda t^\mu = t^{\lambda+\mu}$, which is the "same" as in the 1-variable case. Also, $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is still a unital commutative associative F-algebra. We can therefore define the *untwisted multiloop algebra*, the "several variable" generalization of the untwisted loop algebra of 1.1 as

$$L(\mathfrak{g}) = \mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \tag{1.11}$$

which becomes a Lie algebra with respect to the product

$$[u \otimes t^\lambda, v \otimes t^\mu] = [u, v] \otimes t^{\lambda+\mu}$$

for $u, v \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{Z}^n$. We will meet this Lie algebra again in Example 4.31.

To continue the analogy we let $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ be a family of n commuting finite order automorphisms of $\mathfrak{g}$, say σ_i has order $m_i \in \mathbb{N}_+$. Let $\zeta_i \in F$ be a primitive m_i-th root of 1 (recall that we assumed that F has an ample supply of them). We put

$$\bar{\Lambda} = (\mathbb{Z}/m_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_n\mathbb{Z})$$

and let $\lambda \mapsto \bar{\lambda}$ be the obvious map. The automorphisms σ_i are simultaneously diagonalizable:

$$\mathfrak{g} = \bigoplus_{\bar{\lambda}\in\bar{\Lambda}} \mathfrak{g}_{\bar{\lambda}}, \quad \mathfrak{g}_{\bar{\lambda}} = \{u \in \mathfrak{g} : \sigma_i(u) = \zeta_i^{\lambda_i} u, 1 \le i \le n\}. \tag{1.12}$$

As in the one-variable case, the decomposition (1.12) is a $\bar{\Lambda}$-grading: $[\mathfrak{g}_{\bar{\lambda}}, \mathfrak{g}_{\bar{\mu}}] \subset \mathfrak{g}_{\bar{\lambda}+\bar{\mu}}$ for $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}$. It follows from this that

$$L(\mathfrak{g}, \boldsymbol{\sigma}) = \bigoplus_{\lambda\in\Lambda} \mathfrak{g}_{\bar{\lambda}} \otimes Ft^{\lambda} \tag{1.13}$$

is a subalgebra of $\mathfrak{g}\otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, called the *multiloop algebra associated to* $\mathfrak{g}$ *and* $\boldsymbol{\sigma}$. If all $\sigma_i = \mathrm{Id}_{\mathfrak{g}}$ we will (of course) call it an *untwisted multiloop algebra.* Multiloop algebras are investigated in the papers [ABFP1], [ABFP2], [ABP1], [ABP2] and [ABP3].

Following our procedure in section 1.1 we should now make a central extension to get a bigger Lie algebra $\tilde{\mathcal{L}}$ and then add some derivations:

$$\begin{tikzcd}
\tilde{\mathcal{L}} = \mathcal{L} \oplus C \arrow[r, hook, "\text{add derivations}"] \arrow[d, "\text{central extension}"'] & \hat{\mathcal{L}} = \tilde{\mathcal{L}} \rtimes D \\
\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma}) \arrow[ur, squiggly] &
\end{tikzcd} \tag{1.14}$$

To define the Lie algebra product on $\tilde{\mathcal{L}}$ we would use a 2-cocycle $\psi : \mathcal{L} \times \mathcal{L} \to C$ where C is some vector space and then put

$$[l_1 \otimes c_1, l_1 \otimes c_2]_{\tilde{\mathcal{L}}} = [l_1, l_2]_{\mathcal{L}} \oplus \psi(l_1, l_2) \tag{1.15}$$

for $l_i \in \mathcal{L}$ and $c_i \in C$. The Lie algebra $\hat{\mathcal{L}}$ should be a semidirect product with D acting on $\tilde{\mathcal{L}}$ by derivations.

But here is where the problems start, or things become interesting depending on one's taste. In the one-variable case the 2-cocycle ψ of (1.5) was the only possible choice up to scalars, i.e., the universal central extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$ had a 1-dimensional centre $C = Fc$. This is no longer true in the case of several variables. It is not so surprising that there exists a 2-cocycle with values in F^n: We can simply use the same formula as in (1.5).

Exercise 1.4. Let $\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma})$ be a multiloop algebra and embed $\Lambda \subset F^n$ canonically. Then $\psi : \mathcal{L} \times \mathcal{L} \to F^n$, given by

$$\psi(u \otimes t^{\lambda}, v \otimes t^{\mu}) = \delta_{\lambda+\mu,0}\, \kappa(u, v)\, \lambda\,, \tag{1.16}$$

is a 2-cocycle of $\mathcal{L}$.

However, this is still not the "biggest" possible. Rather, the centre of the universal central extension is infinite-dimensional and the so-called *universal* 2-*cocycle*, i.e., the 2-cocycle used in (1.15) to describe the universal central extension $\hat{\mathcal{L}}$ of $\mathcal{L}$, is described in the following result.

Theorem 1.5 ([Ne6]). *Let $\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma})$ be a multiloop algebra. We embed $\Lambda \subset F^n$ canonically, put $\Gamma = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and let $C = \bigoplus_{\gamma\in\Gamma} C_\gamma$ where $C_\gamma = F^n/F\gamma$. Then the universal 2-cocycle is $\psi_{\mathfrak{u}} : \mathcal{L} \times \mathcal{L} \to C$ for which the γ-component of $\psi_{\mathfrak{u}}$ is*

$$\psi_{\mathfrak{u}}(u \otimes t^\lambda, y \otimes t^\mu)_\gamma = \kappa(u, v)\delta_{\lambda+\mu,-\gamma}\,\bar{\lambda} \in C_\gamma. \tag{1.17}$$

Observe that (1.16) is just the 0-component of (1.17). The theorem is well-known in the untwisted case (all $\sigma_i = \mathrm{Id}_{\mathfrak{g}}$, so $\Gamma = \Lambda$), in which it can be deduced from the description of the universal central extension of the Lie algebra $\mathfrak{g} \otimes A$ where A is any unital commutative associative F-algebra, see [Kas] and [MRK]. (In these references the centre C of the universal central extension is described as Ω_A/dA where Ω_A is the module of Kähler differentials, which is also the same as the first cyclic homology group $\mathrm{HC}_1(A)$.)

In the untwisted case, the universal central extension $\hat{\mathcal{L}}$ was termed the *n-toroidal Lie algebra* based on $\mathfrak{g}$. The reader should however be warned that this terminology is not standard. It is sometimes used for the Lie algebra $\tilde{\mathcal{L}}$ with the 2-cocycle of exercise 1.4, and sometimes also for the Lie algebras of the form $\hat{\mathcal{L}} = \mathcal{L} \oplus C \oplus D$ for an appropriate subalgebra D of derivations, e.g. in [DFP].

Thus, there are many possibilities for C in the diagram (1.14), and it is not clear which one is the best possible choice. (In fact, we will later allow any central extension).

Assuming that we have settled for some C, which D should we take? For simplicity we will discuss this only in the untwisted case. If $n = 1$ we added the degree derivation d described in (1.8). This is far from being an arbitrary derivation. The full derivation algebra of the Lie algebra $\mathfrak{g} \otimes A$ for A is described in [BM, Th. 1]:

$$\begin{aligned} \mathrm{Der}_F(\mathfrak{g} \otimes A) &= \big(\mathrm{Der}_F(\mathfrak{g}) \otimes A\big) \oplus \big(F\,\mathrm{Id} \otimes \mathrm{Der}_F(A)\big) \\ &= \quad \mathrm{IDer}(\mathfrak{g} \otimes A) \oplus F\,\mathrm{Id} \otimes \mathrm{Der}_F(A). \end{aligned} \tag{1.18}$$

where $\mathrm{Der}_F(\mathfrak{g}) \otimes A$ and $F \otimes \mathrm{Der}_F(A) = F\,\mathrm{Id} \otimes \mathrm{Der}_F(A)$ act on $\mathfrak{g} \otimes A$ in the obvious way.

Since $\mathfrak{g} \otimes A$ is perfect, up to a canonical isomorphism, this is then also the derivation algebra of the universal central extension of $\mathfrak{g} \otimes A$ (see for example [BM, Th. 2.2]). From

$$\mathrm{Der}\, F[t^{\pm 1}] = F[t^{\pm 1}]d$$

we see that we added a rather special derivation, one which can be used to define the Λ-grading of $\mathcal{L}$ (see also Ex. 1.7).

We can do something similar in multi-variable case. Define the i-th *degree derivation* ∂_i of $L(\mathfrak{g}) \oplus C$ by

$$\partial_i(u \otimes t^\lambda \oplus c) = \lambda_i\, u \otimes t^\lambda \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda = \mathbb{Z}^n \tag{1.19}$$

and put

$$\mathcal{D} = \mathrm{span}_F\{\partial_i : 1 \le i \le n\},$$

the space of *degree derivations*. Possible (interesting) choices for D are:

(1) $D = \mathcal{D}$,

(2) $F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\mathcal{D}$ (in physics parlance: "all vector fields"), and

(3) $\bigoplus_{\lambda\in\Lambda} Ft^\lambda\{\sum_{i=1}^n s_i\partial_i : \sum_i s_i = 0\}$ (the "divergence 0 vector fields").

It will turn out that for the Lie algebras which we are going to study in the next chapters, the choices (1) and (3) are the correct ones. In addition, there will be a surprise: semidirect products in (1.14) will not be enough!

Exercise 1.6. Recall that a bilinear form $(\cdot|\cdot)$ on a Lie algebra L is called *invariant* if $([l_1, l_2] \mid l_3) = (l_1 \mid [l_2, l_3])$ holds for all $l_i \in L$. Show:

(a) The set $\mathrm{IF}(L)$ of invariant bilinear forms on L is a vector space with respect to the obvious scalar multiplication and addition defined by $(\beta_1 + \beta_2)(l_1, l_2) = \beta_1(l_1, l_2) + \beta_2(l_1, l_2)$ for $\beta_i \in \mathrm{IF}(L)$.

(b) If L is perfect, any invariant bilinear form is symmetric.

(c) Let S be a unital associative F-algebra. A bilinear form b on S is called *invariant* if $b(s_1s_2, s_3) = b(s_1, s_2s_3) = b(s_2, s_3s_1)$ for $s_i \in S$.

- (i) The set $\mathrm{IF}(S)$ of invariant bilinear forms on S is a vector space with respect to the obvious operations.
- (ii) Any linear form $\lambda \in S^*$ with $\lambda([S, S]) = 0$ gives rise to an invariant bilinear form b_λ on S, defined by $b_\lambda(s_1, s_2) = \lambda(s_1s_2)$.
- (iii) The map $(S/[S, S])^* \to \mathrm{IF}(S)$, given by $\lambda \mapsto b_\lambda$, is a vector space isomorphism.

(d) Let L be a perfect Lie algebra with a 1-dimensional space $\mathrm{IF}(L)$, say $\mathrm{IF}(L) = F\kappa$. Also, let S be a unital associative commutative F-algebra. We consider $L \otimes S$ as Lie algebra with respect to the product $[l_1 \otimes s_1, l_2 \otimes s_2] = [l_1, l_2] \otimes s_2s_2$, cf. (1.5). For $\lambda \in \mathrm{IF}(S)$ define a bilinear form $\kappa \otimes \lambda$ on $L \otimes S$ by

$$(\kappa \otimes \lambda)\,(l_1 \otimes s_2,\, l_2 \otimes s_2) = \kappa(l_1, l_2)\,\lambda(s_1, s_2).$$

Then $\kappa \otimes \lambda \in \mathrm{IF}(L \otimes S)$ and the map $\mathrm{IF}(S) \to \mathrm{IF}(L \otimes S)$, given by $\lambda \mapsto \kappa \otimes \lambda$, is an isomorphism of vector spaces.

Exercise 1.7. Define the i$^{\text{th}}$ degree derivation ∂_i of the Laurent polynomial ring $S = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ by $\partial_i(t^\lambda) = \lambda_i t^\lambda$, so that the ∂_i of (1.19) becomes $\partial_i(u \otimes t^\lambda) = u \otimes \partial_i(t^\lambda) = (\mathrm{Id} \otimes \partial_i)(u \otimes t^\lambda)$ (this double meaning of ∂_i should not create any confusion). Show:

(a) The derivation algebra $\mathrm{Der}_F(S)$ of S is given by

$$\mathrm{Der}_F(S) = S\mathcal{D} = \textstyle\bigoplus_{\lambda \in \mathbb{Z}^n} Ft^\lambda \mathcal{D}$$

where, as above, $\mathcal{D} = \mathrm{span}_F\{\partial_i : 1 \le i \le n\}$. The derivation algebra is a $\mathbb{Z}^n$-graded Lie algebra with Lie algebra product determined by

$$[t^\lambda \partial_i, t^\mu \partial_j] = t^{\lambda+\mu}(\mu_i \partial_j - \lambda_j \partial_i).$$

Thus, for $n = 1$ we obtain the usual *Witt algebra*, see for example [MP, 1.4].

(b) $(t^\lambda \mid t^\mu) = \delta_{\lambda+\mu,0}$ defines a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ on S which is *invariant* in the sense that $(ab|c) = (a \mid bc)$ for all $a, b, c \in S$.

(c) Let $\mathrm{SDer}_F(S)$ be the subalgebra of derivations of S, which are skew-symmetric with respect to the form $(\cdot|\cdot)$ of (b). Then

$$\mathrm{SDer}_F(S) = \textstyle\bigoplus_{\lambda \in \mathbb{Z}^n} Ft^\lambda \big\{ \sum_{i=1}^n s_i \partial_i : \sum_i s_i \lambda_i = 0 \big\}.$$

In particular, for $n = 1$ we get $\mathrm{SDer}_F(F[t^{\pm 1}]) = Fd$ for $d = \partial_1$.

1.3. **Appendix on central extensions of Lie algebras.** Central extensions will turn out to be an important tool in the construction of extended affine Lie algebras. Although this provides one with a bigger and hence potentially more complicated Lie algebra, central extensions turn up naturally in the general theory and the biggest of them (the universal central extension) is in fact quite "nice". For example, universal central extensions often have a simpler presentation and a much richer representation theory than the original Lie algebra. In this appendix we review the necessary background.

Definition 1.8 (Extensions)**.** An *extension of a Lie algebra* L is a surjective homomorphism $f : K \to L$ of Lie algebras. A *homomorphism* from an extension $f : K \to L$ to another extension $f' : K' \to L$ is a Lie algebra homomorphism $g : K \to K'$ satisfying $f = f' \circ g$. In other words, the diagram below is commutative.

$$\begin{tikzcd} K \arrow[rr, "g"] \arrow[dr, "f"'] & & K' \arrow[dl, "f'"] \\ & L & \end{tikzcd} \tag{1.20}$$

We will use *abelian extensions*, i.e., extensions $f : K \to L$ with $\operatorname{Ker} f$ an abelian ideal in the construction of an extended affine Lie algebra in section 5.4.

Definition 1.9 (central extensions)**.** A *central extension* of L is an extension $f : K \to L$ whose kernel $\operatorname{Ker} f$ is contained in the centre $Z(K)$ of K. A central extension $f : K \to L$ is called a *covering* if K is perfect, i.e., $K = [K, K]$. It is traditional (but not always advisable) to not specify the morphism f and simply say that K *is a central extension of* L or a *covering*.

A central extension $\mathfrak{u} : \mathfrak{L} \to L$ is called a *universal central extension* if there exists a unique homomorphism from $\mathfrak{u} : \mathfrak{L} \to L$ to any other central extension $f : K \to L$ of L. It is obvious from the universal property that two universal central extensions of L are isomorphic as central extensions and hence in particular their underlying Lie algebras are isomorphic. We denote the universal central extension of L by $\mathfrak{u} : \mathfrak{uce}(L) \to L$ or simply $\mathfrak{uce}(L)$.

Theorem 1.10 ([vdK, Prop. 1.3], [G, §1])**.** *A Lie algebra L has a universal central extension if and only if L is perfect. In this case, the universal central extension $\mathfrak{u} : \mathfrak{uce}(L) \to L$ is perfect too, i.e., $\mathfrak{u}$ is a covering.*

The process of taking universal central extensions stops at $\mathfrak{uce}(L)$, due to the following equivalent conditions for a Lie algebra L:

(i) $\mathrm{Id} : L \to L$ is a universal central extension, i.e., $\mathfrak{uce}(L) = L$,

(ii) every central extension $f : K \to L$ is direct product $K = \tilde{L} \times \operatorname{Ker} f$ such that $f|_{\tilde{L}}$ is an isomorphism between $\tilde{L}$ and L.

If (i) and (ii) hold, one calls L *centrally closed*.

Examples 1.11. (a) It is an immediate corollary of the Levi-Malcev Theorem that every finite-dimensional semisimple Lie algebra is centrally closed ([Bou3, VII, §6.8, Cor. 3] or [We, Cor. 7.9.5]).

(b) An example of a universal central extension is the Virasoro algebra: It is the universal central extension of the Witt algebra $\operatorname{Der}_F(F[t^{\pm 1}])$, see for example [MP, I.9, Prop. 4]. Hence the Virasoro algebra is centrally closed, while $\mathrm{Der}_F(F[t^{\pm 1}])$ is not. On the other hand, the higher rank Witt algebra $\mathrm{Der}_F(F[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$, $n > 1$, is centrally closed ([RSS, V, Th. 5.1]).

Definition 1.12 (Central extensions via 2-cocycles.)**.** We have already seen in §1.1 that one can construct central extensions of a Lie algebra L by using 2-*cocycles*, which, we recall, are bilinear maps $\psi : L \times L \to C$ into a vector space C satisfying for all $l, l_1, l_2, l_3 \in L$

$$\psi(l,l) = 0 \quad \text{and} \quad \psi([l_1,l_2],l_3) + \psi([l_2,l_3],l_1) + \psi([l_3,l_1],l_2) = 0. \tag{1.21}$$

The first equation is of course equivalent to $\psi(l_1,l_2) = -\psi(l_2,l_1)$. Given a 2-cocycle $\psi : L \times L \to C$, the algebra

$$K = L \oplus C \quad \text{by} \quad [l_1 \oplus c_1,\, l_2 \oplus c_2]_K = [l_1,l_2]_L \oplus \psi(l_1,l_2) \tag{1.22}$$

($l_i \in L$, $c_i \in C$) is a Lie algebra and $\mathrm{pr}_L : K \to L$, $\mathrm{pr}_L(l \oplus c) = l$, is a central extension of L, which we will denote by $\mathrm{E}(L, C, \psi)$ or $\mathrm{E}(L, \psi)$ for short.

Conversely, given a central extension $f : K \to L$, let $s : L \to K$ be a *section of* f in the category of vector spaces, i.e. a linear map $s : L \to K$ such that $f \circ s = \mathrm{Id}_L$. Such a section always exists: We can choose a subspace L' of K, which is complementary to $C = \mathrm{Ker}\, f$, and take $s = (f|_{L'})^{-1}$ which makes sense since $(f|L') : L' \to L$ is an invertible linear map (but in general not a Lie algebra homomorphism since L' need not be a subalgebra). Given a section s, the map

$$\psi_s : L \times L \to C, \quad \psi_s(l_1,l_2) = [s(l_1), s(l_2)]_K - s([l_1,l_2]_L) \tag{1.23}$$

turns out to be a 2-cocycle. Moreover, the map

$$K \to L \oplus C, \quad x \mapsto f(x) \oplus \big(x - (s \circ f)(x)\big) = f(x) \oplus x_C,$$

where x_C is the C-component of $x \in K$, is an isomorphism from the central extension $f : K \to L$ to the central extension $\mathrm{E}(L, \mathrm{Ker}\, f, \psi_s)$. To summarize, modulo some verifications left as an exercise, we have proven the following well-known result.

Proposition 1.13. *For any* 2*-cocycle* ψ *the construction* (1.22) *is a central extension* $\mathrm{E}(L,\psi)$ *of* L *and, conversely, every central extension* L *is isomorphic as central extension to some* $\mathrm{E}(L,\psi)$*.*

Exercise 1.14. Let $\psi : L \times L \to C$ be a 2-cocycle and let C' be a subspace of C satisfying $\psi(L,L) := \mathrm{span}_F\{\psi(l_1,l_2) : l_i \in L\} \subset C'$. Then $\mathrm{E}(L, C', \psi)$ is also a central extension, and if $\mathrm{E}(L, C, \psi)$ is a covering then $C = \psi(L,L)$.

Examples 1.15. (a) Any Lie algebra L has many uninteresting central extensions. One can simply take the direct product of L with an abelian Lie algebra, i.e., $L \times C$ with product $[(l_1,c_1), (l_2,c_2)] = ([l_1,l_2], 0)$ for $l_i \in L$, $c_i \in C$, and consider the canonical projection $\mathrm{pr}_L : L \times L \to L$, which is a central extension (but not a covering, unless L is perfect and $C = \{0\}$). Observe that the canonical inclusion $\mathrm{inc} : L \to L \times C$ is a section of pr_L, not only in the category of vector spaces, but even in the category of Lie algebras. Its associated 2-cocycle $\psi_{\mathrm{inc}} = 0$.

(b) Let $h : L \to C$ be a linear map into some vector space C. Then $\beta_h : L \times L \to C$, $\beta_h(l_1,l_2) = h([l_1,l_2])$ is a 2-cocycle, a so-called 2-*coboundary.*

The two examples are related in the following exercise.

Exercise 1.16. For a central extension $f : K \to L$ of L with $C = \operatorname{Ker} f$ the following are equivalent:

(i) The extension $f : K \to L$ is split in the category of Lie algebras, i.e, there exists a section $L \to K$ of f, which is a Lie algebra homomorphism.
(ii) For any section s of f the associated 2-cocycle ψ_s is a 2-coboundary.
(iii) There exists a section s of f, for which the associated 2-cocycle ψ_s is a 2-coboundary.
(iv) As central extension, f is isomorphic to the central extension $\mathrm{pr}_L : L \oplus C \to L$.

If these conditions are fulfilled, one calls f a *split extension.*

Exercise 1.17. Let $\psi : L \times L \to C$ be a 2-cocycle and let $\pi : C \to C'$ be a linear map. Show:

(a) $\psi' = \pi \circ \psi$ is a 2-cocycle of L and the map

$$\mathrm{E}(\pi) : \mathrm{E}(L, C, \psi) \to \mathrm{E}(L, C', \psi'), \quad l \oplus c \mapsto l \oplus \pi(c)$$

is a homomorphism of central extensions of L:

$$\begin{tikzcd} \mathrm{E}(L,C,\psi) \arrow[rr, "\mathrm{E}(\pi)"] \arrow[dr, "\mathrm{pr}_L"'] & & \mathrm{E}(L,C',\psi') \arrow[dl, "\mathrm{pr}_L"] \\ & L & \end{tikzcd}$$

(b) If π is surjective, the map $\mathrm{E}(\pi)$ is a central extension of $L' = \mathrm{E}(L, C', \pi \circ \psi)$, which as central extension of L' has the form $\mathrm{E}(L', C'', \psi'')$ for

$$\psi''(l_1 \oplus c_1', l_2 \oplus c_2') = \big((\mathrm{Id} - \gamma \circ \pi) \circ \psi\big)(l_1, l_2),$$

where $\gamma : C' \to C$ is a section of π with $\gamma(C') = C''$.

(c) Conversely, suppose $f' : L' \to L$ is a central extension and $f : \mathrm{E}(L, C, \psi) \twoheadrightarrow L'$ is a surjective homomorphism of central extensions. Then $\pi = f|C$ maps C onto $C' = \operatorname{Ker} f'$ and there exists a unique isomorphism of extensions $\Phi : L' \to \mathrm{E}(L, C', \psi')$, $\psi' = \pi \circ \psi$ such that all triangles in the diagram below commute:

$$\begin{tikzcd} \mathrm{E}(L,C,\psi) \arrow[rr, "f"] \arrow[dr, "\mathrm{E}(\pi)"] \arrow[ddr, "\mathrm{pr}_L"'] & & L' \arrow[dl, "\Phi"'] \arrow[ddl, "f'"] \\ & \mathrm{E}(L,C',\psi') \arrow[d, "\mathrm{pr}_L"] & \\ & L & \end{tikzcd}$$

Exercise 1.18. Let $C = C_1 \oplus C_2$ be a vector space direct sum and denote by $\pi_i : C \to C_i$ the canonical projections. Let $\psi : L \times L \to C$ be a 2-cocycle with the property that $\pi_2 \circ \psi$ is a 2-coboundary. Then for $\psi_1 = \pi_1 \circ \psi$,

$$\mathrm{E}(L, C, \psi) \cong \mathrm{E}(L, C_1, \psi_1) \times C_2$$

as central extensions of L (even as central extensions of the Lie algebra $\mathrm{E}(L, C_1, \psi_1)$).

Example 1.19. Let $(\cdot|\cdot) : L \times L \to F$ be a symmetric bilinear form, which is invariant, see Ex. 1.6. We denote by $\mathrm{SDer}_F(L)$ the subalgebra of $\mathrm{Der}_F(L)$ which consists of all skew-symmetric derivations, where a derivation $d \in \mathrm{Der}_F(L)$ is called *skew-symmetric* if $(d(l_1) \mid l_2) + (l_1 \mid d(l_2)) = 0$ for all $l_1, l_2 \in L$. Observe that

$$\mathrm{IDer}(L) = \{\mathrm{ad}\, l : l \in L\} \triangleleft \mathrm{SDer}_F(L).$$

Let D be a subspace of $\mathrm{SDer}_F(L)$ and let D^* be its dual space. Then the map $\psi_D : L \times L \to D^*$, given by

$$\psi_D(l_1, l_2)(d) = \big(d(l_1) \mid l_2\big) \tag{1.24}$$

for $l_i \in L$ and $d \in D$, is a 2-cocycle of L.

Exercise 1.20. Show: (a) (1.24) defines indeed a 2-cocycle.

(b) ψ_D for $D \subset \mathrm{IDer}(L)$ is a 2-coboundary.

(c) If $\tilde{D}$ is a subspace of D, then the central extension $\mathrm{E}(L, D^*, \psi_D)$ of L factors through the central extension $\mathrm{E}(L, \tilde{D}^*, \psi_{\tilde{D}})$ of L,

$$\mathrm{E}(L, D^*, \psi_D) \twoheadrightarrow \mathrm{E}(L, \tilde{D}^*, \psi_{\tilde{D}}) \twoheadrightarrow L.$$

Definition 1.21 (Graded central extensions)**.** Let Λ be an abelian group, and let $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ be a Λ-graded Lie algebra. We say that $f : K \to L$ is a Λ-*graded central extension of* L if K is a Λ-graded Lie algebra and f is a central extension which is at the same time a homomorphism of Λ-graded algebras: $f(K^\lambda) \subset L^\lambda$ for all $\lambda \in \Lambda$. A Λ-graded central extension $f : K \to L$ is called a Λ-*covering*, if f is a covering, i.e., K is perfect. We note that an arbitrary central extension of a graded Lie algebra need not be a graded central extension.

A *homomorphism* of a Λ-graded central extension $f : K \to L$ to another Λ-graded central extension $f' : K' \to L$ is a homomorphism $g : K \to K'$ of Λ-graded Lie algebras satisfying $f = f' \circ g$, cf. 1.20.

To define graded central extensions of a Λ-graded Lie algebra L via a 2-cocycle, we need (obviously) a Λ-*graded* 2-*cocycle*, i.e., a 2-cocycle $\psi : L \times L \to C$ into a Λ-graded vector space $C = \bigoplus_{\lambda\in\Lambda} C^\lambda$ which is graded of degree 0,

$$\psi(L^\lambda, L^\mu) \subset C^{\lambda+\mu} \quad \text{for all } \lambda, \mu.$$

For a graded 2-cocycle ψ the Lie algebra $K = L \oplus C$ of (1.22) is naturally Λ-graded by

$$K^\lambda = L^\lambda \oplus C^\lambda$$

and the central extension $\mathrm{pr}_L : K \to L$ is a Λ-graded central extension. Conversely, if $f : K \to L$ is a Λ-graded central extension, we can choose a section $s : L \to K$ of the underlying vector spaces of degree 0, meaning $s(L^\lambda) \subset K^\lambda$. The 2-cocycle associated to s in (1.23) is then a graded 2-cocycle. Thus, Prop. 1.13 holds in an analogous way for graded central extensions.

The following proposition also shows that one does not have to introduce a new object of a "graded universal central extension".

Proposition 1.22 ([Ne2, 1.16])**.** *Let $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ be a Λ-graded perfect Lie algebra. Then its universal central extension $\mathfrak{u} : \mathfrak{uce}(L) \to L$ is Λ-graded, hence a Λ-covering. Moreover,* $\mathrm{Ker}\,\mathfrak{u}$ *is a graded subspace of* $\mathfrak{uce}(L)$.

Example 1.23. We also have the graded versions of the Example 1.15 (details left to the reader) and the Example 1.19, whose details follow.

Let $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ be a Λ-graded Lie algebra and let $(\cdot|\cdot)$ be an invariant bilinear form on L, which is Λ-*graded* in the following sense:

$$(L^\lambda \mid L^\mu) = 0 \quad \text{if } \lambda + \mu \neq 0.$$

We define the Λ-graded subalgebra of $\mathrm{grEnd}_F(L)$

$$\mathrm{grSDer}_F(L) = \mathrm{grEnd}_F(L) \cap \mathrm{SDer}_F(L) = \textstyle\bigoplus_{\lambda\in\Lambda} \big(\mathrm{SDer}_F(L)\big)^\lambda \tag{1.25}$$

where $(\mathrm{SDer}_F(L))^\lambda$ consists of all skew-symmetric derivations of degree λ. If $D \subset \mathrm{grSDer}_F(L)$ is a graded subspace of $\mathrm{grSDer}_F(L)$, the 2-cocycle ψ_D of (1.24) is Λ-graded and maps $L \times L$ into $D^{\mathrm{gr}*}$, thus giving rise to a graded central extension $\mathrm{E}(L, D^{\mathrm{gr}*}, \psi_D)$ of L.

Exercise 1.24. Show that the 2-cocycles ψ of (1.5), (1.16) and (1.17) can be obtained in the form (1.24), i.e., find an invariant bilinear form on $\mathcal{L} = L(\mathfrak{g}, \sigma)$ resp. $\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma})$ and a subspace $D \subset \mathrm{SDer}_F(\mathcal{L})$ such that ψ and ψ_D yield isomorphic central extensions of $\mathcal{L}$.

It is not so surprising that the 2-cocycles we used in sections 1.1 and 1.2 can all be obtained in the form ψ_D for $D \subset \mathrm{grSDer}_F(L)$. This is a special case of the following general result.

Theorem 1.25 ([Ne6])**.** *Let $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ be a Λ-graded Lie algebra, which*

(i) *is perfect and finitely generated as Lie algebra,*
(ii) *has finite homogeneous dimension:* $\dim L^\lambda < \infty$ *for all* $\lambda \in \Lambda$*, and*
(iii) *has an invariant nondegenerate Λ-graded symmetric bilinear form.*

(a) *Then* $\mathrm{Der}_F(L) = \mathrm{grDer}_F(L)$ *is Λ-graded and has finite homogeneous dimension, whence the same is true for* $\mathrm{SDer}_F(L)$.

(b) *The universal central extension* $\mathfrak{uce}(L)$ *has finite homogenous dimension with respect to the Λ-grading of* 1.22*. Moreover,*

$$\mathfrak{uce}(L) \cong \mathrm{E}(L, D^{\mathrm{gr}*}, \psi_D)$$

as central extensions of L, where D is any graded subspace of $\mathrm{SDer}_F(L)$ *which complements* $\mathrm{IDer}(L)$ *in* $\mathrm{SDer}_F(L)$*, and ψ_D is the* 2*-cocycle of* (1.24).

Remarks 1.26. (a) Th. 1.5 is an application of Th. 1.25, as is Th. 4.13(c).

(b) The Exercise 1.18 gives some indication why it is sufficient to take a subspace of $\mathrm{SDer}_F(L)$ complementing $\mathrm{IDer}(L)$ and not an arbitrary subspace of $\mathrm{SDer}_F(L)$.

Exercise 1.27. In the setting of Th. 1.25, every Λ-graded central covering of L is isomorphic as central extension to a central extension $\mathrm{E}(L, B^{\mathrm{gr}*}, \psi_B)$ for some graded subspace B of D.

2. Extended affine Lie algebras: Definition and first examples

Rather than constructing Lie algebras in a concrete way as we have done in Lecture 1, in this chapter we will define extended affine Lie algebras by a set of axioms and give examples. We will see that these examples encompass all the examples of Lecture 1 (with the exception of the choice 2. for D in 1.2).

As before we will consider Lie algebras over an arbitrary field F of characteristic 0, but we will no longer assume that F has enough roots of unity (multiloop algebras will not be play a role here), except in §2.4 where $F = \mathbb{C}$).

2.1. **Definition of an extended affine Lie algebra.** An *extended affine Lie algebra*, or EALA for short, is a pair (E, H) consisting of a Lie algebra E over F and subalgebra H satisfying the following axioms (EA1) – (EA6).

(EA1): *E has an invariant nondegenerate symmetric bilinear form* $(\cdot|\cdot)$.

(EA2): *H is nontrivial finite-dimensional toral and self-centralizing subalgebra of E.*

Before we can state the other four axioms, we need to draw some consequences of the axioms (EA1) and (EA2). But first we give explanations of some of the notions used. The term *invariant* (= *associative*) means that $(\cdot|\cdot)$ satisfies $([e_1, e_2] \mid e_3) = (e_1 \mid [e_2, e_3])$ for all $e_i \in E$, and $(\cdot|\cdot)$ is *nondegenerate* if $(e \mid E) = 0 \implies e = 0$. In the context of above, a *toral subalgebra*, sometimes also called an ad-*diagonalizable subalgebra* is a subalgebra H which induces a decomposition of E via the adjoint representation of H:

$$\begin{aligned} E &= \textstyle\bigoplus_{\alpha \in H^*} E_\alpha, \\ E_\alpha &= \{e \in E : [h, e] = \alpha(h)e \text{ for all } h \in H\}. \end{aligned} \tag{2.1}$$

Such a subalgebra is necessarily abelian, whence $H \subset E_0 = \{e \in E : [h, e] = 0 \text{ for all } h \in H\}$. That H is also required to be *self-centralizing* means

$$H = E_0.$$

Now to the consequences of (EA1) and (EA2). Because of invariance of the bilinear form $(\cdot|\cdot)$, we have

$$(E_\alpha \mid E_\beta) = 0 \quad \text{if } \alpha + \beta \neq 0, \tag{2.2}$$

in particular the restriction of the bilinear form to $E_0 = H$ is nondegenerate. Because of this and finite-dimensionality of H, every linear form $\alpha \in H^*$ is represented by a unique $t_\alpha \in H$, defined by the condition that $(t_\alpha \mid h) = \alpha(h)$ holds for all $h \in H$. This allows us to transport the restricted form $(\cdot|\cdot) \mid H \times H$ to a symmetric bilinear form on H^*, also denoted $(\cdot|\cdot)$ and defined by

$$(\alpha \mid \beta) = (t_\alpha \mid t_\beta), \quad \alpha, \beta \in H^*. \tag{2.3}$$

This transport of bilinear forms is a standard procedure in the theory of semisimple Lie algebras, see for example [Hu, §8]. We can now define

$$\begin{aligned} R &= \{\alpha \in H^* : E_\alpha \neq 0\} \quad (\textit{set of roots of } (E, H)), \\ R^0 &= \{\alpha \in R : (\alpha \mid \alpha) = 0\} \quad (\textit{null roots}), \\ R^{\mathrm{an}} &= \{\alpha \in R : (\alpha \mid \alpha) \neq 0\} \quad (\textit{anisotropic roots}). \end{aligned} \tag{2.4}$$

We prefer to call R the set of roots of (E, H) and not the "root system" since we want to restrict the latter term for root systems in the usual sense, see 3.3. We point out that by definition 0 is a root,

$$0 \in R^0 \subset R.$$

This is the customary convention for EALAs and has some notational advantages.

We define the *core of* (E, H) as the subalgebra E_c of E generated by all anisotropic root spaces:

$$E_c = \langle \textstyle\bigcup_{\alpha \in R^{\mathrm{an}}} E_\alpha \rangle_{\mathrm{subalg}}$$

We can now state the remaining four axioms.

(EA3): *For every $\alpha \in R^{\mathrm{an}}$ and $x_\alpha \in E_\alpha$, the operator* $\operatorname{ad} x_\alpha$ *is locally nilpotent on E.*

(EA4): R^{an} *is connected* in the sense that for any decomposition $R^{\mathrm{an}} = R_1 \cup R_2$ with $(R_1 \mid R_2) = 0$ we have $R_1 = \emptyset$ or $R_2 = \emptyset$.

(EA5): *The centralizer of the core E_c of E is contained in E_c*: $\{e \in E : [e, E_c] = 0\} \subset E_c$.

(EA6): *The subgroup $\Lambda = \operatorname{span}_{\mathbb{Z}}(R^0) \subset H^*$ generated by R^0 in $(H^*, +)$ is a free abelian group of finite rank.* In other words, $\Lambda \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ (including $n = 0$!).

The term *locally nilpotent* means that for every $e \in E$ there exists an $n \in \mathbb{N}$, possibly depending on e, such that $(\operatorname{ad} x_\alpha)^n(e) = 0$. The property (EA5) is called *tameness*. The condition $[e, E_c] = 0$ is of course equivalent to $[e, E_\alpha] = 0$ for all $\alpha \in R^{\mathrm{an}}$. The rationale for this axiom is the following. The subalgebra E_c is in fact an ideal of E (Th. 4.14). Hence we have a representation ρ of E on E_c, given by $\rho(e)(x_c) = [e, x_c]$ for $e \in E$ and $x_c \in E_c$. The kernel of the representation ρ is the centralizer of E_c in E. Hence tameness means that $\operatorname{Ker} \rho \subset E_c$. The idea here is that the core E_c should control E. We will make this more precise in section 5.4. The rank of the free abelian group Λ in axiom (EA6) is called the *nullity* of (E, H). It is invariant under isomorphisms. We will describe EALAs of nullity 0 and 1 below.

Although the structure of an EALA requires the existence of an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ in the axiom (EA1), which is then used to define the anisotropic roots, it turns out that this bilinear form is really not so important. Because of this, we have defined an EALA as a pair (E, H) and not as a triple $(E, H, (\cdot|\cdot))$ as it is for example done in [AF]. Consequently, an *isomorphism* from an EALA (E, H) to another EALA (E', H') is a Lie algebra isomorphism $f : E \to E'$ such that $f(H) = H'$. It is immediate that any isomorphism induces a bijection f' between the set of roots R and R' of (E, H) and (E', H') respectively. It then follows that f' maps R^{an} onto R'^{an}, whence also R^0 onto R'^0. One can then show that f' preserves the forms on $X = \operatorname{span}_F(R)$ and $X' = \operatorname{span}_F(R')$ up to scalars.

For $F = \mathbb{C}$ one can define a special class of EALAs. We call a pair (E, H) a *discrete EALA* if it satisfies the axioms (EA1) – (EA5) and in addition

(DE): R is a discrete subset of H^* with respect to the natural topology of the finite-dimensional complex vector space H^*.

It is justified to call a discrete EALA an EALA, since one can show that a discrete EALA also satisfies (EA6). Indeed, this follows from Prop. 3.21 and Th. 3.22. However, not every EALA over $\mathbb{C}$ is a discrete EALA (see [Ne5, 6.17]).

Some historical comments. Although there were some precursors (papers by Saito and Slodowy for nullity 2), it was in the paper [HT] by the physicists Høegh-Krohn and Torrésani that the class of discrete extended affine Lie algebras was introduced, however not under this name. Rather, they were called "irreducible quasi-simple Lie algebras" and later ([BGK, BGKN]) "elliptic quasi-simple Lie algebras". The stated goal of the paper [HT] was applications in quantum gauge theory. The theory developed there did however not stand up to the scrutiny of mathematicians. The errors of [HT] were corrected in the AMS memoir [AABGP] by Allison, Azam, Berman, Gao and Pianzola. There also the name "extended affine Lie algebras" appears for the first time. But not in the sense as defined above. Rather, the authors develop the basic theory of what here are called discrete EALAs. Nevertheless, [AABGP] has become the standard reference even for the more general extended affine Lie algebras, since many of the results presented there for discrete extended affine Lie algebras easily extend to the more general setting. The definition of an extended affine Lie algebra given above is due to the author ([Ne4]) and was motivated by the fact that all the examples presented in [AABGP] did make sense over an arbitrary base field F and not just over $\mathbb{C}$ only. Before [Ne4] the tameness axiom (EA5) was not part of the definition of an EALA. However, as examples show ([BGK, §3] or [Ne5, 6.10]), it seems impossible to classify EALAs without (EA5). After [Ne4], several generalizations of EALAs have been proposed. They are surveyed in [Ne5].

2.2. **Some elementary properties of extended affine Lie algebras.** The following chapters will (hopefully) show that extended affine Lie algebras share many properties with familiar Lie algebras, like finite-dimensional split simple Lie algebras or affine Kac-Moody Lie algebras. Some of these properties are immediate consequences of the axioms. The following (strongly recommended!) exercise gives an incomplete list of such properties.

Exercise 2.1. Let (E, H) be an EALA. We use the notation of above. Show:

(a) For $\alpha, \beta \in R$ we have

$$[E_\alpha, E_\beta] \subset E_{\alpha+\beta}. \tag{2.5}$$

Thus the root space decomposition (2.1) is a grading by the abelian group $\mathrm{span}_{\mathbb{Z}}(R)$.

(b) H is a *Cartan subalgebra*, defined as a nilpotent subalgebra which is self-normalizing: $H = \{e \in E : [e, H] \subset H\}$.

(c) For $\alpha, \beta \in R$ we have $(E_\alpha \mid E_\beta) = 0$ unless $\alpha + \beta = 0$. The restriction of the bilinear form $(\cdot|\cdot)$ to $E_\alpha \times E_{-\alpha}$ is nondegenerate, i.e., if $x_\alpha \in E_\alpha$ satisfies $(x_\alpha \mid E_{-\alpha}) = 0$ then $x_\alpha = 0$. In particular, $R = -R$.

(d) For $\alpha \in R$ and $x_\alpha \in E_\alpha$ and $y_{-\alpha} \in E_{-\alpha}$,

$$[x_\alpha,\, y_{-\alpha}] = (x_\alpha \mid y_{-\alpha})\, t_\alpha. \tag{2.6}$$

In particular, $[E_\alpha, E_{-\alpha}] = Ft_\alpha$, and if $\alpha \in R^{\mathrm{an}}$ then

$$[[E_\alpha, E_{-\alpha}],\, E_\alpha] = E_\alpha. \tag{2.7}$$

(e) The core E_c satisfies

$$E_c = \Big(\oplus_{\alpha \in R^{\mathrm{an}}} E_\alpha \Big) \oplus \Big(\bigoplus\nolimits_{\alpha \in R^0} (E_c \cap E_\alpha) \Big). \tag{2.8}$$

We will now show that EALAs are built out of "little" $\mathfrak{sl}_2$'s and Heisenberg's (albeit in a complicated way).

Proposition 2.2. *Let (E, H) be an extended affine Lie algebra, with anisotropic root R^{an} and null roots R^0.*

(a) *Let $\alpha \in R^{\mathrm{an}}$. Then* $\dim E_\alpha = 1$*, and for any $e_\alpha \in E_\alpha$ there exists $f_\alpha \in E_{-\alpha}$ such that $(e_\alpha, h_\alpha = [e_\alpha, f_\alpha], f_\alpha) \in E_\alpha \times H \times E_{-\alpha}$ is an $\mathfrak{sl}_2$-triple:*

$$E_\alpha \oplus [E_\alpha, E_{-\alpha}] \oplus E_{-\alpha} = Fe_\alpha \oplus Fh_\alpha \oplus Ff_\alpha \cong \mathfrak{sl}_2(F).$$

(b) *Let* $\alpha \in R^0$. *Then for any* $0 \neq x_\alpha \in E_\alpha$ *there exists* $y_\alpha \in E_{-\alpha}$ *such that* $[x_\alpha, y_\alpha] = t_\alpha$ *and*

$$Fx_\alpha \oplus Ft_\alpha \oplus Fy_\alpha \cong \mathfrak{h}_3,$$

the 3*-dimensional Heisenberg algebra.*

It is not true that $\dim E_\alpha = 1$ if $\alpha \in R^0$ (this is already not true in the examples of sections 2.3 and 2.4). But we will show in Th. 4.18 that all root spaces E_α are finite-dimensional in a rather strong way.

The following exercise shows that one can "extend" the 3-dimensional Heisenberg subalgebras in (b) above.

Exercise 2.3. In the setting and notation of Prop. 2.2(b) show that there exists $d_\alpha \in H$ such that

$$[d_\alpha, x_\alpha] = x_\alpha \quad \text{and} \quad [d_\alpha, y_\alpha] = -y_\alpha.$$

Hence

$$Fx_\alpha \oplus Ft_\alpha \oplus Fd_\alpha \oplus Fy_\alpha$$

is a 4-dimensional subalgebra. It is 2-step solvable, not nilpotent and isomorphic to the subalgebra

$$\left\{ \left(\begin{smallmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{smallmatrix}\right) : a, b, c, d \in F \right\}$$

of $\mathfrak{gl}_3(F)$.

And now an exercise which implies that in an EALA one can produce many so-called elementary automorphisms.

Exercise 2.4. Let M be an F-vector space. Once calls an endomorphism $f \in \mathrm{End}_F(M)$ *locally nilpotent* if for every $m \in M$ there exists $n \in \mathbb{N}$, possibly depending on m, such that $f^n(m) = 0$.

(a) Show that the following conditions are equivalent for $f \in \mathrm{End}_F(M)$:

- (i) f is locally nilpotent,
- (ii) for every finitely spanned subspace N of M there exists a finite-dimensional subspace P of M such that $N \subset P$ and $f(P) \subset P$,
- (iii) f is nilpotent on every finite-dimensional and f-invariant subspace of M.

(b) Let $f \in \mathrm{End}_F(M)$ be locally nilpotent and define the *exponential* $\exp f$ *of* f by

$$(\exp f)(m) = \textstyle\sum_{n \in \mathbb{N}} \frac{1}{n!} f^n(m),$$

for $m \in M$ (note that the sum on the right is always finite). Show that $\exp f$ is an invertible endomorphism of M with inverse given by $(\exp f)^{-1} = \exp(-f)$.

(c) Let L be a Lie algebra and let d be a locally nilpotent derivation of L. Show that then $\exp d$ is an automorphism of L.

We will next present some examples of EALAs.

2.3. **Extended affine Lie algebras of nullity** 0. Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$, for example $\mathfrak{sl}_l(F)$ or a finite-dimensional simple Lie algebra over an algebraically closed field. We will show that then $(\mathfrak{g}, \mathfrak{h})$ is an EALA of nullity 0. The facts needed to prove this can be found in [Bou3, VIII, §2] or in [Hu, §8] for F algebraically closed.

(EA1) Up to a scalar, there exists only one invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, the Killing form κ. Hence we can (and will) take $(\cdot|\cdot) = \kappa$.

(EA2) By definition of a splitting Cartan subalgebra, the Lie algebra $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \big(\textstyle\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\big), \quad \mathfrak{g}_0 = \mathfrak{h},$$

where Φ is the root system of $(\mathfrak{g}, \mathfrak{h})$ (which is a reduced root system in the usual sense, see 3.3) and where the root spaces $\mathfrak{g}_\alpha$ are defined as in 2.1. Hence the set of roots R of $(\mathfrak{g}, \mathfrak{h})$ is

$$R = \{0\} \cup \Phi. \tag{2.9}$$

It is a basic fact that $\kappa(t_\alpha, t_\alpha) \neq 0$ for $t_\alpha \in \mathfrak{h}$ representing $\alpha \in \Phi$ via $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. Hence, the anisotropic and null roots are

$$R^{\rm an} = \Phi \quad \text{and} \quad R^0 = \{0\}.$$

(EA3) is now obvious: From $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for $\alpha \in \Phi$ and $\beta \in R$ and finite-dimensionality of $\mathfrak{g}$, it is clear that $\operatorname{ad} x_\alpha$ for $x_\alpha \in \mathfrak{g}_\alpha$ is not only locally nilpotent but even (globally) nilpotent.

(EA4) is another way of saying that Φ is an irreducible root system. This is indeed the case and follows from simplicity of $\mathfrak{g}$.

(EA5) We first need to determine the core $\mathfrak{g}_c$ of $\mathfrak{g}$. By definition, $\mathfrak{g}_c$ is the subalgebra of $\mathfrak{g}$ generated by $\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Since $\mathfrak{h} = \sum_{\alpha\in\Phi} [g_\alpha, \mathfrak{g}_{-\alpha}]$ we have

$$\mathfrak{g}_c = \mathfrak{g}.$$

It is now a tautology that (EA5) holds, i.e., that the centralizer of the core $\mathfrak{g}_c$ is contained in $\mathfrak{g}_c = \mathfrak{g}$. Of course, we know even more: The centralizer of the core equals the centre of $\mathfrak{g}$, and is therefore $\{0\}$.

(EA6) We have $\Lambda = \langle R^0 \rangle = \langle \{0\} \rangle = \{0\}$.

We have now shown:

(2.10) *A finite-dimensional split simple Lie algebra is an EALA of nullity* 0.

We will see in Prop. 3.24 that the converse of (2.10) is true too. We thus know all the nullity 0 examples of EALAs, and can therefore focus on the higher nullity examples. We will answer the case of nullity 1 in the next section.

2.4. **Affine Kac-Moody Lie algebras again.** To justify the name extended *affine Lie algebra*, we will now show that any affine Kac-Moody Lie algebra is an extended affine Lie algebra. To do so, we will need some basic facts about affine Kac-Moody Lie algebras. All of them can be found in Kac's book [Kac]. Since this reference uses $\mathbb{C}$ as base field, we will do the same in this section. But everything we say here holds true for arbitrary algebraically closed fields of characteristic 0. Thus we let

$$\mathcal{L} = \textstyle\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\bar n} \otimes \mathbb{C}t^n$$

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}(\mathfrak{g}, \sigma) = \mathcal{L} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the complex Lie algebra described in (1.9) and (1.10). Recall that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$ and σ is a diagram automorphism of $\mathfrak{g}$. We let $m \in \{1,2,3\}$ be the order of σ, and denote the canonical map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by $n \mapsto \bar{n}$. Recall from (1.1) and (1.2) that σ induces a $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, namely

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \cdots \oplus \mathfrak{g}_{\overline{m-1}}$$

where $\mathfrak{g}_{\bar{n}} = \{x \in \mathfrak{g} : \sigma(x) = \zeta^n x\}$ for a primitive mth root of unity ζ. For example, for $m = 2$ we get a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\mathfrak{g}_{\bar{0}} = \{x \in \mathfrak{g} : \sigma(x) = x\}$ and $\mathfrak{g}_{\bar{1}} = \{x \in \mathfrak{g} : \sigma(x) = -x\}$. We identify $\mathfrak{g}_{\bar{0}} \equiv \mathfrak{g}_{\bar{0}} \otimes \mathbb{C}t^0$.

We now verify the axioms (EA1) – (EA5) and (DE) which, we recall, implies (EA6).

(EA1) We let κ be the Killing form of $\mathfrak{g}$ and define a bilinear form $(\cdot|\cdot)$ on $\hat{\mathcal{L}}$, using the notation of (1.10),

$$\begin{aligned} &\big(u_{\bar{\lambda}} \otimes t^{\lambda} \oplus s_1 c \oplus s_1' d \mid v_{\bar{\mu}} \otimes t^{\mu} \oplus s_2 c \oplus s_2' d\big) \\ &\qquad = \kappa(u_{\bar{\lambda}}, v_{\bar{\mu}})\delta_{\lambda,-\mu} + s_1 s_2' + s_2 s_1'. \end{aligned} \tag{2.11}$$

The form is visibly symmetric. The reader is invited in Exercise 2.6 to show that it is in fact an invariant nondegenerate symmetric bilinear form on $\hat{\mathcal{L}}$, as required in (EA1). In anticipation of the later developments, we point out that $(\cdot|\cdot)$ has the following features:

- $\hat{\mathcal{L}}$ is an orthogonal sum of $\mathcal{L}$ and $\mathbb{C}c \oplus \mathbb{C}d$: $\hat{\mathcal{L}} = \mathcal{L} \perp (\mathbb{C}c \oplus \mathbb{C}d)$,
- $\mathbb{C}c \oplus \mathbb{C}d$ is a hyperbolic plane, i.e., $(c \mid c) = 0 = (d \mid d)$ while $(c \mid d) = 1$.
- The Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$ has a nondegenerate symmetric bilinear form ϵ given by $\epsilon(t^{\lambda}, t^{\mu}) = \delta_{\lambda,-\mu}$. It is *invariant* in the sense that $\epsilon(pq, r) = \epsilon(p, qr)$ for $p, q, r \in \mathbb{C}[t^{\pm 1}]$, and is *graded* in the sense that $\epsilon(t^{\lambda}, t^{\mu}) = 0$ unless $\lambda + \mu = 0$. For $\sigma = \mathrm{Id}_{\mathfrak{g}}$, the bilinear form on the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ is simply the tensor product form $\kappa \otimes \epsilon$, and for a general σ the form is obtained by restriction.

(EA2) To construct a subalgebra H as required in axiom (EA2) we start with a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since σ is a diagram automorphism, it leaves $\mathfrak{h}$ invariant. We let

$$\mathfrak{h}_{\bar{0}} = \mathfrak{h} \cap \mathfrak{g}_{\bar{0}} = \{h \in \mathfrak{h} : \sigma(h) = h\}$$

and put

$$H = \mathfrak{h}_{\bar{0}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

One knows that $\mathfrak{g}_{\bar{0}}$ is a simple Lie algebra with Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ ([Kac, Prop. 7.9]). The grading property implies that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{n}}] \subset \mathfrak{g}_{\bar{n}}$ for $n \in \mathbb{Z}$. Hence $\mathfrak{g}_{\bar{0}}$ acts on $\mathfrak{g}_{\bar{n}}$ by the adjoint action. Let $\Delta_{\bar{n}}$ be the set of weights of the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{n}}$ with respect to $\mathfrak{h}_{\bar{0}}$:

$$\begin{aligned} \mathfrak{g}_{\bar{n}} &= \textstyle\bigoplus_{\gamma \in \Delta_{\bar{n}}} \mathfrak{g}_{\bar{n},\gamma} \\ \mathfrak{g}_{\bar{n},\gamma} &= \{x \in \mathfrak{g}_{\bar{n}} : [h_{\bar{0}}, x] = \gamma(h_{\bar{0}})x \text{ for all } h_{\bar{0}} \in \mathfrak{h}_{\bar{0}}\}. \end{aligned}$$

In particular, $\Delta_{\bar{0}} \setminus \{0\}$ is the root system of $\mathfrak{g}_{\bar{0}}$ with respect to $\mathfrak{h}_{\bar{0}}$ and $\mathfrak{h}_{\bar{0}} = \mathfrak{g}_{\bar{0},0}$.

We extend $\Delta_{\bar{n}} \subset \mathfrak{h}_{\bar{0}}^*$ to a linear form on H by zero, i.e., for $\gamma \in \Delta_{\bar{n}}$ we put

$$\gamma(h_{\bar{0}} \oplus sc \oplus s'd) = \gamma(h_{\bar{0}})$$

and define a linear form δ on H by

$$\delta(h_{\bar{0}} \oplus sc \oplus s'd) = s'.$$

Then for $\gamma \in \Delta_{\bar n}$, $n \in \mathbb{Z}$, we have

$$\begin{aligned}\hat{\mathcal{L}}_{\gamma\oplus n\delta} &= \{u \in \hat{\mathcal{L}} : [h,u] = (\gamma \oplus n\delta)(h)u \text{ for all } h \in H\} \\ &= \begin{cases} \mathfrak{g}_{\bar n,\gamma} \otimes t^n, & \gamma \oplus n\delta \neq 0, \\ H, & \gamma \oplus n\delta = 0, \end{cases}\end{aligned} \tag{2.12}$$

whence $\hat{\mathcal{L}} = \bigoplus_{\alpha\in R} \hat{\mathcal{L}}_\alpha$ has a root space decomposition with respect to H with set of roots

$$R = \{\gamma \oplus n\delta : \gamma \in \Delta_{\bar s},\ \bar n = \bar s, 0 \le s < m\}. \tag{2.13}$$

This establishes (EA2).

To check the other axioms we first need to determine which of the roots in R are the null respectively anisotropic roots. Following the procedure in §2.1, we consider the restriction of the bilinear form $(\cdot|\cdot)$ to H. With obvious notation this is

$$\big(h_{\bar 0} \oplus s_1 c \oplus s_1' d \mid h_{\bar 0}' \oplus s_2 c \oplus s_2' d\big) = \kappa(h_{\bar 0}, h_{\bar 0}') + s_1 s_2' + s_2 s_1'.$$

Since $\kappa|_{\mathfrak{h}_{\bar 0}\times\mathfrak{h}_{\bar 0}}$ is nondegenerate, this is indeed a nondegenerate symmetric bilinear form on H, as it should be. Let $t_\gamma \in \mathfrak{h}_{\bar 0}$ be the element representing $\gamma \in \mathfrak{h}_{\bar 0}^*$: $\kappa(t_\gamma, h_{\bar 0}) = \gamma(h_{\bar 0})$ for all $h_{\bar 0} \in \mathfrak{h}_{\bar 0}$. For the canonical extension of γ to a linear form of H, also denoted by γ, we then get $(t_\gamma \mid h) = \gamma(h)$ for all $h \in H$. Moreover $(c \mid h_{\bar 0} \oplus sc \oplus s'd) = s' = \delta(h_{\bar 0} \oplus sc \oplus s'd)$ shows that δ is represented by $t_\delta = c \in H$. Therefore $\alpha = \gamma \oplus n\delta \in R$ is represented by

$$t_{\gamma\oplus n\delta} = t_\gamma \oplus nc.$$

Now observe $(t_{\gamma\oplus n\delta} \mid t_{\gamma\oplus n\delta}) = (t_\gamma \oplus nc \mid t_\gamma \oplus nc) = \kappa(t_\gamma, t_\gamma)$. It is of course well-known that $\kappa(t_\gamma, t_\gamma) \neq 0$ for $0 \neq \gamma \in \Delta_{\bar 0}$. But one can (easily) show that this also holds for any $0 \neq \gamma \in \Delta_{\bar n}$. We therefore get

$$R^{\mathrm{an}} = \{\gamma \oplus n\delta \in R : \gamma \neq 0\} \quad \text{and} \quad R^0 = \mathbb{Z}\delta, \tag{2.14}$$

which in the theory of affine Kac-Moody algebras are usually called *real* and *imaginary roots*. We are now set for the verification of the remaining axioms.

(EA3) holds in the stronger form: $\operatorname{ad} \hat{\mathcal{L}}_\lambda$, $\alpha \in R^{\mathrm{an}}$, is nilpotent. (We have already seen the same phenomenon in the Example 2.3 of a finite-dimensional split simple Lie algebra. Perhaps the reader wonders if this is true in general. The answer is yes.)

(EA4) The verification of (EA4) is left to the reader.

(EA5) The core of $\hat{\mathcal{L}}$ is $\hat{\mathcal{L}}_c = \big(\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\bar n} \otimes \mathbb{C}t^n\big) \oplus \mathbb{C}c$, and therefore equals the derived algebra $[\hat{\mathcal{L}}, \hat{\mathcal{L}}]$ of $\hat{\mathcal{L}}$. The centralizer of $\hat{\mathcal{L}}_c$ in $\hat{\mathcal{L}}$, in fact the centre of $\hat{\mathcal{L}}$ is $\mathbb{C}c \subset \hat{\mathcal{L}}_c$, see Exercise 1.2.

(DE) In this example the subgroup $\Lambda = \langle R^0\rangle$ equals $R^0 = \mathbb{Z}d$ and is a discrete subset of H^*.

We have now shown one implication of the following result.

Theorem 2.5 ([ABGP])**.** *A complex Lie algebra E is a discrete EALA of nullity* 1 *if and only if E is an affine Kac-Moody Lie algebra.*

Exercise 2.6. Check the following details of the construction above.

(a) (2.11) defines an invariant symmetric bilinear form on $\hat{\mathcal{L}}$.

(b) $\hat{\mathcal{L}}$ has a root space decomposition whose root spaces are given by (2.12) and whose set of roots is (2.13).

(c) $\kappa(t_\gamma, t_\gamma) \neq 0$ for any $0 \neq \gamma \in \Delta_{\bar n}$.
(d) (EA4) holds for $(\hat{\mathcal{L}}, H)$.

2.5. **Higher nullity examples.** We have seen all examples of EALAs of nullity 0 and 1. In this section we will construct examples of higher nullity. To simplify things we consider untwisted algebras (no non-trivial finite order automorphism are involved). We can therefore go back to our standard setting: $\mathfrak{g}$ is a split simple Lie algebra over a field F of characteristic 0.

As in §1.2 let $F[t_1^{\pm 1}, \dots, t_n^{\pm n}]$ be the Laurent polynomial ring in n variables and let

$$L = L(\mathfrak{g}) = \mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm n}]$$

be the associated untwisted multiloop algebra. We have seen in Exercise 1.4 that L has a 2-cocycle $\psi : \mathcal{L} \times \mathcal{L} \to F^n =: \mathcal{C}$, given by (1.16): $\psi(u \otimes t^\lambda, v \otimes t^\mu) = \delta_{\lambda+\mu,0}\, \kappa(u,v)\, \lambda$. We can therefore define the central extension

$$K = L \oplus \mathcal{C}$$

with product (1.22). In (1.19) we have defined degree derivations ∂_i, $i = 1, \dots, n$, of K. Let

$$\mathcal{D} = \operatorname{span}_F\{\partial_1, \dots, \partial_n\} \tag{2.15}$$

and define the Lie algebra E as the semidirect product,

$$E = \big(L(\mathfrak{g}) \oplus \mathcal{C}\big) \rtimes \mathcal{D}.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and put

$$H = \mathfrak{h} \oplus \mathcal{C} \oplus \mathcal{D}.$$

We claim that (E, H) *is an EALA of nullity* n.

(EA1) We will mimic the construction of an invariant nondegenerate symmetric bilinear form in §2.4 and require

- $(L(\mathfrak{g}) \mid \mathcal{C} \oplus \mathcal{D}) = 0$.
- $\mathcal{C} \oplus \mathcal{D}$ is a hyperbolic space with $(\mathcal{C} \mid \mathcal{C}) = 0 = (\mathcal{D} \mid \mathcal{D})$ and
$$(\textstyle\sum_i s_i c_i \mid \sum_i s_i' \partial_i) = \sum_i s_i s_i',$$
where $c_1, \dots, c_n$ is the canonical basis of F^n. Thus $\mathcal{C} \oplus \mathcal{D}$ is the orthogonal sum of the n hyperbolic planes $Fc_i \oplus F\partial_i$.
- On $L(\mathfrak{g})$ the form is the tensor product form of the Killing form κ of $\mathfrak{g}$ and the natural invariant bilinear form on $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Putting all these requirements together, we arrive at the global formula which is completely analogous to (2.11):

$$\begin{aligned} &\big(u \otimes t^\lambda \,\oplus\, \textstyle\sum_i s_i c_i \oplus \sum_j s_j' \partial_j \;\mid\; v \otimes t^\mu \,\oplus\, \sum_i t_i c_i \oplus \sum_j t_j' \partial_j\big) \\ &\qquad = \kappa(u,v)\delta_{\lambda,-\mu} \,+\, \textstyle\sum_i (s_i t_i' + t_i s_i'). \end{aligned} \tag{2.16}$$

(EA2) Let $\mathfrak{h}$ be a splitting Cartan subalgebra and let Φ be the usual root system of $(\mathfrak{g}, \mathfrak{h})$, thus $0 \notin \Phi$. We put $\Delta = \{0\} \cup \Phi$ and then have the root space decomposition $\mathfrak{g} = \bigoplus_{\gamma \in \Delta} \mathfrak{g}_\gamma$ with $\mathfrak{g}_0 = \mathfrak{h}$. We embed $\Delta \hookrightarrow H^*$ by requiring $\gamma \mid \mathcal{C} \oplus \mathcal{D} = 0$ for $\gamma \in \Delta$. Also we embed $\Lambda = \mathbb{Z}^n \hookrightarrow H^*$ by $\lambda(\mathfrak{h} \oplus \mathcal{C}) = 0$ and $\lambda(\partial_i) = \lambda_i$ for

$\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$. Then E has the root space decomposition $E = \bigoplus_{\alpha \in R} E_\alpha$ with root spaces

$$E_{\gamma \oplus \lambda} = \mathfrak{g}_\gamma \otimes t^\lambda \quad (\gamma \oplus \lambda \neq 0), \qquad E_0 = H, \tag{2.17}$$

$$R^{\mathrm{an}} = \Phi \times \Lambda, \qquad R^0 = \Lambda. \tag{2.18}$$

It is now not difficult to verify (EA3) – (EA5) and (DE). Thus:

Lemma 2.7. *The pair (E, H) constructed above is a discrete EALA of nullity n.*

There is however no analogue of Prop. 2.10 and Th. 2.5: There are many more EALAs of nullity $n \geq 2$. We have just seen the "tip of the iceberg"! Other examples can be found in Ch.III of [AABGP], some of them involving heavy-duty nonassociative algebras, like octonion algebras and Jordan algebras over Laurent polynomial rings!

Exercise 2.8. Supply the missing details of the proof that (E, H) above is a discrete EALA of nullity n. In particular, prove:

(a) (2.16) defines an invariant nondegenerate symmetric bilinear form on E.

(b) The root spaces of (E, H) and the anisotropic and null roots are as stated in (2.17) and (2.18).

3. The structure of the roots of an EALA

In this chapter we will describe the structure of the set of roots R of an EALA (E, H), defined in (2.4). We have already seen some examples: R can be a finite irreducible reduced root system, (2.9), R can be an affine root system (2.13), i.e., the set of roots of an affine Kac-Moody Lie algebra, or R can be of the form $R = S \times \mathbb{Z}^n$ where $S \setminus \{0\}$ is a finite irreducible reduced root system (2.18). Thus any description of the general case has to encompass all these different examples.

It turns out that the roots of an EALA form an extended affine root system and that the latter is naturally described as a special case of affine reflection systems. We therefore first introduce the latter, describe their structure and then specialize later to extended affine root systems. Affine reflection systems are themselves special cases of reflection systems, whose theory is developed in [LN].

3.1. **Affine reflection systems: Definition.** Throughout this section we work with a triple $(R, X, (\cdot|\cdot))$ where

- X is a finite-dimensional vector space over a field F of characteristic 0,
- $(\cdot|\cdot)$ is a symmetric bilinear form on X and
- $R \subset X$.

For any such triple $(R, X, (\cdot|\cdot))$ we define

$$X^0 = \{x \in X : (x \mid X) = 0\}, \text{ the radical of } (\cdot|\cdot),$$
$$R^0 = \{\alpha \in R : (\alpha|\alpha) = 0\}, \quad (\textit{null roots})$$
$$R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha) \neq 0\}, \quad (\textit{anisotropic roots})$$
$$\langle x, \alpha^\vee \rangle = 2\frac{(x|\alpha)}{(\alpha|\alpha)}, \quad (x \in X \text{ and } \alpha \in R^{\mathrm{an}})$$
$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha. \tag{3.1}$$

By definition we therefore have $R = R^0 \cup R^{\rm an}$. The map $s_\alpha : X \to X$ is a *reflection in* α, i.e., $s_\alpha^2 = \mathrm{Id}_X$ and $\{x \in X : s_\alpha(x) = -x\} = F\alpha$. It is also orthogonal with respect to $(\cdot|\cdot)$: $(s_\alpha(x) \mid s_\alpha(y)) = (x \mid y)$ for all $x, y \in X$.

We call $(R, X, (\cdot|\cdot))$, or just R for short, an *affine reflection system* if

(AR1): *$0 \in R$ and R spans X,*
(AR2): *$s_\alpha(R) = R$ for all $\alpha \in R^{\rm an}$,*
(AR3): *for every $\alpha \in R^{\rm an}$ the set $\langle R, \alpha^\vee\rangle$ is finite and contained in $\mathbb{Z}$, and*
(AR4): *$R^0 = R \cap X^0$.*

An affine reflection system is said to be

- *reduced* if for every $\alpha \in R^{\rm an}$ and $c \in F$: $c\alpha \in R^{\rm an} \iff c = \pm 1$,
- *connected* if for any decomposition $R^{\rm an} = R_1 \cup R_2$ with $(R_1 \mid R_2) = 0$ we have $R_1 = \emptyset$ or $R_2 = \emptyset$.

The *nullity* of (R, X) is the rank of the torsion-free abelian group $\mathbb{Z}[R^0] = \mathrm{span}_{\mathbb{Z}}(R^0)$ generated by R^0 in $(X, +)$. Thus, by definition,

$$\text{nullity of } (R, X) = \dim_{\mathbb{Q}}(\mathbb{Z}[R^0] \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_F(\mathbb{Z}[R^0] \otimes_{\mathbb{Z}} F).$$

Since the vector space $\mathbb{Z}[R^0] \otimes_{\mathbb{Z}} F$ maps onto $\mathrm{span}_F(R^0)$, the nullity of (R, Z) is bounded below by $\dim_F \mathrm{span}_F(R^0)$. It is in general not equal to it. But this is of course so for nullity 0: (R, X) has nullity 0 if and only if $R^0 = \{0\} \iff \dim_F \mathrm{span}_F(R^0) = 0$.

Remarks 3.1. - For a large part of the theory it is not necessary that X be finite-dimensional, see [LN]. But assuming this right from the start, simplifies the presentation.

- We need the bilinear form $(\cdot|\cdot)$ to define R^0 and the reflections. But although we will sometimes write $(R, X, (\cdot|\cdot))$, we will not consider $(\cdot|\cdot)$ as part of the structure of an affine reflection system. For example, in the definition of an isomorphism below we will not require that the bilinear forms are preserved. See [LN], where this point of view is emphasized.

- The requirement $0 \in R$ is in line with the previous chapter, in which 0 was considered a root of an EALA. This conflicts with the traditional approach to root systems in which 0 is not a root, see for example [Bou2], [Hu] or [Kac]. The question whether 0 is a root or is not a root, has lead to heated debates. In the author's opinion, there are some advantages of considering 0 as a root, which however can only be fully seen when one develops the theory for affine reflection systems. But perhaps the reader can be convinced by the natural) example $(R, X) = (\{0\}, \{0\})$ of an affine reflection system.

- The condition $\langle \beta, \alpha^\vee\rangle \in \mathbb{Z}$ in axiom (AR3) makes sense since every field of characteristic 0 contains (an isomorphic copy of) the field of rational numbers, which allows us to identify $\mathbb{Z} \equiv \mathbb{Z}1_F$.

- By definition $\langle X^0, \alpha^\vee\rangle = 0$ for all $\alpha \in R^{\rm an}$. Hence $s_\alpha(x^0) = x^0$ for $x^0 \in X^0$. Also, the inclusion $R \cap X^0 \subset R^0$ in (AR4) is always true. Therefore the axioms (AR2)–(AR4) can be replaced by the following conditions

(AR2)′ $s_\alpha(R^{\rm an}) = R^{\rm an}$ for all $\alpha \in R^{\rm an}$,
(AR3)′ for every $\alpha \in R^{\rm an}$ the set $\langle R^{\rm an}, \alpha^\vee\rangle \subset \mathbb{Z}$ is finite,
(AR4)′ $R^0 \subset X^0$.

This new set of axioms makes it (even more) clear that the conditions on R^0 are rather weak: We (may) need R^0 to span X from (AR1), we need $R^0 \subset X^0$ for

(AR4)$'$ and we need $0 \in R$, which is no condition since one can always add 0 to R^0. We will see this phenomena re-appearing in the examples, e.g., in Example 3.2, and in the definition of an extension datum in 3.9.

- The definition of a connected affine reflection system is the same as the axiom (EA4) in the definition of an EALA.
- The definition of an affine reflection system given in [LN] is not the same as the one given here. The equivalence of two definitions follows from [LN, Prop. 5.4].

An *isomorphism* from an affine reflection system $(R, X, (\cdot|\cdot))$ to another affine reflection system $(R', X'(\cdot|\cdot)')$ is a vector space isomorphism $f : X \to X'$ satisfying

$$f(R^{\mathrm{an}}) = R'^{\mathrm{an}} \quad \text{and} \quad f(R^0) = R'^0$$

If such a map exists, $(R, X, (\cdot|\cdot))$ and $(R', X, (\cdot|\cdot)')$ are called *isomorphic*. One can show, as a corollary of the Structure Theorem 3.10, that an isomorphism f also satisfies $f \circ s_\alpha = s_{f(\alpha)} \circ f$ for all $\alpha \in R^{\mathrm{an}}$, equivalently,

$$\langle x, \alpha^\vee \rangle = \langle f(x), f(\alpha)^\vee \rangle$$

for all $x \in X$ and $\alpha \in R^{\mathrm{an}}$. This is always fulfilled if f is an isometry for $(\cdot|\cdot)$ and $(\cdot|\cdot)'$ respectively. But in general an isomorphism is not necessarily an isometry. For example, one can always multiply the bilinear form $(\cdot|\cdot)$ by a non-zero scalar without changing $\langle x, \alpha^\vee \rangle$.

Since a reflection s_α is an isometry, it follows from (AR2) and (AR4) that s_α leaves R^{an} and R^0 invariant and is thus an automorphism of (R, X). The subgroup $W(R)$ of the automorphism group of (R, X) generated by all reflections s_α, $\alpha \in R^{\mathrm{an}}$, is (obviously) called the *Weyl group* of (R, X). (It will not play a big role in this chapter.)

3.2. **Examples of affine reflection systems.** We will now give some immediate examples of affine reflection systems.

Example 3.2 (*The real part of an affine reflections system*)**.** Let $(R, X, (\cdot|\cdot))$ be an affine reflection system. Then

$$\begin{aligned} &\mathrm{Re}(R) = \{0\} \cup R^{\mathrm{an}}, \quad \mathrm{Re}(X) = \mathrm{span}_F(R^{\mathrm{an}}), \\ &(\cdot|\cdot)_{\mathrm{Re}} = (\cdot|\cdot)_{\mathrm{Re}(X) \times \mathrm{Re}(X)} \end{aligned}$$

defines an affine reflection system, called the *real part of* (R, X), with

$$\mathrm{Re}(R)^{\mathrm{an}} = R^{\mathrm{an}}, \quad \mathrm{Re}(R)^0 = \{0\},$$

in particular $\mathrm{Re}(R)$ has nullity 0.

Observe that $(\cdot|\cdot)_{\mathrm{Re}}$ need not be nondegenerate, see Example 3.4 for an example.

The fact that one can "throw away" the non-zero null roots and still have an affine reflection system indicates that one has little control over the null roots in a general affine reflection system. This will be made even more evident in the concept of an extension datum 3.9, used in the general Structure Theorem 3.10 for affine reflection systems. It is therefore natural to define subclasses of affine reflection system by imposing conditions on the null roots. For example, we will do so when we define extended affine root systems in 3.4.

In [Ne5, 3.6] the author claimed that an affine reflection system of nullity 0 is a finite root system. The example above show that this is far from being true. But what remains true is the converse, also claimed in [Ne5, 3.6]: A finite root system is an affine reflection system of nullity 0, as we will show now.

Example 3.3 (*Finite root systems*)**.** Let Φ be a (finite) root system à la Bourbaki [Bou2, VI, §1.1]. Recall that this means that Φ is a subset of an F-vector space Y satisfying the axioms (RS1)–(RS3) below.

(RS1): Φ is finite, $0 \notin \Phi$ and Φ spans Y.

(RS2): For every $\alpha \in \Phi$ there exists a linear form $\alpha^\vee \in Y^*$ such that $\alpha^\vee(\alpha) = 2$ and $s_\alpha(\Phi) = \Phi$, where s_α is the reflection of Y defined by $s_\alpha(y) = y - \alpha^\vee(y)\alpha$,

(RS3): for every $\alpha \in \Phi$ the set $\alpha^\vee(\Phi)$ is contained in $\mathbb{Z}$.

Observe that the reflection s_α defined in (RS2) satisfies $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(y) = y$ for $\alpha^\vee(y) = 0$. It therefore seems to depend on α and the linear form $\alpha^\vee$. However, since Φ is finite, there exists at most one reflection s with $s(\Phi) = \Phi$ and $s(\alpha) = -\alpha$ ([Bou2, VI, §1.1, Lemme 1]). It is therefore not necessary to indicate $\alpha^\vee$ in the notation of s_α.

Note that we do not assume that Φ is reduced. This more general concept of a root system is necessary for the Structure Theorem of affine root systems (3.10). The reader who is only familiar with the theory of reduced finite root systems, as for example developed in [Hu, Ch. III], can perhaps be comforted by the fact that the difference is not very big. Indeed, every finite root system is a direct sum of connected (= irreducible) root systems and there is only one irreducible non-reduced root system of rank l, namely

$$\mathrm{BC}_l = \mathrm{B}_l \cup \mathrm{C}_l = \{\pm\varepsilon_i : 1 \le i \le l\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i, j \le l\}$$

where here and in the following $\varepsilon_1, \dots, \varepsilon_l$ is the standard basis of F^l. (Note $0 \in \mathrm{BC}_l$ in anticipation of the convention introduced below.)

In the context of finite-dimensional Lie algebras, non-reduced root systems arise naturally as the roots of a finite-dimensional semisimple Lie algebra L with respect to a maximal ad-diagonalizable subalgebra $H \subset L$ which is not self-centralizing, hence not a Cartan algebra. In particular, non-reduced root systems do not occur over an algebraically closed field. However, they do occur in the context of infinite-dimensional Lie algebras, even over algebraically closed fields, see Ex. 3.6.

Given a finite root system (Φ, Y), define

$$S = \{0\} \cup \Phi \quad \text{and} \quad (x \mid y) = \textstyle\sum_{\alpha\in\Phi} \alpha^\vee(x)\,\alpha^\vee(y) \tag{3.2}$$

for $x, y \in Y$. Then $(\cdot|\cdot)$ is a nondegenerate symmetric bilinear form on Y with respect to which all reflections s_α are isometric ([Bou2, VI, §1.1, Prop. 3]). Moreover, $(\alpha \mid \alpha)$ is a positive integer for every $\alpha \in \Phi$ (viewing $\mathbb{Q} \subset F$ canonically) and

$$\langle y, \alpha^\vee\rangle = \alpha^\vee(y) = 2\,\tfrac{(y|\alpha)}{(\alpha|\alpha)}$$

for all $y \in Y$. Hence s_α as defined in (RS2) is also given by the formula (3.1). We have $S^0 = \{0\} = X^0 = X^0 \cap S$. Since $\langle \Phi, \alpha^\vee\rangle \subset \mathbb{Z}$ we have shown that

$(S, Y, (\cdot|\cdot))$ *as defined in* (3.2) *is a finite affine reflection system of nullity* 0.

We will characterize finite root systems within the category of affine reflection systems in Cor. 3.11.

In the following we will always assume that a finite root system contains 0. We will usually use the symbol S for a finite root system, and put

$$S^\times = S \setminus \{0\} = \Phi.$$

We will also need the following subsets of roots of a finite root system S:

$S_{\rm div}$ is the set of divisible roots, where $\alpha \in S$ is called *divisible* if $\alpha/2 \in S$. In particular $0 \in S_{\rm div}$. We put $S^\times_{\rm div} = S_{\rm div} \cap S^\times = S_{\rm div} \setminus \{0\}$.

$S_{\rm ind}$ $= S \setminus S^\times_{\rm div}$, the subsystem of *indivisible roots*.

We also need the fact that there exists a unique symmetric bilinear form $(\cdot|\cdot)_u$ on Y which is invariant under the Weyl group $W(S)$ and which satisfies $2 \in \{(\alpha|\alpha)_u : 0 \ne \alpha \in C\} \subset \{2,4,6,8\}$ for every connected component C of S. This follows easily from [Bou2, Prop. 7]. Observe that

$$S^\times_{\rm div} = \{\alpha \in S : (\alpha|\alpha)_u = 8\}.$$

We use $(\cdot|\cdot)_u$ to define short and long roots:

$S_{\rm sh}$ $= \{\alpha \in S : (\alpha|\alpha)_u = 2\}$ is the set of *short roots*.

$S_{\rm lg}$ $= \{\alpha \in S : (\alpha|\alpha)_u \in \{4,6\}\}$ is the set of *long roots* in S.

Thus $S_{\rm lg} = S \setminus (S_{\rm sh} \cup S_{\rm div})$. For example, for $S = {\rm BC}_l$ we have

$$\begin{aligned} {\rm BC}_{l,{\rm sh}} &= \{\pm\varepsilon_i : 1 \le i \le l\},\\ {\rm BC}^\times_{l,{\rm div}} &= \{\pm 2\varepsilon_i : 1 \le i \le l\},\\ {\rm BC}_{l,{\rm lg}} &= \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i \ne j \le l\}, \end{aligned}$$

in particular ${\rm BC}_{1,{\rm lg}} = \emptyset$, and if S is *simply laced*, i.e., $S^\times = S_{\rm sh}$, then $S_{\rm div} = \{0\}$ and $S_{\rm lg} = \emptyset$.

Example 3.4 (*Untwisted affine reflection systems*)**.** Let $(S, Y, (\cdot|\cdot)_Y)$ be a finite root system. Hence $0 \in S$ and $\Phi = S \setminus \{0\}$, as stipulated in Example 3.3. Also, let Z be an n-dimensional F-vector space, say with a basis $\varepsilon_1, \dots, \varepsilon_n$. We define

$$\begin{aligned} X &= Y \oplus Z,\\ \Lambda &= \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n \ \subset Z,\\ R &= \textstyle\bigcup_{\xi\in S}\{\xi \oplus \lambda : \lambda \in \Lambda\} \ \subset Y \oplus Z,\\ (x_1 \mid x_2)_X &= (y_1 \mid y_2)_Y \quad \text{for } x_i = y_i \oplus z_i \text{ with } y_i \in Y \text{ and } z_i \in Z. \end{aligned}$$

By construction we then have

$$X^0 = Z, \quad R^0 = \Lambda, \quad R^{\rm an} = \textstyle\bigcup_{\xi\in\Phi} \xi \oplus \Lambda$$

where of course $\xi \oplus \Lambda = \{\xi \oplus \lambda : \lambda \in \Lambda\}$. For $\alpha = \xi \oplus \lambda \in R^{\rm an}$ with $\xi \in S$ and $\lambda \in \Lambda$ the reflection s_α satisfies

$$s_\alpha(y \oplus z) = s_\xi(y) \oplus (z - \langle y, \xi^\vee\rangle \lambda). \tag{3.3}$$

We will leave it to the reader to verify that

(3.4) (R, X) *is an affine reflection system of nullity* n.

Observe that (R, X) is the set of roots of the EALA constructed in 2.5, see in particular (2.18).

Observe that ${\rm span}_F(R^{\rm an}) = X = {\rm Re}(X)$ in case $S \ne \{0\}$. This shows that the form $(\cdot|\cdot)_{\rm Re}$ of the real part $\Re(R)$ of R need not be nondegenerate.

Exercise 3.5. Show the claim in (3.4), and also that (R, X) is reduced resp. connected if and only if (S, Y) is so.

Example 3.6 (*Affine root systems*)**.** By definition, an *affine root system* is the set of roots of an affine Kac-Moody Lie algebra, which we studied in §1.1 and then again in §2.4, where we showed that an affine Kac-Moody algebra is an EALA of nullity 1. Our goal here is not surprising. We want to show that

(3.5) *an affine root system is an affine reflection system of nullity* 1.

Let us first collect the data necessary to prove this. We use the notation established in 2.4. Thus, $\hat{\mathcal{L}} = \hat{\mathcal{L}}(\mathfrak{g}, \sigma)$ is an affine Kac-Moody Lie algebra over $\mathbb{C}$, σ is a diagram automorphism of the simple finite-dimensional Lie algebra $\mathfrak{g}$ of order $m \in \{1, 2, 3\}$, and $\Delta_{\bar{s}}$ denotes the set of weights of the $(\mathfrak{g}_{\bar{0}}, \mathfrak{h}_{\bar{0}})$-module $\mathfrak{g}_{\bar{s}} \subset \mathfrak{g}$, $s = 0, \dots, m-1$. One knows that $\Delta_{\bar{0}}$ is a reduced irreducible root system in $\mathfrak{h}_{\bar{0}}^* =: Y$. The roots of $\hat{\mathcal{L}}$ with respect to $H = \mathfrak{h}_{\bar{0}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ are

$$R = \{\gamma \oplus n\delta : \gamma \in \Delta_{\bar{s}},\, \bar{n} = \bar{s},\, 0 \le s < m\},$$

see (2.13), hence

$$X = \operatorname{span}_{\mathbb{C}}(R) = Y \oplus \mathbb{C}\delta.$$

The bilinear form $(\cdot|\cdot)_X$ used to determine the (an)isotropic roots in R has the form

$$(x_1 \mid x_2)_X = (y_1 \mid y_2)_Y$$

where $x_i = y_i \oplus a_i\delta$ with $y_i \in Y$ and $a_i \in \mathbb{C}$, and where $(\cdot|\cdot)_Y$ is the nondegenerate symmetric bilinear form on Y, obtained by transporting the Killing form $\kappa\,|_{\mathfrak{h}_{\bar{0}} \times \mathfrak{h}_{\bar{0}}}$ from $\mathfrak{h}_{\bar{0}}$ to Y. It follows that

$$X^0 = \mathbb{C}\delta \quad \text{and} \quad R^{\mathrm{an}} = \{\gamma \oplus n\delta \in R : \gamma \neq 0\}.$$

We can now verify the axioms (AR1)–(AR4).

(AR1) holds by definition. (AR2) is a consequence of [Kac, Prop. 3.7(b)]. Concerning (AR3), it follows from the structure of $(\cdot|\cdot)_X$ that

(3.6) $\langle x, \alpha^\vee \rangle = \langle y, \gamma^\vee \rangle$ for $x = y \oplus a\delta \in X$ and $\alpha = \gamma \oplus n\delta \in R^{\mathrm{an}}$.

This implies that $\langle R, \alpha^\vee \rangle$ is a finite set since

$$S = \Delta_{\bar{0}} \cup \dots \cup \Delta_{\overline{m-1}}$$

is a finite set (S is actually a finite root system; for $m > 1$ see the table below). Moreover $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$ because $\hat{\mathcal{L}}$ is an integrable $\hat{\mathcal{L}}$-module ([Kac, Lemma 3.5]). Thus (AR3) holds, and (AR4) follows from (3.6) and $(\gamma \mid \gamma) = 0 \Leftrightarrow \gamma = 0$ for $\gamma \in S$. This proves (3.5).

To motivate the definition of extension data in Def. 3.9 and the Structure Theorem 3.10 for affine reflection systems, we will now look at R and S more closely. In the untwisted case, i.e., $m = 1$, we have of course

$$\Delta_{\bar{0}} = S, \quad R = S \times \mathbb{Z}\delta \quad (m = 1).$$

Thus R is an untwisted affine root system of nullity 1, a special case of the Example 3.4. For $m = 2, 3$ the structure of $\Delta_{\bar{s}}$ and S is summarized in the table below.

Proofs can be extracted from [Kac, 7.9, 7.8, 8.3].

(3.7)

$(\mathfrak{g}, m)$	$\Delta_{\bar 0}$	$\Delta_{\bar 1}$	S
$(\mathrm{A}_{2l}, 2), l \geq 1$	A_1 or $\mathrm{B}_l (l \geq 2)$	$\Delta_{\bar 0} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq l\}$	BC_l
$(A_{2l-1}, 2), l \geq 2$	C_l	$\{0\} \cup \mathrm{C}_{l,\mathrm{sh}}$	C_l
$(\mathrm{D}_{l+1}, 2), l \geq 3$	B_l	$\{0\} \cup \mathrm{B}_{l,\mathrm{sh}}$	B_l
$(\mathrm{E}_6, 2)$	F_4	$\{0\} \cup \mathrm{F}_{4,\mathrm{sh}}$	F_4
$(\mathrm{D}_4, 3)$	G_2	$\{0\} \cup \mathrm{G}_{2,\mathrm{sh}}$	G_2

We can now rewrite R. For subsets $T \subset S$ and $\Xi \subset \mathbb{Z}\delta$ we put

$$T \oplus \Xi = \{\tau \oplus n\delta : \tau \in T,\, n\delta \in \mathbb{Z}\delta\}$$

and abbreviate $\mathrm{B}_1 = \mathrm{A}_1 = \{0, \pm\alpha\}$. For $m = 2$ we get

$$\begin{aligned} R &= (\Delta_{\bar 0} \oplus 2\mathbb{Z}\delta) \cup \big(\Delta_{\bar 1} \oplus (1 + 2\mathbb{Z})\delta\big) \\ &= \begin{cases} (\{0\} \oplus \mathbb{Z}\delta) \cup \big((\mathrm{B}_l \setminus \{0\} \oplus \mathbb{Z}\delta\big) \cup \big(\mathrm{BC}^{\times}_{\mathrm{div}} \oplus (1 + 2\mathbb{Z})\delta\big), & \mathfrak{g} = \mathrm{A}_{2l} \\ (\{0\} \oplus \mathbb{Z}\delta) \cup \quad (S_{\mathrm{sh}} \oplus \mathbb{Z}\delta) \quad \cup \quad (S_{\mathrm{lg}} \oplus 2\mathbb{Z}\delta), & \mathfrak{g} \neq \mathrm{A}_{2l}\,. \end{cases} \end{aligned}$$

For $(\mathfrak{g}, m) = (\mathrm{D}_4, 3)$ one knows $\Delta_{\bar 1} = \Delta_{\bar 2} = \{0\} \cup \mathrm{G}_{2,\mathrm{sh}}$, whence

$$\begin{aligned} R &= (\Delta_{\bar 0} \oplus 3\mathbb{Z}\delta) \cup \big(\Delta_{\bar 1} \oplus (1 + 3\mathbb{Z})\delta\big) \cup \big(\Delta_{\bar 2} \oplus (2 + 3\mathbb{Z})\delta\big) \\ &= (\{0\} \oplus \mathbb{Z}\delta) \cup (S_{\mathrm{sh}} \oplus \mathbb{Z}\delta) \cup (S_{\mathrm{lg}} \oplus 3\mathbb{Z}\delta). \end{aligned}$$

In all three cases R has a simultaneous description in terms of the root system S and subsets $\Lambda_{\mathrm{sh}}, \Lambda_{\mathrm{lg}}, \Lambda_{\mathrm{div}} \subset \mathbb{Z}\delta$ as

$$R = R^0 \cup (S_{\mathrm{sh}} \oplus \Lambda_{\mathrm{sh}}) \cup (S_{\mathrm{lg}} \oplus \Lambda_{\mathrm{lg}}) \cup (S^{\times}_{\mathrm{div}} \oplus \Lambda_{\mathrm{div}}) \tag{3.8}$$

where

$$\Lambda_{\mathrm{sh}} = \mathbb{Z}\delta = R^0, \quad \Lambda_{\mathrm{div}} = (1 + 2\mathbb{Z})\delta, \quad \Lambda_{\mathrm{lg}} = \begin{cases} \mathbb{Z}\delta, & \mathfrak{g} = \mathrm{A}_{2l},\, m = 2, \\ 2\mathbb{Z}\delta, & \mathfrak{g} \neq \mathrm{A}_{2l},\, m = 2, \\ 3\mathbb{Z}\delta, & m = 3. \end{cases} \tag{3.9}$$

If we define Λ_ξ for $\xi \in S$ by $\Lambda_\xi \in \{\Lambda_0 = R^0, \Lambda_{\mathrm{sh}}, \Lambda_{\mathrm{lg}}, \Lambda_{\mathrm{div}}\}$ according to ξ belong to the corresponding subset of S, then (3.8) becomes

$$R = \textstyle\bigcup_{\xi \in S} \xi \oplus \Lambda_\xi. \tag{3.10}$$

Note that we also recover [Kac, Th. 5.6(b)]:

$$R \cap X^0 = R \cap \mathbb{Z}\delta.$$

Example 3.7 (*Type* A_1 *generalized*)**.** We consider a final example of an affine reflection system to motivate the definition of an extension datum in 3.9 below.

Let Z be a finite-dimensional F-vector space and define the vector space X and a symmetric bilinear form on X by

$$X = F\alpha \oplus Z, \quad (a_1\alpha \oplus z_1 \mid a_2\alpha \oplus z_2) = a_1 a_2,$$

where $0 \neq \alpha$ and $a_i \in F$. We define $R \subset X$ in terms of three non-empty subsets $\Lambda_0, \Lambda_\alpha, \Lambda_{-\alpha} \subset Z$ as follows:

$$R = \Lambda_0 \cup (\alpha \oplus \Lambda_\alpha) \cup (-\alpha \oplus \Lambda_{-\alpha}). \tag{3.11}$$

It is then immediate that

$$X^0 = Z, \quad R^0 = \Lambda_0, \quad R^{\rm an} = (\alpha \oplus \Lambda_\alpha) \cup (-\alpha \oplus \Lambda_{-\alpha}).$$

We will now discuss under which conditions $(R, X, (\cdot|\cdot))$ is an affine reflection system. Let us start with (AR2). For $s_i \in \{\pm 1\}$, $\mu \in \Lambda_{s_1\alpha}$ and $\lambda \in \Lambda_{s_2\alpha}$ we have

$$\langle s_1\alpha \oplus \mu, (s_2\alpha \oplus \lambda)^\vee \rangle = 2\, s_1 s_2,$$
$$s_{s_2\alpha\oplus\lambda}(s_1\alpha \oplus \mu) = -s_1\alpha \oplus (\mu - 2s_1s_2\lambda)$$

Hence, all reflections $s_{s_2\alpha\oplus\lambda}$ leave R invariant if and only if $\mu - 2s_1s_2\lambda \in \Lambda_{-s_1\alpha}$ for μ, λ as above, i.e., in obvious short form $\Lambda_{s_1\alpha} - 2s_1s_2\Lambda_{s_2\alpha} \subset \Lambda_{-s_1\alpha}$. In particular,

$$\begin{array}{ll} \Lambda_\alpha - 2\Lambda_\alpha \subset \Lambda_{-\alpha}, \quad \Lambda_{-\alpha} - 2\Lambda_{-\alpha} \subset \Lambda_\alpha & \text{for } s_1 = s_2, \\ \Lambda_{-\alpha} + 2\Lambda_\alpha \subset \Lambda_\alpha \quad \text{for } s_1 = -1 = -s_2. & \end{array}$$

For $\lambda \in \Lambda_\alpha$ we therefore get $\lambda - 2\lambda = -\lambda \in \Lambda_{-\alpha}$, whence $-\Lambda_\alpha \subset \Lambda_{-\alpha}$ and, analogously, $\Lambda_{-\alpha} \subset -\Lambda_\alpha$. We therefore obtain

$$\Lambda_{-\alpha} = -\Lambda_\alpha, \tag{3.12}$$

or with the notation of above $\Lambda_{s_1\alpha} = s_1\Lambda_\alpha$. It is now easy to see that (AR2) is equivalent to the two conditions (3.12) and

$$2\Lambda_\alpha - \Lambda_\alpha \subset \Lambda_\alpha. \tag{3.13}$$

It then follows that R is an affine reflection system if and only if

(i) (3.12) and (3.13) hold,
(ii) $0 \in \Lambda_0$, and
(iii) $Z = \operatorname{span}_F(\Lambda_0 \cup \Lambda_\alpha \cup \Lambda_{-\alpha})$.

Observe the similarity with the previous examples: R has the form

$$R = \textstyle\bigcup_{\xi \in S} \xi \oplus \Lambda_\xi$$

where $S = \{0, \pm\alpha\}$ is a finite root system and $(\Lambda_\xi : \xi \in S)$ is a family of subsets in X^0. However, in the previous examples the Λ_ξ were subgroups of $(Z, +)$ while here we only have the condition (3.13). Does this imply that Λ_α is a subgroup? The answer is no! For example, in $Z = F$ the subset $\Lambda_\alpha = 1 + 2\mathbb{Z} \subset F$ satisfies (3.13).

A subset A of an abelian group $(Z, +)$ is called a *reflection subspace* if $2a_1 - a_2 \in A$ for all $a_i \in A$ (see [L] or [Ne5, 3.3] for a justification for this terminology). Hence, (3.13) just says that Λ_α is a reflection subspace. The structure of two special types of reflection subspaces is described in Exercise 3.8 below.

While in general Λ_α is far from being a subgroup, one can always "re-coordinatize" R to at least get $0 \in \Lambda_\alpha$. Namely, for a fixed $\lambda \in \Lambda_\alpha$ we have $\alpha + \Lambda_\alpha = (\alpha + \lambda) + (\Lambda_\alpha - \lambda)$. Hence, with $\tilde\alpha = \alpha + \lambda$ and $\Lambda_{\tilde\alpha} = \Lambda_\alpha - \lambda$, we obtain

$$R = \Lambda_0 \cup (\tilde\alpha + \Lambda_{\tilde\alpha}) \cup \big(-(\tilde\alpha + \Lambda_{\tilde\alpha})\big), \tag{3.14}$$

where now $\Lambda_{\tilde\alpha}$ not only satisfies (3.13) but also $0 \in \Lambda_{\tilde\alpha}$. In other words, $\Lambda_{\tilde\alpha}$ is a pointed reflection subspace as defined in Lemma 3.8 and therefore also satisfies $\Lambda_{\tilde\alpha} = -\Lambda_{\tilde\alpha}$.

The process of re-coordinatization works well in this example. The reason is that the finite root system S in (3.14) is reduced. Re-coordinatization will not work if S is not reduced, as for example in the case $(\mathfrak{g}, m) = (\mathrm{A}_{2l}, 2)$ of Example 3.6. This "explains" why in the property (ED2) of an extension datum in 3.9 we require $0 \in \Lambda_\xi$ only for an indivisible root $\xi \in S$.

Exercise 3.8. Let A be a subset of an abelian group $(Z,+)$. As above we put $2A - A = \{2a_1 - a_2 : a_i \in A\}$. We denote by $\Lambda = \operatorname{span}_{\mathbb{Z}}(A)$ the $\mathbb{Z}$-span of A in Z. A subset $A \subset Z$ is called *symmetric* if $A = -A$.

(a) The following equivalent conditions characterize symmetric reflection subspaces $A \subset Z$:
  (i) $2A - A \subset A$ and $A = -A$,
  (ii) $2\lambda + a \in A$ for every $\lambda \in \Lambda$ and $a \in A$,
  (iii) A is a union of cosets modulo 2Λ,
  (iv) $a_1 - 2a_2 \in A$ for all $a_i \in A$.

(b) The following are equivalent for $A \subset Z$:
  (i) $0 \in A$ and $A - 2A \subset A$,
  (ii) $0 \in A$ and $2A - A \subset A$,
  (iii) $2\mathbb{Z}[A] \subset A$ and $2\mathbb{Z}[A] - A \subset A$,
  (iv) A is a union of cosets modulo $2\mathbb{Z}[A]$, including the trivial coset $2\mathbb{Z}[A]$.
  In this case A is called a *pointed* reflection subspace.

(c) Every pointed reflection subspace is symmetric.

(d) If A is a symmetric reflection subspace then $A + A$ is a pointed reflection subspace.

3.3. **The Structure Theorem of affine reflection systems.** After the many examples in 3.2, the following definition should not be too surprising.

Definition 3.9. Let S be a finite root system as defined in 3.3. Recall $S^\times = S\setminus\{0\}$ and $S_{\rm ind} = \{0\} \cup \{\alpha \in S : \alpha/2 \notin S\} = S \setminus S^\times_{\rm div}$. Also, let Z be a finite-dimensional F-vector space. An *extension datum of type* (S, Z), sometimes simply called an *extension datum*, is a family $(\Lambda_\xi : \xi \in S)$ of subset $\Lambda_\xi \subset Z$ satisfying the axioms (ED1)–(ED3) below.

(ED1): For $\eta, \xi \in S^\times$, $\mu \in \Lambda_\eta$ and $\lambda \in \Lambda_\xi$ we have $\mu - \langle \eta, \xi^\vee\rangle\xi \in \Lambda_{s_\xi(\eta)}$, in obvious short form

$$\Lambda_\eta - \langle \eta, \xi^\vee\rangle \Lambda_\xi \subset \Lambda_{s_\xi(\eta)}.$$

(ED2): $0 \in \Lambda_\xi$ for $\xi \in S_{\rm ind}$, and $\Lambda_\xi \neq \emptyset$ for $\xi \in S \setminus S_{\rm ind} = S^\times_{\rm div}$.
(ED3): $Z = \operatorname{span}_F \left(\bigcup_{\xi\in S} \Lambda_\xi\right)$.

The axiom (ED1) is trivially true for $\eta = 0$ since $\langle \eta, \xi^\vee\rangle = 0$ and $s_\xi(0) = 0$. Also, if $S^\times_{\rm div} = \emptyset$, then there is no Λ_ξ for $\xi \in S^\times_{\rm div}$ and so the second condition in (ED2) is trivially fulfilled. (ED3) simply serves to determine Z. If it does not hold, one can simply replace Z by $\operatorname{span}_{\mathbb{Z}}(\bigcup_{\xi\in S}\Lambda_\xi)$.

The definition of an extension datum above is a special case of the notion of an extension datum for a pre-reflection system, introduced in [LN, 4.2]. (The reader will note that the axiom (ED1) in [LN] simplifies since in our setting the subset $S^{\rm re}$ of [LN] is $S^{\rm re} = S^{\rm an} = S \setminus \{0\}$.) The Structure Theorem 3.10 below is proven in [LN, Th. 4.6] for extensions of pre-reflection systems. Affine reflection systems are special types of such extensions, namely finite-dimensional extensions of finite root systems.

The rationale for the concept of an extension datum is the following Structure Theorem for affine reflection systems.

Theorem 3.10 (Structure Theorem for affine reflection systems).
(a) *Let $(S, Y, (\cdot|\cdot)_Y)$ be a finite root system and let $\mathfrak{L} = (\Lambda_\xi : \xi \in S)$ be an extension datum of type (S, Z). Define $(R, X, (\cdot|\cdot)_X)$ by*

$$\begin{aligned} X &= Y \oplus Z \\ R &= \textstyle\bigcup_{\xi \in S} \xi \oplus \Lambda_\xi \ \subset \ Y \oplus Z = X, \\ (y_1 \oplus z_1 \mid y_2 \oplus z_2)_X &= (y_1 \mid y_2)_Y \end{aligned}$$

for $y_i \in Y$ and $z_i \in Z$. Then $(R, X, (\cdot|\cdot)_X)$ is an affine reflection system, denoted $\mathcal{A}(S, \mathfrak{L})$, with

$$R^0 = \Lambda_0, \quad X^0 = Z \quad \text{and} \quad R^{\text{an}} = \textstyle\bigcup_{0 \neq \xi \in S} \xi \oplus \Lambda_\xi.$$

For $\alpha = \xi \oplus \lambda \in R^{\text{an}}$ and $x = y \oplus z \in X$ the reflection s_α is given by

$$s_\alpha(x) = s_\xi(y) \oplus (z - \langle y, \xi^\vee \rangle \lambda)$$

(b) *Conversely, let $(R, X, (\cdot|\cdot)_X)$ be an affine reflection system.*

- (i) *Let $f : X \to X/X^0 =: Y$ be the canonical map, put $S = f(R)$ and let $(\cdot|\cdot)_Y$ be the induced bilinear form on Y, that is $(f(x_1) \mid f(x_2))_Y = (x_1 \mid x_2)_X$. Then $(S, Y, (\cdot|\cdot)_Y)$ is a finite root system, the so-called* quotient root system *of (R, X).*
- (ii) *There exists a linear map $g : Y \to X$ satisfying $f \circ g = \mathrm{Id}_Y$ and $g(S_{\text{ind}}) \subset R$.*
- (iii) *For g as in* (ii) *and $\xi \in S$ define $\Lambda_\xi \subset \operatorname{Ker}(f) =: Z$ by*

$$R \cap f^{-1}(\xi) = g(\xi) \oplus \Lambda_\xi. \tag{3.15}$$

Then $\mathfrak{L} = (\Lambda_\xi : \xi \in S)$ is an extension datum of type (S, Z).

- (iv) *(R, X) is isomorphic to the affine reflection system $\mathcal{A}(S, \mathfrak{L})$ constructed in* (a).

Let us note that it is not reasonable to expect $g(S) \subset R$ in (b.ii) above, since R may be reduced while S is not, see for example the case $(\mathfrak{g}, \mathrm{A}_{2l})$ in 3.6. The quotient root system S is uniquely determined, but not so the extension datum, see [LN, Th. 4.6(c)].

Corollary 3.11. *An affine reflection system $(R, X, (\cdot|\cdot))$ is nondegenerate in the sense that $(\cdot|\cdot)$ is nondegenerate if and only if R is a finite root system.*

Proof. If $(R, X, (\cdot|\cdot))$ is an affine reflection system with a nondegenerate form $(\cdot|\cdot)$, then $\{0\} = X^0 = \operatorname{Ker} f$, so f is the identity. We have seen the other direction in Example 3.3. □

Corollary 3.12 ([LN, Cor. 5.5])**.** *Let $(R, X, (\cdot|\cdot))$ be an affine reflection system over $F = \mathbb{R}$. Then there exists a positive semidefinite symmetric bilinear form $(\cdot|\cdot)_\geq$ on X such that $(R, X, (\cdot|\cdot)_\geq)$ is an affine reflection system with the same (an)isotropic roots and reflections.*

The morale of the Structure Theorem is

$$\boxed{\text{affine reflection system} \qquad = \qquad \text{finite root system + extension datum}}$$

Thus properties of an affine reflection system can be described in terms of properties of its quotient root system and the associated extension datum. Some examples of this philosophy are given in the Proposition 3.13 and the Exercise 3.14 below.

Proposition 3.13 ([LN, Cor. 5.2]). *Let R be an affine reflection system, let S be its quotient root system and let $(\Lambda_\xi : \xi \in S)$ be the associated extension datum. We define*

$$\Lambda_{\mathrm{diff}} = \bigcup_{0 \neq \xi \in S} \Lambda_\xi - \Lambda_\xi.$$

Then $\mathbb{Z}\Lambda_{\mathrm{diff}} = \Lambda_{\mathrm{diff}}$. Moreover:

(a) *R is* tame *in the sense that $R^0 \subset R^{\mathrm{an}} - R^{\mathrm{an}}$ if and only if $R^0 \subset \Lambda_{\mathrm{diff}}$.*

(b) *All* root strings

$$\mathbb{S}(\beta, \alpha) = R \cap (\beta + \mathbb{Z}\alpha), \quad (\beta \in R, \alpha \in R^{\mathrm{an}})$$

are unbroken, *i.e., $\mathbb{Z}(\beta, \alpha) = \{n \in \mathbb{Z} : \beta + n\alpha \in R\}$ is either a finite interval in $\mathbb{Z}$ or equals $\mathbb{Z}$, if and only if $\Lambda_{\mathrm{diff}} \subset R^0$.*

(c) *A tame affine reflection system with unbroken root strings is symmetric.*

Exercise 3.14. Let R be an affine reflection system, let S be its quotient root system and let $(\Lambda_\xi : \xi \in S)$ be the associated extension datum. We use the notation of Prop. 3.13. Prove:

(a) R is reduced if and only if for all $0 \neq \xi \in S$ with $2\xi \in S$ we have

$$\Lambda_{2\xi} \cap 2\Lambda_\xi = \emptyset.$$

In particular, S need not be reduced for R to be reduced!

(b) R is connected iff S is connected (= irreducible).

(c) R is symmetric, i.e., $R = -R$, iff Λ_0 is symmetric.

(d) For all $\alpha \in R^{\mathrm{an}}$ and $\beta \in R$ the α-string through β, i.e., $\mathbb{S}(\beta, \alpha)$ has length $|\mathbb{S}(\beta, \alpha)| \leq 5$.

(e) Let $(\alpha, \beta) \in R^{\mathrm{an}} \times R$ and define $d, u \in \mathbb{N}$ by put $-d = \min \mathbb{Z}(\beta, \alpha)$ and $u = \max \mathbb{Z}(\beta, \alpha)$. Then $u - d = \langle \beta, \alpha^\vee \rangle$.

We will now describe how the examples of affine reflection systems of section 3.2 fit into the general scheme of the Structure Theorem above.

Examples 3.15. (a) Let $\mathfrak{L} = (\Lambda_\xi : \xi \in S)$ be an extension datum of type (S, Z). Observe that the only conditions on Λ_0 are $0 \in \Lambda_0$ from (ED2) and that Λ_0 together with the other Λ_ξ's spans Z from (ED3). This is in line with our earlier observation that one has little control over the null roots R^0 of an affine reflection system. Following the Example 3.2 we define a new extension datum $\mathrm{Re}(\mathfrak{L}) = (\mathrm{Re}(\Lambda_\xi) : \xi \in S)$ of type $(S, \mathrm{Re}(Z))$ by

$$\mathrm{Re}(Z) = \mathrm{span}_F \left(\bigcup_{0 \neq \xi \in S} \Lambda_\xi \right), \quad \mathrm{Re}(\Lambda_\xi) = \begin{cases} \{0\} & \text{for } \xi = 0, \\ \Lambda_\xi & \text{for } \xi \neq 0. \end{cases}$$

If $\mathfrak{L}$ is the extension datum associated to the affine reflection system (R, X), then $\mathrm{Re}(\mathfrak{L})$ is the extension datum associated to the affine reflection system $\mathrm{Re}(R)$.

(b) All $\Lambda_\xi = \{0\}$, whence $Z = \{0\}$, defines a *trivial* extension datum for any root system S. It is "used" when we view S as an affine reflection system, as done in Example 3.3.

(c) Let Λ be a subgroup of a finite-dimensional vector space Z such that $\mathrm{span}_F(\Lambda) = Z$. Then for any finite root system S the family $(\Lambda_\xi \equiv \Lambda : \xi \in S)$ is an extension datum of type (S, Z). It is used to construct the untwisted affine reflection systems of Example 3.4.

(d) Let R be an affine root system. We have seen that R is an affine reflection system. Its quotient root system S and associated extension datum $(\Lambda_\xi : \xi \in S)$ are described in Example 3.6 using the very same symbols, see the formulas (3.10) and (3.15).

(e) The family $\tilde{\mathfrak{L}} = (\Lambda_0, \Lambda_{\tilde\alpha}, \Lambda_{-\tilde\alpha})$ in Example 3.7 is an extension datum, but not necessarily $\mathfrak{L} = (\Lambda_0, \Lambda_\alpha, \Lambda_{-\alpha})$ since 0 need not lie in $\Lambda_{\pm\alpha}$. In fact, replacing $\mathfrak{L}$ by $\tilde{\mathfrak{L}}$ was the rationale for the re-coordinatization in 3.7.

To describe the classification of affine reflection systems we need some more properties of the subsets Λ_ξ of an extension datum. They are given in the following exercise (just do it!). Recall from Exercise 3.8 that a reflection subspace A is called symmetric if $A = -A$ and is called pointed if $0 \in A$.

Exercise 3.16. Let $(\Lambda_\eta : \eta \in S)$ be an extension datum of type (S, Z). Show:

(a) Every Λ_ξ for $0 \neq \xi \in S$ is a symmetric reflection subspace and is even a pointed reflection subspace if $\xi \in S^\times_{\rm ind}$.

(b) For $w \in W(S)$, the Weyl group of S, we have

$$\Lambda_\xi = \Lambda_{w(\xi)}. \tag{3.16}$$

In particular, $\Lambda_\xi = \Lambda_{-\xi} = -\Lambda_\xi$.

(c) Whenever $0 \neq \xi \in S$ and $2\xi \in S$, then

$$\Lambda_{2\xi} \subset \Lambda_\xi.$$

(d) $\mathbb{Z}\Lambda_\xi \subset \Lambda_\xi$ for $\xi \in S^\times_{\rm ind}$.

Let $(\Lambda_\xi : \xi \in S)$ be an extension datum where S is irreducible. Then $W(S)$ acts transitively on the roots of the same length ([Bou2, VI, §1.3, Prop. 11]), i.e., on $\{0\}$, $S_{\rm sh}$, $S_{\rm lg}$ and $S_{\rm div}$ (some of these sets might be empty). Because of (3.16), there are therefore at most four different subsets Λ_0, $\Lambda_{\rm sh}$, $\Lambda_{\rm lg}$ and $\Lambda_{\rm div}$ among the Λ_ξ, defined by

$$\Lambda_\xi = \begin{cases} \Lambda_0, & \xi = 0; \\ \Lambda_{\rm sh}, & \xi \in S_{\rm sh}; \\ \Lambda_{\rm lg}, & \xi \in S_{\rm lg}; \\ \Lambda_{\rm div}, & \xi \in S^\times_{\rm div}. \end{cases} \tag{3.17}$$

Of course, $\Lambda_{\rm lg}$ or $\Lambda_{\rm div}$ only exists if the corresponding subset of roots exits. The assertions below referring to $\Lambda_{\rm lg}$ or $\Lambda_{\rm div}$ should be interpreted correspondingly.

We have seen in Exercise 3.16 (did you do it?), that the subsets $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are pointed reflection subspaces and that $\Lambda_{\rm div}$ is a symmetric reflection subspace. Assuming only these properties, does however not give an extension datum, since only parts of the axiom (ED1) are fulfilled, namely those with $\eta = \pm\xi$. We also need to evaluate what happens for $\eta \neq \pm\xi$ with $\langle \eta, \xi^\vee \rangle \neq 0$. We will do this in the following examples.

Examples 3.17. Let S be an irreducible root system. We suppose that we are given a pointed reflection subspace $\Lambda_{\rm sh}$ of a finite-dimensional vector space, and if $S_{\rm lg} \neq \emptyset$ or $S^\times_{\rm div} \neq \emptyset$ then also a pointed reflection subspace $\Lambda_{\rm lg}$ and a symmetric reflection subspace $\Lambda_{\rm div}$. We define Λ_ξ, $\xi \in S$, by (3.17) and ask, when is the family $\mathfrak{L}_{\rm min}$ defined in this way an extension datum in $Z = \operatorname{span}\big(\bigcup_{\xi \in S} \Lambda_\xi\big)$? Note that we only have to check (ED1). We will consider some examples of S.

(a) $S = \mathrm{A}_1$: In this case there are no further conditions, so $\mathfrak{L}_{\min}$ describes all possible extension data for A_1 with $\Lambda_0 = \{0\}$.

(b) $S = \mathrm{A}_2$: In this case there exists roots η, ξ with $\langle \eta, \xi^\vee \rangle = 1$, namely those for which $\angle(\eta, \xi) = \frac{\pi}{3}$. Evaluating (ED1) for those gives $\Lambda_{\rm sh} - \Lambda_{\rm sh} \subset \Lambda_{\rm sh}$, forcing $\Lambda_{\rm sh}$ to be a subgroup of Z. Thus, $\mathfrak{L}_{\min}$ is an extension datum for $S = \mathrm{A}_2$ iff $\Lambda_{\rm sh}$ is a subgroup.

(c) S simply laced, $\operatorname{rank} S \geq 2$: The argument in (b) works whenever $S_{\rm sh}$ contains roots η, ξ with $\langle \eta, \xi^\vee \rangle = 1$. Since this is the case here, we get that $\mathfrak{L}_{\min}$ is an extension datum iff $\Lambda_{\rm sh}$ is a subgroup of Z.

(d) $S = \mathrm{B}_2 = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}$: Here we have pointed reflection subspaces $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$. Since non-zero roots of the same length are either proportional or orthogonal, (ED1) is fulfilled for them. Because (ED1) is invariant under sign changes, we are left to evaluate the case of two roots η, ξ of different lengths forming an obtuse angle of $\frac{3\pi}{4}$, for example $\eta = \varepsilon_1$, $\xi = \varepsilon_2 - \epsilon_1$.

If $\langle \eta, \xi^\vee \rangle = -2$, then η is long, ξ is short and so (ED1) becomes $\Lambda_{\rm lg} + 2\Lambda_{\rm sh} \subset \Lambda_{\rm lg}$. If η is short, ξ is long, we have $\langle \eta, \xi^\vee \rangle = -1$ and thus get the condition $\Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$. To summarize: *$\mathfrak{L}_{\min}$ is an extension datum for $S = \mathrm{B}_2$ iff $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are pointed reflection subspaces satisfying*

$$\Lambda_{\rm lg} + 2\Lambda_{\rm sh} \subset \Lambda_{\rm lg} \quad \textit{and} \quad \Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}. \tag{3.18}$$

Note that (3.18) implies $2\Lambda_{\rm sh} \subset \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$.

(e) $S = \mathrm{B}_l$, $l \geq 3$. Recall $S = \{\pm\epsilon_i : 1 \leq i \leq l\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq l\}$. Since the short roots in S are either proportional or orthogonal, (ED1) is fulfilled for all short roots η, ξ. But there exist long roots $\eta, \xi \in S$ with $\angle(\eta, \xi) = \frac{\pi}{3}$, whence $\langle \eta, \xi^\vee \rangle = 1$ and so (ED1) reads $\Lambda_{\rm lg} - \Lambda_{\rm lg} \subset \Lambda_{\rm lg}$. This forces $\Lambda_{\rm lg}$ to be a subgroup. As for $S = \mathrm{B}_2$, (ED1) for roots of different lengths leads to the condition (3.18). It is then easy to check that $\mathfrak{L}_{\min}$ *is an extension datum for $S = \mathrm{B}_l$, $l \geq 3$, iff $\Lambda_{\rm sh}$ is a pointed reflection subspace, $\Lambda_{\rm lg}$ is a subgroup and* (3.18) *holds.*

Continuing in this way, one arrives at the following.

Theorem 3.18 (Structure of extension data). *Let S be an irreducible finite root system and define a family $\mathfrak{L}_{\min}$ as in (3.17) with $\Lambda_0 = \{0\}$. Then $\mathfrak{L}_{\min}$ is an extension datum if and only if $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are pointed reflection subspaces, $\Lambda_{\rm div}$ is a symmetric reflection subspace and the following conditions, depending on S, hold.*

(i) *S is simply laced,* $\operatorname{rank} S \geq 1$ *: No further condition for $S = \mathrm{A}_1$, but $\Lambda_{\rm sh}$ is a subgroup if* $\operatorname{rank} S \geq 2$.

(ii) $S = \mathrm{B}_l (l \geq 2)$, $\mathrm{C}_l (l \geq 3)$, F_4 : *$\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ satisfy*

$$\Lambda_{\rm lg} + 2\Lambda_{\rm sh} \subset \Lambda_{\rm lg} \quad \textit{and} \quad \Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}.$$

Moreover,

- *$\Lambda_{\rm lg}$ is a subgroup if $S = \mathrm{B}_l$, $l \geq 3$ or $S = \mathrm{F}_4$, and*
- *$\Lambda_{\rm sh}$ is a subgroup if $S = \mathrm{C}_l$ or $S = F_4$.*

(iii) $S = \mathrm{G}_2$: *$\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are subgroups satisfying*

$$\Lambda_{\rm lg} + 3\Lambda_{\rm sh} \subset \Lambda_{\rm lg} \quad \textit{and} \quad \Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}.$$

(iv) $S = \mathrm{BC}_1$: *$\Lambda_{\rm sh}$ and $\Lambda_{\rm div}$ satisfy*

$$\Lambda_{\rm div} + 4\Lambda_{\rm sh} \subset \Lambda_{\rm div} \quad \textit{and} \quad \Lambda_{\rm sh} + \Lambda_{\rm div} \subset \Lambda_{\rm sh}.$$

(v) $S = \mathrm{BC}_l (l \geq 2)$: *$\Lambda_{\rm sh}$, $\Lambda_{\rm lg}$ and $\Lambda_{\rm div}$ satisfy*

$$\begin{aligned} \Lambda_{\mathrm{lg}} + 2\Lambda_{\mathrm{sh}} &\subset \Lambda_{\mathrm{lg}}, & \Lambda_{\mathrm{sh}} + \Lambda_{\mathrm{lg}} &\subset \Lambda_{\mathrm{sh}}, \\ \Lambda_{\mathrm{div}} + 2\Lambda_{\mathrm{lg}} &\subset \Lambda_{\mathrm{div}}, & \Lambda_{\mathrm{lg}} + \Lambda_{\mathrm{div}} &\subset \Lambda_{\mathrm{lg}}, \\ \Lambda_{\mathrm{div}} + 4\Lambda_{\mathrm{sh}} &\subset \Lambda_{\mathrm{div}}, & \Lambda_{\mathrm{sh}} + \Lambda_{\mathrm{div}} &\subset \Lambda_{\mathrm{sh}}. \end{aligned}$$

In addition, if $l \geq 3$ then Λ_{lg} is a subgroup.

The inclusions $\Lambda_{\mathrm{div}} + 4\Lambda_{\mathrm{sh}} \subset \Lambda_{\mathrm{div}}$ and $\Lambda_{\mathrm{sh}} + \Lambda_{\mathrm{div}} \subset \Lambda_{\mathrm{sh}}$ in case (v) above are consequences of the other inclusions. Since 0 lies in Λ_{sh} and also in Λ_{lg} if it exists, the displayed inclusions in the Structure Theorem above imply

$$\Lambda_{\mathrm{div}} \subset \Lambda_{\mathrm{lg}} \subset \Lambda_{\mathrm{sh}}. \tag{3.19}$$

The details of this theorem are given in [AABGP, II, §2] for the special case of extended affine root systems and then in [Y3] in general (it follows from the Structure Theorem 3.10 that an affine reflection system is the same as a "root system extended by a torsion-free abelian group of finite rank" in the sense of [Y3]). The reference [AABGP] also contains a classification of discrete extension data for extended affine root systems of low nullity.

Exercise 3.19. Without looking at [AABGP] or [Y3], work out some of the cases above.

3.4. **Extended affine root systems.** Let us come back to the beginning of this chapter. Our goal was to describe the structure of the set of roots occurring in an extended affine Lie algebra. After all the preparations in 3.1–3.3, this is now easy.

We start with the same setting as in 3.1, i.e., X is a finite-dimensional vector space over a field F of characteristic 0, R is a subset of X and $(\cdot|\cdot)$ is a symmetric bilinear form on X. As in 3.1 we define $R^0 = \{\alpha \in R : (\alpha|\alpha) = 0\}$, $R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha) \neq 0\}$ and

$$\langle x, \alpha^\vee \rangle = 2\frac{(x|\alpha)}{(\alpha|\alpha)}, \quad (x \in X \text{ and } \alpha \in R^{\mathrm{an}}).$$

Definition 3.20. A triple $(R, X, (\cdot|\cdot))$ as above is called an *extended affine root system* or EARS for short, if the following seven axioms (EARS1)–(EARS7) are fulfilled.

(EARS1): $0 \in R$ and R spans X,

(EARS2): R has *unbroken finite root strings*, i.e., for every $\alpha \in R^{\mathrm{an}}$ and $\beta \in R$ there exist $d, u \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\} \quad \text{and} \quad d - u = \langle \beta, \alpha^\vee \rangle.$$

(d stands for "down" and u for "up".)

(EARS3): $R^0 = R \cap X^0$.

(EARS4): R is reduced as defined in 3.1: for every $\alpha \in R^{\mathrm{an}}$ we have $F\alpha \cap R^{\mathrm{an}} = \{\pm\alpha\}$.

(EARS5): R is connected in the sense of 3.1: whenever $R^{\mathrm{an}} = R_1 \cup R_2$ with $(R_1 \mid R_2) = 0$, then $R_1 = \emptyset$ or $R_2 = \emptyset$.

(EARS6): R is tame, i.e., $R^0 \subset R^{\mathrm{an}} + R^{\mathrm{an}}$.

(EARS7): The abelian group $\mathrm{span}_{\mathbb{Z}}(R^0)$ is free of finite rank.

In analogy with the concept of discrete EALAs we call $(R, X, (\cdot|\cdot))$ for $F = \mathbb{C}$ or $F = \mathbb{R}$ a *discrete* extended affine root system if (EARS1)–(EARS6) hold and in addition

(DE): R is a discrete subset of X, equipped with the natural topology.

As for EALAs, a discrete extended affine root system necessarily satisfies (EARS7), see Proposition 3.21(c) below, so that it is justified to call it an EARS.

We will immediately connect EARS to affine reflection systems:

Proposition 3.21. (a) *A pair* (R,X) *satisfying* (EARS1)–(EARS3) *is an affine reflection system. In particular:*

(i) *An extended affine root system is an affine reflection system which is reduced, connected, symmetric, tame and which has unbroken root strings.*

(ii) *If* $F=\mathbb{R}$ *we can assume that* $(\cdot|\cdot)$ *is positive semidefinite.*

(b) *Let* (R,X) *be an affine reflection system with quotient root system* S *and extension datum* $\mathcal{L}$. *Then* (R,X) *is an extended affine root system if and only if*

(i) S *is irreducible, hence* $\mathcal{L}=(\Lambda_0,\Lambda_{\rm sh},\Lambda_{\rm lg},\Lambda_{\rm div})$,

(ii) $\Lambda_0=\Lambda_{\rm sh}+\Lambda_{\rm sh}$, $\Lambda_{\rm div}\cap 2\Lambda_{\rm sh}=\emptyset$, *and*

(iii) (EARS7) *holds.*

(c) *For an extended affine root system* (R,X) *over* $F=\mathbb{R}$ *or* $F=\mathbb{C}$ *the following are equivalent:*

(i) R *is discrete;*

(ii) R^0 *is a discrete subset of* X*;*

(iii) $\operatorname{span}_{\mathbb{Z}}(R^0)$ *is a discrete subgroup of* X.

In this case, all reflection subspaces Λ_ξ *of* (b) *are discrete too, and* $\operatorname{span}_{\mathbb{Z}}(R^0)$ *is a free abelian group of finite rank.*

Proof. (a) Obviously (AR1) = (EARS1) and (AR4) = (EARS3). The axiom (AR2), i.e., $s_\alpha(R)=R$, follows from (EARS2): $s_\alpha(\beta)=\beta+(u-d)\alpha$ and $-d\le u-d\le u$. Also (AR3) is immediate from (EARS2).

In any affine reflection system the root strings $R\cap(\beta+\mathbb{Z}\alpha)$ for $(\beta,\alpha)\in R\times R^{\rm an}$ are finite. This is Exercise 3.14(d) and an immediate consequence of the Structure Theorem 3.10. Also, a tame affine reflection system with unbroken roots strings is necessarily symmetric by Prop. 3.13. The characterization of an EARS in (i) is now clear. (ii) follows from Cor. 3.12.

(b) follows from Prop. 3.13 and Exercise 3.14, since for a connected R = irreducible S the formula (3.19) implies $\Lambda_{\rm diff}=\Lambda_{\rm sh}+\Lambda_{\rm sh}$.

(c) (i) ⇒ (ii) is obvious. Suppose (ii) holds. We know from (b) that $R^0=\Lambda_0=\Lambda_{\rm sh}+\Lambda_{\rm sh}$. Since $\Lambda_{\rm sh}$ is a pointed reflection subspace, so is Λ_0 (Exercise 3.8). Hence $2\operatorname{span}_{\mathbb{Z}}(\Lambda_0)\subset\Lambda_0$ is discrete. But then so is $\operatorname{span}_{\mathbb{Z}}(\Lambda_0)$. Thus (ii) ⇒ (iii).

By (3.19) and (b.ii), $\Lambda_{\rm div}\subset\Lambda_{\rm lg}\subset\Lambda_{\rm sh}\subset\Lambda_0$. Hence, if (iii) holds, then all Λ_ξ are discrete subsets of X. But then so is is R, as a finite union of discrete subsets. This shows (iii) → (i).

It is well-known fact that every discrete subgroup of a finite-dimensional real vector space is free of finite rank. □

A quick comparison of [AABGP, Definition 2.1] and our Definition 3.20 together with Prop. 3.21(a) will convince the reader that an extended affine root system in the sense of [AABGP] is the same as a discrete extended affine root system over $\mathbb{R}$ in our sense. The reason for the generalization and the change of name is the same as the one justifying our more general notion of extended affine Lie algebras: We are considering EALAs over arbitrary fields of characteristic 0 and the set of roots of an EALA will not be an extended affine root system in the sense of [AABGP].

Finally, here is the result which brings us back to EALAs.

Theorem 3.22. *Let (E, H) be an EALA over F and let $R \subset H^*$ be its set of roots. Put $X = \operatorname{span}_F(R)$ and let $(\cdot|\cdot)_X$ be the restriction of the bilinear form (2.3) to X. Then $(R, X, (\cdot|\cdot)_X)$ is an extended affine root system. If $F = \mathbb{C}$, then (E, H) is a discrete EALA if and only if $(R, X, (\cdot|\cdot)_X)$ is a discrete extended affine root system.*

Remarks 3.23. (a) For (E, H) a discrete EALA over $F = \mathbb{C}$, the theorem is proven in [AABGP, I, Th. 2.16], using discreteness. The generalization to arbitrary EALAs is due to the author, see [Ne4, Prop. 3]. It has been further generalized to other classes of Lie algebras, the so-called invariant affine reflection algebras, see [Ne5, Th. 6.6 and Th. 6.8]. Special cases have also been proven in [Az1] and [MY]. That R is symmetric, is an easy exercise, namely Exercise 2.1(c).

(b) In view of the theorem above, one can ask if every extended affine root system is the set of roots of some extended affine Lie algebra. This is however not the case, see [AG, Th. 6.2] for a detailed discussion of this question.

As a first application of this theorem, we can now completely characterize EALAs of nullity 0.

Proposition 3.24. *The following are equivalent:*

(i) *(E, H) is an EALA of nullity 0,*
(ii) *(E, H) is an EALA with a finite-dimensional E,*
(iii) *E is a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra H.*

In this case, E equals its core and the set of roots R coincides with the quotient root system S of R and is an irreducible reduced finite root system.

4. The core and centreless core of an EALA

In the previous chapter we have studied affine reflection systems per se. The rationale for doing so became clear only in the end, when we saw in Th. 3.22 that the set of roots R of an EALA (E, H) is an extended affine root system, a special type of an affine reflection system.

In this chapter we start by drawing consequences of the Structure Theorem 3.10 of affine reflection systems and the description of extended affine root systems in Prop. 3.21. The examples in §2.4 and §2.5 indicate that the core E_c and centreless core $E_{cc} = E_c/Z(E_c)$ of an extended affine Lie algebra (E, H) really are the "core" of the matter. We will show in Th. 4.14 and in Cor. 4.16 that both are so-called Lie tori, a new class of Lie algebras which we will introduce in 4.1. We will present some basic properties of Lie tori in 4.2 and describe some examples in 4.4 and 4.5.

With some justification, this chapter could therefore also be entitled "On Lie tori". But the reader can be re-assured that we are not getting side-tracked too much: In the next chapter we will see that Lie tori are precisely what is needed to construct EALAs.

4.1. **Lie tori: Definition.**

Lie tori are special objects in the following category of graded Lie algebras.

Definition 4.1. Let (S, Y) be a finite irreducible, but not necessarily reduced root system, as defined in Example 3.3. We denote by $\mathcal{Q}(S) = \operatorname{span}_{\mathbb{Z}}(S) \subset Y$ the root lattice of S. To avoid some degeneracies we will always assume that $S \neq \{0\}$. Let Λ be an abelian group.

A $(\mathcal{Q}(S), \Lambda)$-*graded* Lie algebra is a Lie algebra L with compatible $\mathcal{Q}(S)$- and Λ-gradings. It is convenient (and helpful) to use subscripts for the $\mathcal{Q}(S)$-grading and superscripts for the Λ-grading. Thus,

$$L = \bigoplus_{q \in \mathcal{Q}(S)} L_q = \bigoplus_{\lambda \in \Lambda} L^\lambda$$

are $\mathcal{Q}(S)$- and Λ-gradings of L, and compatibility means

$$L = \bigoplus_{q \in \mathcal{Q}(S),\, \lambda \in \Lambda} L_q^\lambda \quad \text{for } L_q^\lambda = L_q \cap L^\lambda.$$

Hence for $\lambda, \mu \in \Lambda$ and $p, q \in \mathcal{Q}(S)$

$$L^\lambda = \bigoplus_{q \in \mathcal{Q}(S)} L_q^\lambda, \quad L_q = \bigoplus_{\lambda \in \Lambda} L_q^\lambda \quad \text{and} \quad [L_q^\lambda, L_p^\mu] \subset L_{q+p}^{\lambda+\mu}.$$

Thus, L has three gradings, by $\mathcal{Q}(S)$, Λ and $\mathcal{Q}(S) \oplus \Lambda$ whose interplay will be crucial in the following. Corresponding to these three different gradings are three *support sets* : $\operatorname{supp}_{\mathcal{Q}(S)} L = \{q \in \mathcal{Q}(S) : L_q \neq 0\}$, $\operatorname{supp}_\Lambda L = \{\lambda \in L : L^\lambda \neq 0\}$, and $\operatorname{supp}_{\mathcal{Q}(S)\oplus\Lambda} L = \{(q, \lambda) \in (\mathcal{Q}(S), \Lambda) : L_q^\lambda \neq 0\}$.

Definition 4.2. We keep the notation of the Def. 4.1. A *Lie torus of type* (S, Λ) is a $(\mathcal{Q}(S), \Lambda)$-graded Lie algebra L over F, a field of characteristic 0, satisfying the axioms (LT1)–(LT3) below.

(LT1): $\operatorname{supp}_{\mathcal{Q}(S)} L \subset S$, hence $L = \bigoplus_{\xi \in S} L_\xi$.

(LT2): If $L_\xi^\lambda \neq 0$ and $\xi \neq 0$, then there exist $e_\xi^\lambda \in L_\xi^\lambda$ and $f_\xi^\lambda \in L_{-\xi}^{-\lambda}$ such that

$$L_\xi^\lambda = F e_\xi^\lambda, \quad L_{-\xi}^{-\lambda} = F f_\xi^\lambda, \tag{4.1}$$

and for $x_\tau \in L_\tau$ we have

$$[[e_\xi^\lambda, f_\xi^\lambda], x_\tau] = \langle \tau, \xi^\vee \rangle x_\tau. \tag{4.2}$$

(LT3): (a) $L_\xi^0 \neq 0$ if $\xi \in S_{\text{ind}}^\times$, i.e., $0 \neq \xi \in S$ and $\xi/2 \notin S$.
(b) As a Lie algebra, L is generated by $\bigcup_{0 \neq \xi \in S} L_\xi$.
(c) $\Lambda = \operatorname{span}_{\mathbb{Z}}(\operatorname{supp}_\Lambda L)$.

We will say that L is a Lie torus if L is a Lie torus for some pair (S, Λ).

A Lie torus is called *invariant*, if L has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ which is *graded* in the sense that

$$(L_\xi^\lambda \mid L_\tau^\mu) = 0 \quad \text{if } \lambda + \mu \neq 0 \text{ or } \xi + \tau \neq 0. \tag{4.3}$$

Two Lie tori L and $\tilde{L}$, both of type (S, Λ), are called *graded-isomorphic* if there exists a Lie algebra isomorphism $f : L \to \tilde{L}$ such that $f(L_\xi^\lambda) = \tilde{L}_\xi^\lambda$ for all $(\xi, \lambda) \in S \times \Lambda$. Thus, a graded-isomorphism is an isomorphism in the category of graded Lie algebras. But we will use the term "graded-isomorphism" to emphasize that Lie tori are graded algebras.

Remarks 4.3. (a) Let L be a Lie torus. Hence, by (LT1), $L = \bigoplus_{\xi \in S} L_\xi = \bigoplus_{\xi \in S,\, \lambda \in \Lambda} L_\xi^\lambda$. We will determine $\operatorname{supp}_{\mathcal{Q}(S)} L$ in Cor. 4.6 below. The axiom (LT2) implies that

$$\dim L_\xi^\lambda = 1 \quad \text{if } 0 \neq \xi \text{ and } L_\xi^\lambda \neq 0 \tag{4.4}$$

and that

$$(e_\xi^\lambda, h_\xi^\lambda, f_\xi^\lambda) \quad \text{with} \quad h_\xi^\lambda = [e_\xi^\lambda, f_\xi^\lambda] \in L_0^0 \tag{4.5}$$

is an $\mathfrak{sl}_2$-triple. The condition (LT3.a) together with (LT2) ensures that a Lie torus has enough $\mathfrak{sl}_2$-triples.

The other two conditions in (LT3) are not really serious; they just serve to normalize things: If (LT3.c) does not hold, one can simply replace Λ by $\operatorname{span}_{\mathbb{Z}}(\operatorname{supp}_\Lambda L)$. Also,

$$\text{(LT3.b)} \iff L_0^\lambda = \textstyle\sum_{0\neq\xi\in S}\sum_{\mu\in\Lambda}[L_\xi^\mu,\, L_{-\xi}^{\lambda-\mu}] \tag{4.6}$$

for all $\lambda \in \Lambda$. If one has a Lie algebra, for which all axioms except (4.6) hold, one can replace the subspaces L_0^λ by the right hand side of (4.6) and then gets a Lie torus. Observe that (4.6) for $\lambda = 0$ together with (LT2) yields

$$L_0^0 = \textstyle\sum F h_\xi^\lambda \tag{4.7}$$

where the sum in (4.7) is taken over all pairs (ξ, λ) for which h_ξ^λ exists, i.e., those with $L_\xi^\lambda \neq 0$ and $\xi \neq 0$.

(b) A Lie torus is a special type of a so-called division-(S,Λ)-graded Lie algebras, or more generally of a root-graded Lie algebra. This and also the different approaches to root-graded Lie algebras are discussed in [Ne5, §5]. Lie tori were first defined by Yoshii in [Y3, Y4], using the notion of a root-graded Lie algebra. The definition above is due to the author [Ne3].

Viewing a Lie torus as a special type of a root-graded Lie algebra is the approach used in the classification of Lie tori.

(c) Why was a Lie torus christened a "Lie torus"? The historically correct answer is: Because of pure analogy with already existing names like a quantum torus, defined in 4.25, or an alternative or Jordan torus. All of these are graded algebras, in which every non-zero homogeneous element is invertible. If one interprets the elements e and f of the $\mathfrak{sl}_2$-triple (4.5) as invertible elements of L, then a Lie torus is a graded Lie algebra in which most of the non-zero homogenous elements are invertible. It is certainly unusual to speak of "invertible elements" in a Lie algebra. But the examples below will provide some justification to that: We will see that the invertible elements of L are given by invertible elements of its coordinate algebra.

Besides the analogy with the already existing concepts of "tori" in categories of (non)associative algebras, the fact that a *toroidal* Lie algebra is a Lie *torus*, see 4.31, reinforces the choice of the name Lie torus.

4.2. **Some basic properties of Lie tori.** Throughout this section L is a Lie torus of type (S, Λ). We use the notation of Def. 4.2. We describe some basic properties of Lie tori and prove some of them, in particular those for which there does not yet exist a published proof.

We first show that the homogeneous subspaces of the $\mathcal{Q}(S)$-grading of L are weight spaces for the ad-diagonalizable subalgebra

$$\mathfrak{h} = \operatorname{span}_F\{h_\xi^0 : \xi \in S_{\mathrm{ind}}^\times\}.$$

Lemma 4.4. *The subspaces L_τ, $\tau \in S$, are given by*

$$L_\tau = \{l \in L : [h_\xi^0, l] = \langle \tau, \xi^\vee\rangle l \textit{ for all } \xi \in S_{\mathrm{ind}}^\times\}. \tag{4.8}$$

Proof. The inclusion from left to right holds by (4.2). For the proof of the other inclusion we write $l \in L$ as $l = \sum_\alpha l_\alpha$ with $l_\alpha \in L_\alpha$. Then l satisfies $[h_\xi^0, l] = \langle \tau, \xi^\vee\rangle l$ for all $\xi \in S_{\mathrm{ind}}^\times$ if and only if for every $\alpha \in S$ we have $\langle \alpha - \tau, \xi^\vee\rangle l_\alpha = 0$ for all $\xi \in S_{\mathrm{ind}}^\times$. Since $\operatorname{span}_F(S_{\mathrm{ind}}) = Y$ and the bilinear form on Y associated with the root system S is nondegenerate, for every pair $(\alpha, \tau) \in S^2$ with $\alpha \neq \tau$ there exists $\xi \in S^\times_{\rm ind}$ with $\langle \alpha - \tau, \xi^\vee\rangle \neq 0$. Hence, any l belonging to the set on the right hand side of (4.8) has $l_\alpha = 0$ for $\alpha \neq \tau$, proving $l \in L_\tau$. □

Proposition 4.5. *For every $(\xi, \lambda) \in \operatorname{supp}_{\mathcal{Q}(S)\oplus\Lambda} L$ with $\xi \neq 0$ the map*

$$\varphi_\xi^\lambda = \exp\big(\operatorname{ad}(e_\xi^\lambda)\big)\, \exp\big(\operatorname{ad}(-f_\xi^\lambda)\big)\, \exp\big(\operatorname{ad}(e_\xi^\lambda)\big)$$

is a well-defined automorphism of the Lie algebra L with the property

$$\varphi_\xi^\lambda(L_\tau^\mu) = L_{s_\xi(\tau)}^{\mu - \langle \mu, \xi^\vee\rangle\lambda}. \tag{4.9}$$

Moreover, for every $w \in W(S)$, the Weyl group of S, there exists an automorphism φ_w of the Lie algebra L such that

$$\varphi_w(L_\tau^\mu) = L_{w(\tau)}^\mu$$

for all $\tau \in S$ and $\mu \in L$.

This proposition can be proven in the same way as [AABGP, Prop. 1.27].

Corollary 4.6 ([ABFP2, Lemma 1.10])**.** *The $\mathcal{Q}(S)$-support of L satisfies*

$$\operatorname{supp}_{\mathcal{Q}(S)} L = \begin{cases} S & \text{if } S \text{ is reduced,} \\ S \text{ or } S_{\rm ind} & \text{if } S \text{ is non-reduced.}\end{cases}$$

As a consequence of this corollary, $\operatorname{supp}_{\mathcal{Q}(S)} L$ is always a finite irreducible root system. It is immediate that L is also a Lie torus of type $(\operatorname{supp}_{\mathcal{Q}(S)}, \Lambda)$. Without loss of generality we can therefore assume that $S = \operatorname{supp}_{\mathcal{Q}(S)} L$ if this is convenient.

Proposition 4.7. *For $\xi \in S$ define*

$$\Lambda_\xi = \{\lambda \in \Lambda : L_\xi^\lambda \neq 0\},$$

so that $\operatorname{supp}_\Lambda L = \bigcup_{\xi\in S} \Lambda_\xi$. Then the family $(\Lambda_\xi : \xi \in S)$ satisfies the axioms (ED1) *and* (ED2) *of* Def. 3.9*,*

$$\Lambda_\eta - \langle \eta, \xi^\vee\rangle \Lambda_\xi \subset \Lambda_{s_\xi(\eta)} \tag{ED1}$$

$$0 \in \Lambda_\xi \text{ for } \xi \in S_{\rm ind} \text{ and } \Lambda_\xi \neq \emptyset \text{ for } \xi \in S^\times_{\rm div} \tag{ED2}$$

Hence Λ_ξ is a pointed reflection subspace for $\xi \in S^\times_{\rm ind}$, a symmetric reflection subspace for $\xi \in S^\times_{\rm div}$ and

$$\Lambda_\xi = \Lambda_{w(\xi)} \quad \text{for all } w \in W(S). \tag{4.10}$$

Defining $\Lambda_{\rm sh}$, $\Lambda_{\rm lg}$ and $\Lambda_{\rm div}$ as in 3.17*, we have*

$$\Lambda_{\rm sh} \supset \Lambda_{\rm lg} \supset \Lambda_{\rm div}, \tag{4.11}$$

$$\operatorname{supp}_\Lambda L = \Lambda_0 = \Lambda_{\rm sh} + \Lambda_{\rm sh}, \tag{4.12}$$

$$\emptyset = 2\Lambda_{\rm sh} \cap \Lambda_{\rm div}, \tag{4.13}$$

$$\Lambda = \operatorname{span}_{\mathbb{Z}}(\Lambda_{\rm sh}). \tag{4.14}$$

The support families $(\Lambda_\xi : \xi \in S)$ are the same for L and $L/Z(L)$.

Proof. (ED1) is a consequence of Prop. 4.5 and (ED2) of (LT3.a). It then follows as in section 3 that Λ_ξ, $\xi \in S^\times$, are pointed respectively symmetric reflection subspaces such that (4.10) and (4.11) hold. (4.12) and (4.13) are proven in [Y3, Th. 5.1] and [ABFP2, Lemma 1.1.12]. (4.14) follows from (4.11) and (4.12). The last claim is also proven in [Y3, Th. 5.1]. □

Proposition 4.8. *Define*

$$\mathfrak{g} = \textit{subalgebra generated by } \{L_\xi^0 : \xi \in S_{\rm ind}^\times\},$$
$$\mathfrak{h} = \operatorname{span}_F\{h_\xi^0 : \xi \in S_{\rm ind}^\times\}.$$

(a) *Then $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$.*

(b) *The root system $S_{\rm ind}$ and the root system of $(\mathfrak{g}, \mathfrak{h})$ are canonically isomorphic. Namely, for every $\xi \in S_{\rm ind}^\times$ there exists a unique $\tilde{\xi} \in \mathfrak{h}^*$, defined by $\tilde{\xi}(h_\eta^0) = \langle \xi, \eta^\vee \rangle$ for $\eta \in S_{\rm ind}^\times$, such that the map $\xi \mapsto \tilde{\xi}$ extends to an isomorphism between the root system $S_{\rm ind}$ and the root system of $(\mathfrak{g}, \mathfrak{h})$.*

Proof. This is a special case of a result for arbitrary root-graded Lie algebras, see [Ne1, Remark 2 of §2.1] and [Ne5, Prop. 5.9]. It is essentially a corollary to the Chevalley-Serre presentation of finite-dimensional split simple Lie algebras. For Lie tori it was announced in [Ne3, §3]. The details of the proof are given in [ABFP2, Prop. 1.2.2]. □

The following Ex. 4.10 lists some more basic properties of Lie tori. You will need Ex. 4.9(a) in part (d) of 4.10.

Exercise 4.9. (a) Let K be a perfect Lie algebra. Then $K/Z(K)$ is a perfect and centreless Lie algebra.

(b) Let E be a Lie algebra with an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$, and let K be an ideal of E with $K = [E, K]$. Then $\{z \in K : (z \mid K) = 0\} = Z(K)$.

Exercise 4.10. Let L be a Lie torus of type (S, Λ). Show:

(a) L_0^λ is given by the formula (4.6).

(b) L is perfect.

(c) The centre satisfies $Z(L) = \bigoplus_{\lambda\in\Lambda} Z(L)^\lambda$ for $Z(L)^\lambda = Z(L) \cap L_0^\lambda$.

(d) Let $Y = \bigoplus_{\lambda\in\Lambda} Y_0^\lambda$, $Y_0^\lambda = Y \cap L_0^\lambda$, be a graded subspace of $Z(L)$. Then L/Y is a Lie torus with respect to the subspaces $(L/Y)_\xi^\lambda = L_\xi^\lambda / Y_\xi^\lambda$, where for $\xi \neq 0$ we put $Y_\xi^\lambda = \{0\}$ and thus have $(L/Y)_\xi^\lambda \cong L_\xi^\lambda$ as vector spaces. In particular, $L/Z(L)$ is a centreless Lie torus.

(e) For $\lambda, \mu \in \Lambda_\xi$, $\xi \in S^\times$, we have $h_\xi^\lambda \equiv h_\xi^\mu \mod Z(L)$.

(f) $L^0 = \mathfrak{g} \oplus Z(L)^0$ and $L_0^0 = \mathfrak{h} \oplus Z(L)^0$.

(g) Let I be a Λ-graded ideal of L, whence $I = \bigoplus_{\lambda\in\Lambda} I^\lambda$ for $I^\lambda = I \cap L^\lambda$. Then either $I = L$ or $I \subset Z(L)$. In particular, a centreless Lie torus is graded-simple with respect to the Λ-grading of L.

Since a Lie torus is perfect by part (b) of the exercise above, it has a universal central extension (Th. 1.10).

Theorem 4.11 ([Ne3, §5], [Ne6])**.** *Let $\mathfrak{u} : \mathfrak{uce}(L) \to L$ be a universal central extension of a Lie torus $L = \bigoplus_{\xi,\lambda} L_\xi^\lambda$ of type (S, Λ). Then $\mathfrak{uce}(L)$ is also a Lie torus of type (S, Λ), say $\mathfrak{uce}(L) = \bigoplus_{\xi,\lambda} \mathfrak{uce}(L)_\xi^\lambda$, and $\mathfrak{u}$ maps $\mathfrak{uce}(L)_\xi^\lambda$ onto L_ξ^λ.*

Remark 4.12. It follows from this theorem and the exercise above that in order to describe Lie tori up to graded isomorphism, one can proceed in two steps:

(A) Classify centreless Lie tori, up to graded isomorphism. We will discuss some examples in 4.4 and 4.5.

(B) Describe the universal central extension of the centreless Lie tori from (A). They are unique up to isomorphism. We will not say anything about this here. The reader can find some results in [BGK, BGKN, Ne4, Ne6] for Lie tori arising from EALAs and in [ABG1, ABG2, BS, Ne6] for general root-graded Lie algebras.

Once (A) and (B) completed, an arbitrary Lie torus of type (S, Λ) is then obtained as $\mathfrak{uce}(L)/C$ where L is taken from the list in (A) and where C is a graded subspace of the centre of $\mathfrak{uce}(L)$.

The following results only holds for special types of Lie tori.

Theorem 4.13 ([Ne3, Th. 5], proven in [Ne6])**.** *Let $L = \bigoplus_{\xi \in S,\, \lambda \in \Lambda} L_\xi^\lambda$ be a Lie torus of type (S, Λ) where Λ is a finitely generated abelian group.*

(a) *Then L is finitely generated as Lie algebra and has bounded homogeneous dimension with respect to the $\mathcal{Q}(S) \oplus \Lambda$-grading of L.*

(b) *Moreover, the Lie algebra $\operatorname{Der}_F(L) = \operatorname{grDer}_F(L)$ where $\operatorname{grDer}_F(L)$ is naturally $\mathcal{Q}(S) \oplus \Lambda$-graded and has bounded homogeneous dimension with respect to this grading.*

(c) *If L is invariant, its universal central extension is isomorphic to the central extension $\mathrm{E}(L, D^{\mathrm{gr}*}, \psi_D)$ where D is any graded complement of $\operatorname{IDer}(L)$ in $\operatorname{SDer}_F(L)$.*

Part (c) of this theorem is an immediate corollary of parts (a) and (b) and of Th. 1.25.

4.3. **The core of an EALA.** We will now connect extended affine Lie algebras and Lie tori, and first introduce some notation. Let (E, H) be an EALA with set of roots R. We have seen in Th. 3.22 that R is an extended affine root system, hence an affine reflection system. We can therefore apply the Structure Theorem 3.10. Recall the following data describing the structure of R:

- $X = \operatorname{span}_F(R) \subset H^*$, $X^0 = \{x \in X : (x \mid X) = 0\} = \{x \in X : (x \mid R) = 0\}$, $f : X \to X/X^0 = Y$ the canonical projection,
- $S = f(R)$ the quotient root system, a finite irreducible but possibly non-reduced root system, and $S_{\mathrm{ind}} = \{\alpha \in S : \alpha/2 \notin S\} \cup \{0\}$,
- $g : Y \to X$ a linear map satisfying $f \circ g = \mathrm{Id}_Y$ and $g(S_{\mathrm{ind}}) \subset R$,
- $(\Lambda_\xi : \xi \in S)$ the associated extension datum, defined by $R \cap f^{-1}(\xi) = g(\xi) \oplus \Lambda_\xi$ and $\Lambda_\xi \subset X^0$,
- $\Lambda = \operatorname{span}_{\mathbb{Z}}\big(\bigcup_{\xi \in S} \Lambda_\xi\big)$, a free abelian group of finite rank (this is axiom (EARS7)).

Hence

$$\begin{aligned} R &= \textstyle\bigcup_{\xi \in S} \big(g(\xi) \oplus \Lambda_\xi\big) \subset g(Y) \oplus X^0, \\ R^{\mathrm{an}} &= \textstyle\bigcup_{\xi \in S^\times} \big(g(\xi) \oplus \Lambda_\xi\big), \\ R^0 &= 0 \oplus \Lambda_0 = R \cap X^0. \end{aligned}$$

Theorem 4.14 ([AG] for $F = \mathbb{C}$)**.** *Let $K = E_c$ be the core of an EALA (E, H). We use the notation of above and define subspaces*

$$K_\xi^\lambda = K \cap E_{g(\xi) \oplus \lambda} = \begin{cases} E_{g(\xi)\oplus\lambda} & \xi \neq 0, \\ K \cap E_{0 \oplus \lambda}, & \xi = 0. \end{cases} \tag{4.15}$$

(a) *Then* $K = \bigoplus_{\xi,\,\lambda} K_\xi^\lambda$ *is a Lie torus of type* (S,Λ)*, where* Λ *is free abelian of finite rank.*

(b) K *is a perfect ideal of* E*.*

(c) *Let* $(\cdot|\cdot)$ *be a nondegenerate invariant bilinear form on* E*, whose existence is guaranteed by the axiom* (EA1)*. Then the radical of the restricted bilinear form* $(\cdot|\cdot)|_{K\times K}$ *equals the centre* $Z(K)$*, that is*

$$\{z \in K : (z \mid K) = 0\} = Z(K) = \bigoplus_{\lambda\in\Lambda} Z(K) \cap K_0^\lambda. \tag{4.16}$$

Remark 4.15. The subspaces K_ξ^λ in (4.15) and hence the Lie torus structure of K depend on the section g. A different choice of g leads to a so-called *isotope* of K, see [AF] and [Ne5, Prop. 6.4].

Corollary 4.16. *We use the notation of* Th. 4.14*, and put* $E_{cc} = K/Z(K) = L$*, the* centreless core of (E,H)*. Then* L *is an invariant centreless Lie torus of type* (S,Λ) *with respect to the homogeneous subspaces*

$$L_\xi^\lambda = K_\xi^\lambda / \big(Z(K) \cap K_\xi^\lambda\big)$$

and the bilinear form $(\cdot|\cdot)_L$ *defined by*

$$(\bar{x} \mid \bar{y})_L = (x \mid y)$$

where $x, y \in K$*,* $\bar{x}$ *and* $\bar{y}$ *are the canonical images in* L *and* $(\cdot|\cdot)$ *is the bilinear form of* 4.14(c)*.*

Remark 4.17. Yoshii ([Y4]) has shown that any Lie torus of type (S,Λ) with Λ a torsion-free abelian group admits a non-zero graded invariant symmetric bilinear form. This implies that the existence of a nondegenerate such form on E_{cc}. However, Yoshii's proof uses the existence of invariant nondegenerate symmetric bilinear forms on Jordan tori ([NY]) and hence relies on the classification of Jordan tori.

We can now show that all root spaces of an EALA are finite-dimensional in a strong form.

Proposition 4.18 ([Ne4, Prop. 3])**.** *An EALA has finite bounded dimension. The same is true for its core and centreless core.*

Proof. Let $K = E_c$ be the core of the EALA (E,H). By Th. 4.14, K is a Lie torus of type (S,Λ), where Λ is a free abelian group of finite rank. Hence, by Th. 4.13, one knows that K has finite bounded dimension with respect to its double grading, say $\dim K_\xi^\lambda \le M_1$ for all pairs (ξ,λ). By the same reference, one also knows that the Lie algebra $\mathrm{Der}_F(K)$ of all F-linear derivations of K has a double grading by $\mathcal{Q}(S)$ and Λ,

$$\mathrm{Der}_F(K) = \bigoplus_{\xi\in S,\,\lambda\in\Lambda} (\mathrm{Der}_F K)_\xi^\lambda,$$

where $(\mathrm{Der}_F K)_\xi^\lambda$ is the subspace of those derivations mapping K_τ^μ to $K_{\xi+\tau}^{\lambda+\mu}$, and that $\mathrm{Der}_F(K)$ has finite bounded dimension with respect to this grading, say $\dim_F(\mathrm{Der}_F K)_\xi^\lambda \le M_2$ for all pairs (ξ,λ).

Since K is an ideal, we have a Lie algebra homomorphism $\rho : E \to \mathrm{Der}_F(K)$, given by $\rho(e) = \mathrm{ad}\, e|_K$. It is homogenous of degree 0, i.e., $\rho(E_\alpha) \subset (\mathrm{Der}_F K)_\xi^\lambda$ for $\alpha = g(\xi) \oplus \lambda$ as in (4.15). Moreover, by the tameness axiom (EA5) for an EALA we know that $\mathrm{Ker}\,\rho \subset K$ (whence $\mathrm{Ker}\,\rho = Z(K)$, but we won't need this). It now follows that $\dim E_\alpha = \dim \operatorname{Ker}(\rho|_{E_\alpha}) + \dim \rho(E_\alpha) \le \dim K_\xi^\lambda + \dim(\operatorname{Der}_F K)_\xi^\lambda \le M_1 + M_2$. □

4.4. **Lie tori of type A_l, $l \ge 3$.** As explained in Rem. 4.12, in classifying Lie tori one can restrict one's attention to the case of centreless Lie tori, at least modulo the solution of problem (B) in 4.12. In this section we will describe centreless Lie tori of type A.

The reader will expect that this will have something to do with trace-0-matrices. This turns out to be correct, but only with the proper interpretation of "trace-0". It will not be sufficient to consider trace-0-matrices over F. We will see in 4.28 that they will only lead to nullity 0-examples. Rather, one must allow matrices with entries from a possibly non-commutative algebra.

To avoid some degeneracies, in this section we let N be a natural number with $N \ge 3$. We start with an arbitrary associative unital F-algebra A. In particular, A need not be commutative, and hence

$$[A, A] = \operatorname{span}_F\{a_1a_2 - a_2a_1 : a_1, a_2 \in A\}$$

is in general non-zero. As usual, $\mathfrak{gl}_N(A)$ is the Lie algebra of all $N \times N$ matrices with entries in A and Lie algebra product $[x, y] = xy - yx$, the usual commutator of the matrices x and y. We define the *special linear Lie algebra* $\mathfrak{sl}_N(A)$ as the derived algebra

$$\mathfrak{sl}_N(A) = [\mathfrak{gl}_N(A), \mathfrak{gl}_N(A)].$$

of $\mathfrak{gl}_N(A)$. In particular, $\mathfrak{sl}_N(A)$ is an ideal of $\mathfrak{gl}_N(A)$. To analyze the structure of $\mathfrak{sl}_N(A)$ we use the matrix units E_{ij}, i.e., the $N \times N$ matrices with 1 at the position (ij) and 0 at all other positions. They satisfy the basic multiplication rule

$$[aE_{ij},\, bE_{mn}] = \delta_{jm}\, ab\, E_{in} - \delta_{ni}\, ba\, E_{mj} \tag{4.17}$$

where δ_* is the usual Kronecker delta. We put $E_N = \sum_{i=1}^N E_{ii}$. Some properties of the Lie algebra $\mathfrak{sl}_N(A)$ are listed in the following (very worthwhile) exercise.

Exercise 4.19. (a) $\mathfrak{sl}_N(A) = \{x \in \mathfrak{gl}_N(A) : \operatorname{tr}(x) \in [A, A]\}$.

(b) As a vector space, $\mathfrak{sl}_N(A)$ decomposes as

$$\begin{aligned} \mathfrak{sl}_N(A) &= \mathfrak{sl}_N(A)_0 \oplus \big(\textstyle\bigoplus_{i\ne j} AE_{ij}\big), \quad \text{where} \\ \mathfrak{sl}_N(A)_0 &= \mathfrak{sl}_N(A) \cap \big(\textstyle\bigoplus_{i=1}^N AE_{ii}\big) \\ &= \textstyle\sum_{i\ne j}[AE_{ij}, AE_{ji}] = \sum_{i\ne j} \operatorname{span}_F\{abE_{ii} - baE_{jj} : a, b \in A\} \\ &= \{cE_N : c \in [A, A]\} \oplus \big(\textstyle\bigoplus_{i=1}^{N-1}\{a(E_{ii} - E_{i+1,i+1}) : a \in A\}\big) \end{aligned} \tag{4.18}$$

(c) For a commutative A:

$$\mathfrak{sl}_N(A) = \{x \in \mathfrak{gl}_N(A) : \operatorname{tr}(x) = 0\} = \mathfrak{sl}_N(F) \otimes_F A.$$

(d) The centre of $\mathfrak{sl}_N(A)$ is $Z(\mathfrak{sl}_N(A)) = \{zE_N : z \in Z(A) \cap [A, A]\}$ where $Z(A) = \{z \in A : za = az \text{ for all } a \in A\}$ is the centre of A.

(e) For $a, b \in A$ and $i \ne j$,

$$(aE_{ij}, E_{ii} - E_{jj}, bE_{ji})$$

is an $\mathfrak{sl}_2$-triple if and only if a is invertible and $b = a^{-1}$.

The exercise shows that the structure of a general $\mathfrak{sl}_N(A)$ is quite similar to that of $\mathfrak{sl}_N(F)$. In particular, the decomposition (4.18) is a $\mathcal{Q}(\mathrm{A}_l)$-grading where

$$\mathrm{A}_l = \{\varepsilon_i - \varepsilon_j : 1 \le i, j \le N\}, \quad l = N-1.$$

is the root system of type A_l and

$$\mathfrak{sl}_N(A)_{\varepsilon_i - \varepsilon_j} = AE_{ij} \quad \text{for } i \ne j.$$

In fact, $\mathfrak{sl}_N(A)$ is the prototype of an A_l-graded Lie algebra (see [BerM]). At this level of generality we are far from the structure of a Lie torus. Most importantly, we are missing a compatible Λ-grading of $\mathfrak{sl}_N(A)$. We will use gradings of A, defined as follows.

Definition 4.20. Let $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ be a unital associative Λ-graded F-algebra. Then A is called an *associative torus of type* Λ if it satisfies (AT1)–(AT3) below.

(AT1): if every non-zero A^λ contains an invertible element,
(AT2): $\dim A^\lambda \le 1$ for all $\lambda \in \Lambda$, and
(AT3): $\mathrm{span}_{\mathbb{Z}}(\mathrm{supp}_\Lambda A) = \Lambda$.

One calls A simply an *associative torus* if A is an associative torus of type Λ for some abelian group Λ. See 4.23 for a short discussion of associative tori.

These definitions are justified by the following exercise, describing when $\mathfrak{sl}_N(A)$ is a Lie torus of type (A_l, Λ).

Exercise 4.21. (a) The Lie algebra $\mathfrak{sl}_N(A)$ has a Λ-grading compatible with the $\mathcal{Q}(\mathrm{A}_l)$-grading (4.18) if and only if A is Λ-graded. In this case, the compatible Λ-grading of $\mathfrak{sl}_N(A)$ is given by $\mathfrak{sl}_N(A) = \bigoplus_{\lambda\in\Lambda} \mathfrak{sl}_N(A)^\lambda$ where $\mathfrak{sl}_N(A)^\lambda$ consists of matrices in $\mathfrak{sl}_N(A)$, which have all their entries in A^λ.

(b) With respect to the compatible gradings of (a), the Lie algebra $\mathfrak{sl}_N(A)$ is a Lie torus of type $(\mathrm{A}_{N-1}, \Lambda)$ if and only if A is an associative torus of type Λ.

(c) The Lie torus $\mathfrak{sl}_N(A)$ is invariant with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{sl}}$ given by $(\sum_{i,j} x_{ij}E_{ij} \mid \sum_{p,q} y_{pq}E_{pq})_{\mathfrak{sl}} = \sum_{i,j}(x_{ij}y_{ji})_0$ where a_0 for $a \in A$ denotes the A^0-component of a.

But we not only have an example of a Lie torus of type (A_l, Λ), we actually have all centreless examples.

Theorem 4.22. *Let $l \ge 3$. A Lie algebra L is a centreless Lie torus of type* (A_l, Λ) *if and only if L is graded-isomorphic to $\mathfrak{sl}_{l+1}(A)$ for A an associative torus of type Λ. In this case, L is an invariant Lie torus.*

Proof. This is a special case of the Coordinatization Theorem of A_l-graded Lie algebras ([BerM, Recognition Theorem 0.7]): A centreless Lie algebra L is A_l-graded $(l = N-1)$ if and only if L is $\mathcal{Q}(\mathrm{A}_l)$-graded-isomorphic to $\mathfrak{sl}_N(A)/Z(\mathfrak{sl}_N(A))$ for some associative F-algebra A. If L is a Lie torus, it follows as in the Exercise 4.21 above that A is an associative torus. But $Z(\mathfrak{sl}_N(A)) = \{0\}$ for an associative torus ([NY, (3.3.2)] and Exercise 4.19). □

Besides [BerM], related results are proven in [BGK, Th. 2.65] (see Cor. 4.27 below), [GN, 2.11 and 3.4] and [Y2, Prop. 2.13].

Review 4.23 (Associative tori versus twisted group algebras)**.** In view of Th. 4.22 it is of interest to know more about associative tori. First of all, the identity 1_A of an associative torus A satisfies $1_A \in A^0$. Hence $a^{-1} \in A^{-\lambda}$ for every invertible

$a \in A^\lambda$. Moreover, since the product of two invertible elements in an associative algebra is again invertible, it follows that $\operatorname{supp}_\Lambda A$ is a subgroup of Λ, whence (AT3) is equivalent to

(AT3)′: $\operatorname{supp}_\Lambda A = \Lambda$.

Next, choose a family $(u_\lambda : \lambda \in \Lambda)$ of invertible elements $u_\lambda \in A^\lambda$. This is then in particular an F-basis of A so that the algebra structure of A is completely determined by the equations

$$u_\lambda u_\mu = c(\lambda, \mu) u_{\lambda+\mu} \tag{4.19}$$

for $\lambda, \mu \in \Lambda$ and suitable non-zero scalars $c(\lambda, \mu) \in F$. It is not necessary that $c(\lambda, \mu) = 1$, see for example Ex. 4.26. Rather, given an F-vector space with basis $(u_\lambda : \lambda \in \Lambda)$ one can define a multiplication on A by (4.19), and this multiplication is associative if and only if

$$c(\lambda, \mu)\, c(\lambda + \mu, \nu) = c(\mu, \nu)\, c(\lambda, \mu + \nu) \tag{4.20}$$

In this case, the algebra is an associative torus of type Λ. It is clear from the construction that, conversely, any associative torus is obtained in this way from a family $(c(\lambda, \mu))_{\lambda,\mu}$ of non-zero scalars. The algebras constructed in this way are called *twisted group algebras*. The reader with some knowledge in group cohomology will recognize that the families $(c(\lambda, \mu))$ satisfying (4.20) are precisely the 2-cocycles of Λ with values in $F \backslash \{0\}$. One can show that two families define graded-isomorphic tori if and only if their cohomology classes coincide.

Example 4.24 (Group algebra)**.** Although, as we have pointed out, the $c(\lambda, \nu)$ need not equal 1 in general, the family for which all $c(\lambda, \mu) = 1$ satisfies (4.20) and so yields an example of a Λ-torus, called the *group algebra of* Λ and denoted $F[\Lambda]$. In particular, this implies that associative tori exist for all Λ, and hence Lie tori exist for all types (A_l, Λ), $l \geq 2$.

The Lie tori that arise as cores of an EALA have type (S, Λ) where Λ is a free abelian group of finite rank. This condition on Λ follows from the axiom (EA6) or, equivalently from the axiom (EARS7). We therefore discuss this special case now.

Definition 4.25. Let $\mathbf{q} = (q_{ij})$ be an $n \times n$ matrix such that the entries $q_{ij} \in F$ satisfy $q_{ii} = 1 = q_{ij} q_{ji}$ for all $1 \leq i, j \leq n$. The *quantum torus* associated to $\mathbf{q}$ is the associative algebra $\mathbb{F}_\mathbf{q}$ presented by the generators t_i, t_i^{-1}, $1 \leq i \leq n$ subject to the relations

$$t_i t_i^{-1} = t_i^{-1} t_i, \quad \text{and} \quad t_i t_j = q_{ij}\, t_j t_i \text{ for all } 1 \leq i, j \leq n.$$

For example, if all $q_{ij} = 1$, then $\mathbb{F}_\mathbf{q} = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n variables. Thus, a general $\mathbb{F}_\mathbf{q}$ is a non-commutative version of $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the coordinate ring of the n-dimensional algebraic torus $(F \setminus \{0\})^n$, which explains the name "quantum torus".

Exercise 4.26. Let $\mathbb{F}_\mathbf{q}$ be a quantum torus. Show:

(a) $\mathbb{F}_\mathbf{q} = \bigoplus_{\lambda \in \mathbb{Z}^n} F t^\lambda$ for $t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$.

(b) The t^λ satisfy the multiplication rule $t^\lambda t^\mu = c(\lambda, \mu) t^{\lambda+\mu}$ with

$$c(\lambda, \mu) = \textstyle\prod_{1 \leq j < i \leq n} q_{ij}^{\lambda_i \mu_j}.$$

(c) $\mathbb{F}_\mathbf{q}$ is an associative torus of type $\mathbb{Z}^n$.

(d) Every associative torus of type $\mathbb{Z}^n$ is graded-isomorphic to some quantum torus $\mathbb{F}_{\mathbf{q}}$.

(e) The centre $Z(\mathbb{F}_{\mathbf{q}}) = \{z \in \mathbb{F}_{\mathbf{q}} : [z, \mathbb{F}_{\mathbf{q}}] = 0\}$ of the associative algebra $\mathbb{F}_{\mathbf{q}}$ is a graded subspace of $\mathbb{F}_{\mathbf{q}}$, namely $Z(\mathbb{F}_{\mathbf{q}}) = \bigoplus_{\gamma \in \Gamma} F t^\gamma$, where Γ is a subgroup of $\mathbb{Z}^n$ given by

$$\begin{aligned}\Gamma &= \{\gamma \in \mathbb{Z}^n : c(\gamma,\mu) = c(\mu,\gamma) \text{ for all } \mu \in \mathbb{Z}^n\}\\ &= \{\gamma \in \mathbb{Z}^n : \textstyle\prod_{j=1}^n q_{ij}^{\gamma_j} = 1 \text{ for } 1 \le i \le n\}.\end{aligned}$$

Moreover, the following are equivalent:

(i) All q_{ij} are roots of unity.
(ii) $[\mathbb{Z}^n : \Gamma] < \infty$.
(iii) $\mathbb{F}_{\mathbf{q}}$ is finitely generated as a module over its centre $Z(\mathbb{F}_{\mathbf{q}})$.

Combining this exercise with Th. 4.22 we obtain the classification of the cores of EALAs of type A:

Corollary 4.27 ([BGK, Th. 2.65]). *The Lie algebra $\mathfrak{sl}_N(\mathbb{F}_{\mathbf{q}})$ for a quantum torus $\mathbb{F}_{\mathbf{q}}$ is a centreless Lie torus of type $(\mathrm{A}_{N-1}, \mathbb{Z}^n)$. Conversely, any centreless Lie torus of type $(\mathrm{A}_{N-1}, \mathbb{Z}^n)$ with $N \ge 4$ is graded-isomorphic to some $\mathfrak{sl}_N(\mathbb{F}_{\mathbf{q}})$.*

The attentive reader will have noticed that we didn't say anything about Lie tori of type (A_1, Λ) and (A_2, Λ) in Th. 4.22 and Cor. 4.27. Of course, $\mathfrak{sl}_3(A)$ of an associative Λ-torus will be centreless Lie tori of type (A_2, Λ). The limitation $N \ge 4$ in Th. 4.22 is justified, since for $N = 3$ there are more examples: One needs coordinate algebras A which are no longer associative but only alternative. Moreover, one needs to replace the matrix algebra $\mathfrak{sl}_3(A)$ by something more general, a Tits-Kantor-Koecher algebra or an abstractly defined Lie algebra, see [BGKN] for details.

Analogous remarks apply for the A_1-case, in which the coordinates come from certain Jordan algebras (called *Jordan tori*) and in which $\mathfrak{sl}_2$ has to be replaced by a Jordan algebra. One now has classification theorems for Lie tori of all types. The references up to 2007 for each type are listed at the beginning of §7 in [AF]. An additional recent reference is [NT] for $S = \mathrm{B}_2$.

4.5. **Some more easy examples of Lie tori.** We describe some more easy examples, where easy means that they do not require some knowledge of non-associative algebras, like Jordan algebras, alternative or structurable algebras.

Example 4.28 ($\Lambda = \{0\}$). Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \bigoplus_{\xi \in S} g_\xi$ where $\mathfrak{g}_0 = \mathfrak{h}$ and S is the root system of $(\mathfrak{g}, \mathfrak{h})$, a finite reduced root system. Since $\mathfrak{g}$ is simple, S is also irreducible. Using standard properties of finite-dimensional split simple Lie algebras, it is easy to check that $\mathfrak{g} = \bigoplus_{\xi \in S} \mathfrak{g}_\xi$ *is a Lie torus of type* $(S, \{0\})$.

Conversely, if L is a Lie torus of type $(S, \{0\})$, then L is a finite-dimensional split simple Lie algebra. Indeed, $L = \mathfrak{g}$ in the notation of Prop. 4.8.

Note that this fits nicely the picture of EALAs of nullity 0, which we have characterized in Prop. 3.24 as finite-dimensional split simple Lie algebras.

Example 4.29. As in the previous Example 4.28 let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$ and root system S. We would like to consider a Lie algebra of the form $\mathfrak{g} \otimes A$ where A is an associative algebra. For $\mathfrak{g}$ of type A we could take non-commutative "coordinates" A to get a Lie torus, see §4.4. However, for $\mathfrak{g}$ not of type A the algebra A must be commutative in order to get a Lie algebra.

Therefore, we let $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ be a *commutative* associative torus of type Λ and consider $\mathfrak{g} \otimes A$, which becomes a Lie algebra (over F) by $[u_1 \otimes a_1, u_2 \otimes a_2] = [u_1, u_2] \otimes a_1 a_2$. It is a *centreless Lie torus of type* (S, Λ) *with respect to the homogeneous subspaces*

$$(\mathfrak{g} \otimes A)^\lambda_\xi = \mathfrak{g}_\xi \otimes A^\lambda.$$

Note that the support of the $\mathcal{Q}(S) \oplus \Lambda$-graded Lie algebra $\mathfrak{g} \otimes A$ is the set

$$\operatorname{supp}_{\mathcal{Q}(S) \oplus \mathbb{Z}^n} \mathfrak{g} \otimes A = S \times \Lambda,$$

and that $\mathfrak{g} \otimes A$ is an invariant Lie torus with respect to the bilinear form

$$(x \otimes a^\lambda \mid y \otimes b^\mu) = \kappa(x, y)\, (a^\lambda b^\mu)_0$$

where κ is the Killing form of $\mathfrak{g}$ and c_0 for $c \in A$ is the 0-component of c. This example works for any type of S. But it yields all examples only for special types of S.

Theorem 4.30. *Any centreless Lie torus of type* (S, Λ) *for* S *of type* D_l, $l \geq 4$ *or* E_l, $l = 6, 7, 8$, *is graded-isomorphic to an example as in* 4.29 *for* $\mathfrak{g}$ *of the corresponding type and* A *a commutative associative torus of type* Λ.

Proof. The proof is analogous to the proof of Th. 4.22: One applies the Coordinatization Theorem of [BerM] to get that L has the form $\mathfrak{g} \otimes A$ for some commutative associative F-algebra. One then has to discuss when such a Lie algebra is a Lie torus. This is the case exactly when A is a torus. □

Example 4.31 (Untwisted multiloop algebras)**.** For EALAs it is of interest to describe the centreless Lie tori of type (S, Λ) with Λ a free abelian group of finite rank, say of rank n. Hence $\Lambda \cong \mathbb{Z}^n$. It is immediate that $\mathfrak{g} \otimes A$ is a Lie torus of type $(S, \mathbb{Z}^n)$ if and only if A is a commutative quantum torus, i.e., a Laurent polynomial ring in several, say n variables. In other words, these are the untwisted multiloop algebra of (1.11),

$$L(\mathfrak{g}) = \mathfrak{g} \otimes_F F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

Hence by Th. 4.11 the universal central extension of $L(\mathfrak{g})$, the toroidal Lie algebras of §1.2 are also Lie tori. Finally, Th. 4.30 has the following corollary.

Corollary 4.32 ([BGK])**.** *Any centreless Lie torus of type* $(S, \mathbb{Z}^n)$ *with* $S = \mathrm{D}_l$, $l \geq 4$ *or* $S = \mathrm{E}_l$, $l = 6, 7, 8$, *is graded-isomorphic to an untwisted multiloop algebra* $L(\mathfrak{g})$ *as in Example* 4.31.

Perhaps the reader now expects that the next example will be the general multiloop algebras $L(\mathfrak{g}, \boldsymbol{\sigma})$ defined in (1.13). However, an arbitrary multiloop algebra is in general not a Lie torus, see [ABFP2, Th. 3.3.1] and [Na, Th. 5.1.4] for a characterization of centreless Lie tori which are multiloop algebras. But this phenomenon does not occur in nullity 1.

Exercise 4.33. Verify that the loop algebra $L(\mathfrak{g}, \sigma)$ of (1.3) is an invariant Lie torus of type $(S, \mathbb{Z})$ where S is the root system of Table 3.7.

5. The construction of all EALAs

Recall Th. 4.14: If (E,H) is an EALA, its core E_c and its centreless core E_{cc} are Lie tori, the latter being an invariant Lie torus. Moreover, if (S,Λ) is the type of E_c and E_{cc} then Λ is a free abelian group of finite rank. Thus:

$$\begin{tikzcd} \text{core } E_c \text{ (Lie torus)} \arrow[d, rightsquigarrow] & \text{EALA } (E,H) \arrow[l, leftsquigarrow] \\ \text{centreless core } E_{cc} \text{ (invariant Lie torus)} & \end{tikzcd}$$

In this chapter we will reverse the process, starting from an invariant Lie torus we will construct an EALA.

To motivate the construction it is useful to look again at the construction in Example 1.1 and 2.4 of an affine Kac-Moody Lie algebra. It can be summarized as follows: We start with a twisted loop algebra $\mathcal{L} = L(\mathfrak{g},\sigma)$, which as we now know is an invariant Lie torus (Ex. 4.33). We then take a central extension $\tilde{\mathcal{L}}$, which in this example is the universal central extension and hence by Th. 4.11 again a Lie torus (of course, one can also verify this directly in this example). Finally, we add some (not all) derivations to $\tilde{L}$ to get an affine Kac-Moody Lie algebra, and all affine Kac-Moody Lie algebras are obtained in this way (Kac's Realization Theorem 1.3). To summarize, using the EALA terminology:

$$\begin{tikzcd} \begin{array}{c}\text{central extension of } L\\ \text{(another Lie torus)}\end{array} \arrow[r, rightsquigarrow, "\begin{array}{c}\text{add}\\ \text{derivations}\end{array}"] & \text{EALA } (E,H) \\ \text{invariant Lie torus } L \arrow[u, rightsquigarrow] & \end{tikzcd}$$

To do something like this in general, one faces the following two problems.

(A) An invariant Lie torus has in general many central extension. For example, the untwisted multiloop algebra $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is an invariant Lie torus by Ex. 4.31. If $n \geq 2$, its universal central extension has an infinite-dimensional centre, a result we already mentioned in §1.2, see in particular Th. 1.5. Hence, there are many possible central extensions. Should we only consider the universal central extension?

(B) Which derivations should we add? Already in the affine case did we not add all derivations, as follows for example from Ex. 1.7!

It turns out that the two problems are closely related, and we will solve both at the same time. Rather than taking a 2-step approach, we will take one big step, by taking what one may call an *affine extension* (after all, the result will be an extended affine Lie algebra). In fact, affine extensions are a special case of so-called *double extension*, see for example [Bor].

(5.1)
$$\begin{tikzcd} \text{central extension of } L \arrow[r, dotted] & \text{EALA } (E,H) \\ \text{invariant Lie torus } L \arrow[u, dotted] \arrow[ur, rightsquigarrow, "\begin{array}{c}\text{affine}\\ \text{extension}\end{array}"'] & \end{tikzcd}$$

The key idea is based on the construction of a 2-cocycle in Ex. 1.19: Any subspace D of skew-symmetric derivations will give rise to a 2-cocycle and hence to a central extension. But not only do we get examples of central extensions. By Th. 1.25 and Ex. 1.27, up to isomorphism all central coverings of L are of the form $\mathrm{E}(L, D, \psi_D)$ for some graded subspace D of $\mathrm{SDer}(L)$ with $D \cap \mathrm{IDer}(L) = \{0\}$. Observe that we can indeed apply this theorem: The invariant Lie torus L

(i) is perfect by Ex. 4.10 and is finitely generated as Lie algebra by Th. 4.13 (recall that Λ is free of finite rank, where (S, Λ) is the type of L),
(ii) has finite homogeneous dimension, even bounded homogeneous dimension also by Th. 4.13, and
(iii) has an invariant nondegenerate Λ-graded symmetric bilinear form, by definition of an invariant Lie torus.

But we need more than just a central covering. For example, the axiom (EA2) requires that we construct an ad-diagonalizable subalgebra H for which the subspaces L_ξ^λ, $\xi \neq 0$, are root spaces, as can be seen from Th. 4.14. By Lemma 4.4 we can realize the subspaces L_ξ as root spaces of some natural subalgebra $\mathfrak{h} \subset L$. But we do not have a result, which describes the subspaces L^λ in a similar fashion, i.e., as root spaces of some toral subalgebra. There is in fact no natural choice of a subalgebra to do so. Rather, we will distinguish these subspaces "externally", i.e., via an action of some non-inner derivation algebra. The required formalism to do this, is described in the next section. This has nothing to do with Lie algebras. Rather, it is a topic in the theory of graded vector spaces, and we will therefore describe it in this setting.

5.1. **Degree maps.** In this section, V is vector space over a field F, which could be of arbitrary characteristic until Prop. 5.3(b). Also, Λ denotes an arbitrary abelian group. We recall that a Λ-grading of V is simply a direct vector space decomposition of V by a family $(V^\lambda : \lambda \in \Lambda)$ of subspaces $V^\lambda \subset V$. Our goal in this section is to present a method describing the homogeneous subspaces V^λ of a given Λ-grading of V as the joint eigenspaces of a subspace of diagonalizable endomorphisms.

To motivate the construction, let us first look at the converse, namely inducing a grading of V via the action of endomorphisms. We will say that a subspace $T \subset \mathrm{End}_F(V)$ is a *subspace of simultaneously diagonalizable endomorphisms*, if

$$V = \textstyle\bigoplus_{\lambda \in T^*} V^\lambda \quad \text{for} \qquad (5.2)$$
$$V^\lambda = \{v \in V : t(v) = \lambda(t)v \text{ for all } t \in T\}.$$

In this case, T obviously consists of pairwise commuting diagonalizable endomorphisms. Conversely, it is well-known that a finite-dimensional subspace of pairwise commuting diagonalizable endomorphisms is a subspace of simultaneously diagonalizable endomorphisms (this is no longer true if T is infinite-dimensional). Observe that the decomposition (5.2) is a grading of V by the group $\mathrm{span}_\mathbb{Z}(\mathrm{supp}_{T^*} V)$ where $\mathrm{supp}_{T^*} V = \{\lambda \in T^* : V^\lambda \neq 0\}$. Our goal is to realize a given grading of V in this way.

To do so, we will use the F-vector space

$$\mathrm{D}(\Lambda) = \mathrm{Hom}_\mathbb{Z}(\Lambda, F)$$

consisting of all maps $\theta : \Lambda \to F$ which are $\mathbb{Z}$-linear: $\theta(\lambda_1 + \lambda_2) = \theta(\lambda_1) + \theta(\lambda_2)$ for all $\lambda_i \in \Lambda$. This is an F-vector space by defining for $\theta, \theta_i \in \mathrm{D}(\Lambda)$ and $s \in F$ the sum $\theta_1 + \theta_2$ and the scalar multiplication $s\theta$ by $(\theta_1 + \theta_2)(\lambda) = \theta_1(\lambda) + \theta_2(\lambda)$ and $(s\theta)(\lambda) = s(\theta(\lambda))$.

Exercise 5.1. (a) Show $\mathrm{D}(\Lambda) \cong \mathrm{Hom}_F(\Lambda \otimes_{\mathbb{Z}} F, F) = (\Lambda \otimes_F F)^*$. Thus $\mathrm{D}(\Lambda)$ is naturally a dual vector space.

(b) If Λ is free of rank n, say with $\mathbb{Z}$-basis $\varepsilon_1, \dots, \varepsilon_n$, then $\mathrm{D}(\Lambda) = F\partial_1 \oplus \cdots \oplus F\partial_n$ where $\partial_i \in \mathrm{D}(\Lambda)$ is defined by $\partial_i(\sum_j m_j \varepsilon_j) = m_i$. In particular, $\dim_F \mathrm{D}(\Lambda) = n$.

We now suppose that $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ is a Λ-grading of the vector space V. Any $\theta \in \mathrm{D}(\Lambda)$ defines an endomorphism $\partial_\theta \in \mathrm{End}_F(V)$ by

$$\partial_\theta(v^\lambda) = \theta(\lambda)\, v^\lambda \quad \text{for } v^\lambda \in V^\lambda.$$

We put

$$\mathcal{D}(V) = \{\partial_\theta : \theta \in \mathrm{D}(\Lambda)\}$$

and call the elements of $\mathcal{D}(V)$ *degree maps*. If $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ is a Λ-grading of an algebra A, the maps ∂_θ are derivations and $\mathcal{D}(A)$ is called the space of *degree derivations*.

The map $\partial : \mathrm{D}(\Lambda) \to \mathcal{D}(V)$ is clearly F-linear and surjective by definition. Its kernel is $\{\theta \in \mathrm{D}(\Lambda) : \theta(\mathrm{supp}_\Lambda V) = 0\}$. To make ∂ an isomorphism we will

$$\text{(5.3)} \qquad \textit{from now on assume } \mathrm{span}_{\mathbb{Z}}(\mathrm{supp}_\Lambda V) = \Lambda.$$

As we have pointed out at previous occasions, this is not a serious assumptions since one can always replace Λ by $\mathrm{span}_{\mathbb{Z}}(\mathrm{supp}_\Lambda V)$ without changing the given grading. Since now ∂ is an isomorphism, we can define a linear form $\mathrm{ev}_\lambda \in \mathcal{D}(V)^*$ for every $\lambda \in \Lambda$:

$$\mathrm{ev}_\lambda(\partial_\theta) = \theta(\lambda)$$

The F-linear map

$$\mathrm{ev} : \Lambda \to \mathcal{D}(V)^*, \quad \lambda \mapsto \mathrm{ev}_\lambda$$

is called the *evaluation map*. By construction,

$$\text{(5.4)} \qquad V^\lambda \subset \{v \in V : d(v) = \mathrm{ev}_\lambda(d)v \text{ for all } d \in \mathcal{D}(V)\}$$

since for $d = \partial_\theta$ and $v \in V^\lambda$ we have $\partial_\theta(v^\lambda) = \theta(\lambda)v^\lambda = \mathrm{ev}_\lambda(\partial_\theta)v^\lambda$.

Definition 5.2. In the setting of above, i.e., $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ is Λ-graded and (5.3) holds, we will say that a subspace $T \subset \mathcal{D}(V)$ *induces the Λ-grading of V* if

$$V^\lambda = \{v \in V : t(v) = \mathrm{ev}_\lambda(t)v \text{ for all } t \in T\}$$

holds for all $\lambda \in \Lambda$.

Proposition 5.3. *Let $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ be a Λ-grading of the vector space V such that* (5.3) *holds.*

(a) *A subspace $T \subset \mathcal{D}(V)$ induces the Λ-grading of V if the restricted evaluation map*

$$\mathrm{ev}_T : \Lambda \to T^*, \quad \mathrm{ev}_T(\lambda) = \mathrm{ev}_\lambda |_T$$

is injective.

(b) *Suppose F has characteristic 0 and Λ is torsion-free, i.e., $n\lambda = 0$ for some $n \in \mathbb{Z}$ implies $\lambda = 0$. Then Λ embeds into the F-vector space $U = \Lambda \otimes_{\mathbb{Z}} F$ and for every subspace $S \subset \mathrm{D}(\Lambda)$ separating the points of Λ in U the corresponding subspace $T = \partial(S) \subset \mathcal{D}(V)$ induces the Λ-grading of V. In particular, this holds for $\mathcal{D}(V)$ itself.*

5.2. **The centroid of Lie algebras, in particular of Lie tori.** After the intermezzo on how to induce gradings of vector spaces in the previous section 5.1 we now come back to Lie algebras, but not immediately to Lie tori and EALAs. Of course, the topic of this section is motivated by the over-all goal of this chapter: The construction of EALAs from Lie tori using certain subspaces of derivations. The derivations, which in 5.4 will be used in the general construction, are products of degree maps, studied in 5.1, and so-called centroidal transformations, to which this section is devoted.

The basic idea of the centroid of a Lie algebra (or of any algebra for that matter) is that it identifies the largest ring over which the given algebra can be considered as an algebra. For example, if one studies the real Lie algebra L which is $\mathfrak{sl}_n(\mathbb{C})$ considered as a real Lie algebra by restricting the scalars to $\mathbb{R}$, the centroid will be $\cong \mathbb{C}$ and will thus indicate that L can also be considered as a complex Lie algebra.

In general, the centroid will not be a field but only a (commutative) ring. Hence, considering a Lie algebra as algebra over its centroid, necessitates that in the following definition and the Lemma 5.6 after it we will deviate from our standard assumption and consider Lie algebras over rings. The definition of a Lie algebra L defined over a ring, say k, is not surprising: L is a k-module with a k-bilinear map $[.,.] : L \times L \to L$ which is alternating, i.e., $[l, l] = 0$ for all $l \in L$, and satisfies the Jacobi identity.

Definition 5.4 ([J, Ch. X])**.** The *centroid* $\mathrm{Cent}_k(L)$ of a Lie algebra L defined over a ring k is defined as

$$\mathrm{Cent}_k(L) = \{\chi \in \mathrm{End}_k(L) : \chi([l_1, l_2]) = [l_1, \chi(l_2)] \text{ for all } l_1, l_2 \in L\}.$$

Of course, $\chi \in \mathrm{Cent}_k(L) \Leftrightarrow \chi([l_1, l_2]) = [\chi(l_1), l_2]$ for all $l_1, l_2 \in L$. It is important to indicate k in the notation $\mathrm{Cent}_k(L)$ since the centroid depends on the base ring k.

We have $k\,\mathrm{Id}_L \subset \mathrm{Cent}_k(L)$ for every L. One calls L *central* if the map $k \to \mathrm{Cent}_F(L)$, $s \mapsto s\,\mathrm{Id}_L$, is an isomorphism, and one says that L is *central-simple* if L is just that: central and simple.

Let $L = \bigoplus_{\lambda \in \Lambda} L^\lambda$ be a Lie algebra graded by an abelian group Λ. We can then also define the Λ-*graded centroid* as

$$\mathrm{grCent}_k(L) = \mathrm{grEnd}_k(L) \cap \mathrm{Cent}_k(L) = \textstyle\bigoplus_{\lambda \in \Lambda} \mathrm{Cent}_k(L)^\lambda$$

where $\mathrm{Cent}_k(L)^\lambda$ consists of the centroidal transformations which have degree λ: $\chi(L^\mu) \subset L^{\lambda+\mu}$ for all $\mu \in \Lambda$.

Example 5.5. As an immediate example we calculate the centroid of the Lie algebra $L = \mathfrak{sl}_2(\mathbb{C})$, considered as real Lie algebra by restricting the scalars to $\mathbb{R}$. We let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_2(\mathbb{C})$. Then the relations $[h, ce] = 2ce$ and $[h, cf] = -2cf$ for $c \in \mathbb{C}$ show that $\chi(ce)$ and $\chi(cf)$ are uniquely determined by $\chi(h)$. For example, $2\chi(ce) = \chi([h, ce]) = [\chi(h), ce]$. Moreover, $\chi(h) \in \mathbb{C}h$ because $[\chi(h), ch] = \chi([h, ch]) = 0$. Hence $\dim_{\mathbb{R}} \mathrm{Cent}_{\mathbb{R}}(L) \le 2$. On the other side, $\mathbb{C}\,\mathrm{Id}_L \subset \mathrm{Cent}_{\mathbb{R}}(L)$ is clear, whence $\mathbb{C}\,\mathrm{Id}_L = \mathrm{Cent}_{\mathbb{R}}(L)$.

We leave it to the reader to show $\mathrm{Cent}_{\mathbb{R}}(L) = \mathbb{C}\,\mathrm{Id}_L$ for $L = \mathfrak{sl}_n(\mathbb{C})$ considered as real Lie algebra, without using any of the results mentioned below!

The following lemma gives a mathematical meaning to the claims made before the Def. 5.4, and lists the most important properties of the centroid of Lie algebras which are not necessarily Lie tori.

Lemma 5.6 (Folklore)**.** *Let L be a Lie algebra defined over a ring k.*

(a) *The centroid of L is always a unital associative subalgebra of the endomorphism algebra* $\mathrm{End}_k(L)$ *of L. Hence* $\mathrm{Cent}_k(L)$ *is a k-algebra and L becomes a* $\mathrm{Cent}_k(L)$*-module by defining the action of* $\mathrm{Cent}_k(L)$ *on L by $\chi \cdot l = \chi(l)$ for $\chi \in \mathrm{Cent}_k(L)$ and $l \in L$.*

(b) *If the centroid of L is commutative, then with respect to the action of* $\mathrm{Cent}_k(L)$ *on L defined in* (a)*, L is a Lie algebra over the ring* $\mathrm{Cent}_k(L)$*. Moreover, L is central as a Lie algebra over its centroid.*

(c) *If L is perfect, its centroid is commutative and does not depend on the base ring k:* $\mathrm{Cent}_k(L) = \mathrm{Cent}_{\mathbb{Z}}({}_{\mathbb{Z}}L)$ *where ${}_{\mathbb{Z}}L$ is the Lie algebra L with scalars restricted to $\mathbb{Z}$.*

(d) *If L is simple, its centroid is a field and L as a Lie algebra over the field* $\mathrm{Cent}_F(L)$ *is central-simple. In particular:*

- (i) *a finite-dimensional simple Lie algebra over an algebraically closed field F is central-simple, and*
- (ii) *the centroid of a simple real Lie algebra L is either $\cong \mathbb{R}\,\mathrm{Id}$, in which case L is central-simple, or is $\cong \mathbb{C}\,\mathrm{Id}$, in which case L is a simple complex Lie algebra, considered as a real Lie algebra.*

(e) *Suppose L is Λ-graded. Then* $\mathrm{grCent}_k(L)$ *is a Λ-graded subalgebra of the full centroid* $\mathrm{Cent}_k(L)$*. Moreover,* $\mathrm{grCent}_k(L) = \mathrm{Cent}_k(L)$ *if L is finitely generated as an ideal, i.e., there exist $l_1, \ldots, l_n \in L$ such that the ideal generated by $l_1, \ldots, l_n$ is all of L.*

(f) *If $\chi \in \mathrm{Cent}_k(L)$ and $d \in \mathrm{Der}_k(L)$, then $\chi \circ d \in \mathrm{Der}_k(L)$. With respect to this operation,* $\mathrm{Der}_k(L)$ *is a* $\mathrm{Cent}_k(L)$*-module and* $\mathrm{IDer}(L)$ *is a submodule of the* $\mathrm{Cent}_k(L)$*-module* $\mathrm{Der}_k(L)$*.*

The proof of this lemma is a straightforward exercise, which the reader will be asked to do now. The exercise also lists some interesting additional facts on the centroid.

Exercise 5.7. (a) For any $\chi \in \mathrm{Cent}_k(L)$ the kernel $\mathrm{Ker}\,\chi$ and the image $\mathrm{Im}\,\chi$ are ideals of L satisfying $[\mathrm{Ker}\,\chi, \mathrm{Im}\,\chi] = 0$.

(b) Prove Lemma 5.6. For part (c) of the Lemma use (a) above.

(c) If L is perfect, any $\chi \in \mathrm{Cent}_k(L)$ is symmetric with respect to any invariant bilinear form on L.

Here is the result, which describes the centroid of the Lie algebras of interest in this chapter. We will use the notion of an associative torus, introduced in 4.20 and further discussed in 4.23–4.26.

Proposition 5.8 ([BN, Prop. 3.13])**.** *Let $L = \bigoplus_{\xi \in S,\, \lambda \in \Lambda} L_\xi^\lambda$ be a centreless Lie torus of type (S, Λ).*

(a) *With respect to the $(\mathcal{Q}(S) \oplus \Lambda)$-grading of L we have*

$$\mathrm{Cent}_F(L) = \bigoplus\nolimits_{\lambda \in \Lambda} \mathrm{Cent}_F(L)_0^\lambda = \mathrm{grCent}_F(L). \tag{5.5}$$

In particular, $\chi(L_\xi) \subset L_\xi$ for any $\chi \in \mathrm{Cent}_k(L)$ and $\xi \in S$.

(b) *Moreover, with respect to the decomposition* (5.5) *the centroid* $\mathrm{Cent}_F(L)$ *is an associative commutative torus of type Γ, where $\Gamma = \mathrm{supp}_\Lambda \mathrm{Cent}_F(L)$ is a subgroup of Λ, hence a twisted group algebra over Γ.*

(c) *In particular, if Λ is free abelian of finite rank n, the centroid $\mathrm{Cent}_F(L)$ is graded-isomorphic to $F[\Gamma]$, the group algebra of Γ as defined in 4.24, and is thus isomorphic to a Laurent polynomial ring in ν variables, $0 \le \nu \le n$. Moreover, L is a free module over its centroid.*

Proof. Parts (a) and (b) of this proposition are proven in [BN, Prop. 3.13]. Part (c) is ([Ne3, Th. 7]). The first part of (c) follows from (b): A twisted group algebra over a free group is a group algebra. The second part is a special case of a general fact: Any graded module over an associative torus is free. □

Example 5.9. Let $L = \mathfrak{sl}_N(A)$ for A an associative F-algebra, see 4.4, and let $Z(A) = \{z \in A : [z, A] = 0\}$ be the centre of the associative algebra A. Any $z \in Z(A)$ induces a centroidal transformation χ_z defined by mapping $x = (x_{ij}) \in \mathfrak{sl}_N(A)$ to $\chi_z(x) = (zx_{ij})$. It is easily seen ([Ne5, 7.9]) that

$$Z(A) \to \mathrm{Cent}_F(\mathfrak{sl}_N(A)), \quad z \mapsto \chi_z$$

is an isomorphism of F-algebras (the only non-obvious part is surjectivity).

Let us now specialize to the case of a Lie torus $\mathfrak{sl}_N(A)$ of type $(\mathrm{A}_{N-1}, \mathbb{Z}^n)$. Thus, by Cor. 4.27, $A = \mathbb{F}_{\mathbf{q}}$ is a quantum torus. A description of the centre $Z(\mathbb{F}_{\mathbf{q}})$ is given in Ex. 4.26(e) (see [BGK, Prop. 2.44] for a proof): $Z(\mathbb{F}_{\mathbf{q}}) = \bigoplus_{\gamma \in \Gamma} Ft^{\gamma}$ where Γ is the subgroup

$$\Gamma = \{\gamma \in \mathbb{Z}^n : \textstyle\prod_{j=1}^n q_{ij}^{\gamma_j} = 1 \text{ for } 1 \le i \le n\}.$$

of $\mathbb{Z}^n$. The centre of $\mathbb{F}_{\mathbf{q}}$ is therefore isomorphic to a Laurent polynomial ring in, say, ν variables, as claimed in Prop. 5.8(c). To see that the inequalities $0 \le \nu \le n$ stated there are sharp, we consider the quantum torus associated to the matrix

$$\mathbf{q} = \begin{bmatrix} 1 & q \\ q^{-1} & 1 \end{bmatrix}.$$

Specializing the description of Γ above we get

$$\Gamma = \begin{cases} \{0\}, & q \text{ not a root of unity,} \\ m\mathbb{Z} \oplus m\mathbb{Z}, & q \text{ an } m\text{th root of unity.} \end{cases}$$

Hence $\nu = 0$ in the first case and $\nu = 2 = n$ in the second case.

However, the following result says that this is the only case in which the centroidal grading group Γ has smaller rank than Λ.

Theorem 5.10 ([Ne3, Th. 7])**.** *Let L be a centreless Lie torus of type $(S, \mathbb{Z}^n)$ with S not of type* A. *Then $[\mathbb{Z}^n : \Gamma] < \infty$ and L is a free $\mathrm{Cent}_F(L)$-module of finite rank.*

This result, together with the Realization Theorem of [ABFP1] implies that an invariant Lie torus of type $(S, \mathbb{Z}^n)$, $S \ne \mathrm{A}_l$, is graded-isomorphic to a multiloop algebra as defined in (1.13). A characterization of which multiloop algebras are Lie tori is the main result of [ABFP2]. A more general approach to realizing Lie tori as multiloop algebras is developed in [Na].

It is easy to verify Th. 5.10 in case L is a Lie torus of type $(S, \mathbb{Z}^n)$ and S of type D or E. As we have seen in Th. 4.30, in this case $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The centroids of these types of Lie algebras are described in the next example.

Example 5.11. Let $\mathfrak{g}$ be a finite-dimensional central simple Lie algebra. For example, by [BN, Remark 3.6] any finite-dimensional split simple Lie algebra is central and thus central-simple. (Over algebraically closed fields, this also follows from Lemma 5.6(d).) Also, let A be an associative commutative F-algebra.

A straightforward verification shows that for $s \in F$ and $a \in A$ the map $\chi_{s,a}$, defined by $u \otimes b \mapsto su \otimes ab$, is a centroidal transformation of the Lie algebra $\mathfrak{g} \otimes A$. It follows from [ABP3, Lemma 2.3(a)] or [Az2, Lemma 1.2] or [BN, Cor. 2.23] that these are all the maps in the centroid of $\mathfrak{g} \otimes A$:

$$F \operatorname{Id}_{\mathfrak{g}} \otimes A \cong \operatorname{Cent}_F(\mathfrak{g} \otimes A), \quad \text{via } s \otimes a \mapsto \chi_{s,a}.$$

Although this will not be needed in the following, we mention that the centroid of an EALA is known too.

Proposition 5.12. *Let E be an EALA, let $K = E_c$ be its core and put $D = E/K$. Then K is a central Lie algebra, and*

$$\operatorname{Cent}_F(E) = F \operatorname{Id}_E \oplus \mathcal{V}(K), \quad \mathcal{V}(K) = \{\chi \in \operatorname{Cent}_F(E) : \chi(K) = 0\}.$$

As a vector space, the ideal $\mathcal{V}(K)$ of $\operatorname{Cent}_F(E)$ is canonically isomorphic to the D-module homomorphisms $D \to Z(K)$:

$$\mathcal{V}(K) \cong \operatorname{Hom}_D(D, Z(K)).$$

This is proven in [BN, Cor. 4.13]. Observe that the reference to [Ne4, Th.6] in the proof of [BN] can now be replaced by the combination of Th. 4.13(c) and Ex. 1.27.

5.3. **Centroidal derivations of Lie tori.** In this section L is a centreless Lie torus of type (S, Λ). Regarding L as a Λ-graded Lie algebra, the results of section 5.1 apply and provide us with the subspace

$$\mathcal{D} = \mathcal{D}(L) = \{\partial_\theta : \theta \in \mathrm{D}(\Lambda)\}$$

of degree derivations of L. Moreover, we can apply Lemma 5.6(f) and get that $\chi \circ \partial_\theta \equiv \chi \partial_\theta$ is a derivation for any $\chi \in \operatorname{Cent}_F(L)$. We call the elements of

$$\operatorname{CDer}_F(L) = \operatorname{Cent}_F(L)\, \mathcal{D}$$

centroidal derivations. (A notion of centroidal derivations for arbitrary Λ-graded Lie algebra is developed in [Ne5, 4.9].) Recall from Prop. 5.8 that $\operatorname{Cent}_F(L) = \bigoplus_{\gamma \in \Gamma} \operatorname{Cent}_F(L)^\gamma$ is a commutative associative torus of type Γ, where Γ is a subgroup of Λ. Since $\mathcal{D}$ consist of degree 0 endomorphisms, $\operatorname{CDer}(L)$ is Γ-graded,

$$\begin{aligned} \operatorname{CDer}_F(L) &= \textstyle\bigoplus_{\gamma \in \Gamma} \operatorname{CDer}_F(L)^\gamma \quad \text{for} \\ \operatorname{CDer}_F(L)^\gamma &= \operatorname{Cent}_F(L)^\gamma\, \mathcal{D} = \operatorname{Cent}_F(L) \cap \operatorname{End}_F(L)^\gamma. \end{aligned} \tag{5.6}$$

It is then easily seen that $\operatorname{CDer}_F(L)$ is a Γ-graded subalgebra of $\operatorname{Der}_F(L)$. For $\chi^\gamma \in \operatorname{Cent}_F(L)^\gamma$, $\chi^\delta \in \operatorname{Cent}_F(L)^\delta$ and $\theta, \psi \in \mathrm{D}(\Lambda)$ the Lie algebra product of $\operatorname{CDer}_F(L)$ is given by the formula

$$[\chi^\gamma \partial_\theta,\, \chi^\delta \partial_\psi] = \chi^\gamma \chi^\delta \big(\theta(\delta)\, \partial_\psi - \psi(\gamma)\, \partial_\theta\big). \tag{5.7}$$

Thus, $\operatorname{CDer}_F(L)$ is a generalized Witt algebra, see for example [NY, 1.9].

Suppose now that L is an invariant Lie torus, say with respect to the invariant bilinear from $(\cdot|\cdot)$. We can then consider the *skew centroidal derivations*

$$\operatorname{SCDer}_F(L) = \operatorname{SDer}_F(L) \cap \operatorname{CDer}_F(L),$$

defined as the centroidal derivations which are skew-symmetric with respect to $(\cdot|\cdot)$. This is a Γ-graded subalgebra of $\mathrm{CDer}_F(L)$ whose homogenous components are given by

$$\mathrm{SCDer}_F(L)^\gamma = \{\chi^\gamma\partial_\theta : \chi^\gamma \in \mathrm{Cent}_F(L)^\gamma, \theta(\gamma) = 0\}. \tag{5.8}$$

In particular,

$$\mathrm{SCDer}_F(L)^0 = \mathcal{D}$$

is a toral subalgebra of $\mathrm{SCDer}_F(L)$ since $[\partial_\theta,\, \chi^\delta\partial_\psi] = \theta(\delta)\partial_\psi$ by (5.7). It is also of interest to point out that $[\mathrm{SCDer}_F(L)^\gamma,\, \mathrm{SCDer}_F(L)^{-\gamma}] = 0$, which implies that $\mathrm{SCDer}_F(L)$ is a semidirect product,

$$\mathrm{SCDer}_F(L) = \mathcal{D} \ltimes \left(\textstyle\bigoplus_{\gamma\neq 0} \mathrm{SCDer}_F(L)^\gamma\right)$$

of the toral subalgebra $\mathcal{D}$ and the ideal spanned by the homogeneous subspaces of non-zero degree. For the construction of EALAs, the following theorem is fundamental.

Theorem 5.13 ([Ne3, Th. 9]). *Let L be an invariant Lie torus of type (S,Λ) with Λ free of finite rank. Then $\mathrm{Der}_F(L)$ is a semi-direct product,*

$$\begin{aligned} \mathrm{Der}_F(L) &= \mathrm{IDer}_F(L) \rtimes \mathrm{CDer}_F(L), \quad \textit{hence} \\ \mathrm{SDer}_F(L) &= \mathrm{IDer}_F(L) \rtimes \mathrm{SCDer}_F(L), \end{aligned} \tag{5.9}$$

where $\mathrm{IDer}_F(L)$ denotes the ideal of all inner derivations.

Some remarks on the proof of this theorem follow. By Prop. 5.8 the centroid of L is a Laurent polynomial ring. Let K be its field of fractions, a field of rational functions. As a $\mathrm{Cent}_F(L)$-module, L is torsion-free and hence L embeds into the Lie K-algebra

$$\tilde{L} = L \otimes_{\mathrm{Cent}_F(L)} K,$$

the so-called *central closure of* L. If the $\mathrm{Cent}_F(L)$-module L is finitely generated, its central closure is a finite-dimensional central-simple Lie algebra. Hence, in this case $\mathrm{Der}_K(\tilde{L}) = \mathrm{IDer}(\tilde{L})$, from which the theorem easily follows. If however L is not finitely generated as a $\mathrm{Cent}_F(L)$-module, then we know from Th. 5.10 that L is a Lie torus of type A. More precisely, as a consequence of the results in [Y1] and [NY] for type A_1, [BGKN] for type A_2 and Cor. 4.27 for type A_l, $l \geq 3$, such a Lie torus is graded-isomorphic to $\mathfrak{sl}_n(\mathbb{F}_\mathbf{q})$. But in this case the result follows from [BGK, 2.17, 2.53], [BGKN, Th. 1.40] and [NY, Th. 4.11]. We will discuss the special case $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm n}]$ in Ex. 5.14. To avoid any confusion, we note that the splitting (5.9) is not the one proven in [Be, Th. 3.12] for arbitrary root-graded Lie algebras.

The importance of the theorem stems from Th. 1.25: It identifies a natural complement of $\mathrm{IDer}(L)$ in $\mathrm{SDer}_F(L)$. Hence, up to graded-isomorphism, any graded covering of L has the form $\mathrm{E}(L, D^{\mathrm{gr}*}, \psi_D)$ for a graded subspace $D \subset \mathrm{SCDer}_F(L)$. Moreover, since $\mathcal{D} \subset \mathrm{SCDer}_F(L)$, we can require that $D^0 \subset \mathcal{D}$ be not too small and use it to distinguish the homogeneous spaces L^λ by applying Prop. 5.3. This will be our approach in section 5.4. But first some examples.

Example 5.14. Let $L = \mathfrak{g} \otimes A$ where $\mathfrak{g}$ is a split simple finite-dimensional Lie algebra with root system S and where $A = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is a Laurent polynomial ring in n variables. This is an invariant Lie torus of type $(S,\mathbb{Z}^n)$, see the Examples 4.29 and 4.31. We have seen in (1.18) that

$$\mathrm{Der}_F(\mathfrak{g}\otimes A)=\mathrm{IDer}(\mathfrak{g}\otimes A)\oplus(\mathrm{Id}_{\mathfrak{g}}\otimes \mathrm{Der}_F(A)).$$

The reader has (or should have) determined $\mathrm{Der}_F(A)$ in Ex. 1.7: $\mathrm{Der}_F(A)=A\,\mathcal{D}$ where $\mathcal{D}=\mathrm{span}_{\mathbb{Z}}(\{\partial_i : 1\le i\le n\})$ in the notation of the quoted exercise. But by Ex. 5.1, $\mathcal{D}=\mathcal{D}(A)$ is also the space of degree derivations of A. Since the Λ-grading of $L=\mathfrak{g}\otimes A$ is concentrated in the factor A, it follows that $\mathrm{Id}\otimes\mathcal{D}$ is the space of degree derivations of L, whence, by Example 5.11, we have

$$\mathrm{CDer}_F(\mathfrak{g}\otimes A)=\mathrm{Id}_{\mathfrak{g}}\otimes A\,\mathcal{D}=\mathrm{Id}_{\mathfrak{g}}\otimes\mathrm{Der}_F(A).$$

Thus, for the invariant Lie torus $\mathfrak{g}\otimes A$ the decomposition (1.18) is the same as the decomposition (5.9)!

We have seen in Ex. 4.29 that L is an invariant Lie torus with respect to the tensor product form $(\cdot|\cdot)=\kappa\otimes\beta$ where κ is the Killing form of $\mathfrak{g}$ and where β is the bilinear form on A defined by $\beta(t^\lambda,t^\mu)=\delta_{\lambda,-\mu}$. It is then easy to identify $\mathrm{SCDer}_F(L)$ using (5.8). In particular, for $n=1$ we see that $\mathrm{SCDer}(\mathfrak{g}\otimes F[t^{\pm1}])=Fd$, where d is the degree derivation of (1.8). In particular, this together with Th. 1.25 gives a new proof of the theorem, mentioned in 1.1, that the Lie algebra $\tilde{\mathcal{L}}(\mathfrak{g},\sigma)$ of (1.7) is the universal central extension of the twisted loop algebra $\mathcal{L}(\mathfrak{g},\sigma)$.

5.4. **The general construction.** Finally, we can describe the ingredients (L,D,τ) of the general construction:

- $L=\bigoplus_{\xi\in S,\lambda\in\Lambda}L_\xi^\lambda$ is an invariant Lie torus of type (S,Λ) with Λ a free abelian group of finite rank; we put $\Gamma=\mathrm{supp}_\Lambda\,\mathrm{Cent}(L)$, see Prop. 5.8.
- $D=\bigoplus_{\gamma\in\Gamma}D^\gamma\subset\mathrm{SCDer}_F(L)$ is a graded subalgebra such that the evaluation map

$$\mathrm{ev}_{D^0}:\Lambda\to D^{0\,*},\quad \lambda\to\mathrm{ev}_\lambda\,|_{D^0}\ \text{ is injective.}\tag{5.10}$$

- $\tau: D\times D\to D^{\mathrm{gr}*}$ is an *affine cocycle*, i.e., τ is a bilinear map satisfying for all $d,d_i\in D$

$$\tau(d,d)=0\quad\text{and}\quad \textstyle\sum_{\circlearrowleft} d_1\cdot\tau(d_2,d_3)=\sum_{\circlearrowleft}\tau([d_1,d_2],d_3),\tag{5.11}$$

$$\tau(D^0,D)=0,\quad\text{and}\quad \tau(d_1,d_2)(d_3)=\tau(d_2,d_3)(d_1)\tag{5.12}$$

Recall from Prop. 5.3 that the condition (5.10) implies that D^0 induces the Λ-grading of L, i.e.,

$$L^\lambda=\{l\in L: d^0(l)=\mathrm{ev}_\lambda(d^0)l,\ \text{for all } d^0\in D^0\}.\tag{5.13}$$

For example, (5.10) holds for $D=\mathcal{D}=\mathrm{SCDer}_F(L)^0$ or D any graded subalgebra with $D^0=\mathcal{D}$. In (5.11), the symbol $\sum_{\circlearrowleft}$ denotes the cyclic sum: $\sum_{\circlearrowleft}d_1\cdot\tau(d_2,d_3)=d_1\cdot\tau(d_2,d_3)+d_2\cdot\tau(d_3,d_1)+d_3\cdot\tau(d_1,d_2)$ and analogously for $\sum_{\circlearrowleft}\tau([d_1,d_2],d_3)$. Moreover, $d\cdot c$ for $c\in D^{\mathrm{gr}*}$ is the contragradient action of D on the graded dual space $D^{\mathrm{gr}*}$. The condition (5.11) says that τ is an *abelian* 2*-cocycle*, meaning that $D^{\mathrm{gr}*}\oplus D$ is a Lie algebra with respect to the product formula

$$[c_1\oplus d_1,\,c_2\oplus d_2]=\big(d_1\cdot c_2-d_2\cdot c_1+\tau(d_1,d_2)\big)\oplus[d_1,d_2]\tag{5.14}$$

for $c_i\in D^{\mathrm{gr}*}$ and $d_i\in D$. Thus,

$$0\longrightarrow D^{\mathrm{gr}*}\xrightarrow{\mathrm{inc}}D^{\mathrm{gr}*}\oplus D\xrightarrow{\mathrm{pr}_D}D\longrightarrow 0$$

is an abelian extension: $D^{\mathrm{gr}*}$ is an abelian ideal, not necessarily contained in the centre. The conditions in (5.12) will allow us to define a toral subalgebra H and an invariant bilinear form $(\cdot|\cdot)$ below. We note that an affine cocycle is necessarily graded of degree 0:

$$\tau(D^\gamma, D^\delta) \subset (D^{\mathrm{gr}*})^{\gamma+\delta}$$

for $\gamma, \delta \in \Gamma$. There do exist non-trivial affine cocycles, see [BGK, Rem. 3.71] and [ERM].

To data (L, D, τ) as above we associate a Lie algebra

$$E = L \oplus D^{\mathrm{gr}*} \oplus D$$

with product ($l_i \in L$, $c_i \in D^{\mathrm{gr}*}$ and $d_i \in D$)

$$\begin{aligned} [l_1 \oplus c_1 \oplus d_1,\, l_2 \oplus c_2 \oplus d_2] &= \big([l_1, l_2]_L + d_1(l_2) - d_2(l_1)\big) \\ &\big(\psi_D(l_1, l_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)\big) \oplus [d_1, d_2]. \end{aligned} \tag{5.15}$$

Here $[.,.]_L$ is the Lie algebra product of L, $d_i(l_j)$ is the natural action of D on L, and ψ_D is the central 2-cocycle of (1.24). It is immediate from the product formula that

(i) $L \oplus D^{\mathrm{gr}*}$ is an ideal of E, and the canonical projection $L \oplus D^{\mathrm{gr}*} \to L$ is a central extension.

(ii) The Lie algebra $D^{\mathrm{gr}*} \oplus D$ of (5.14) is a subalgebra of E.

The Lie algebra E has a a subalgebra

$$H = \mathfrak{h} \oplus D^{0\,*} \oplus D^0$$

where $\mathfrak{h} = \operatorname{span}_F\{h_\xi^\lambda : \xi \in S^\times, \lambda \in \Lambda\} = \operatorname{span}_F\{h_\xi^0 : 0 \neq \xi \in S_{\mathrm{ind}}\}$. We embed S into the dual space $\mathfrak{h}^*$, using the evaluation map of (5.10), and extend $\xi \in S \subset \mathfrak{h}^*$ to a linear form of H by $\xi(D^{0\,*} \oplus D^0) = 0$. We embed $\Lambda \subset D^{0\,*}$, using the evaluation map of Prop. 5.3, and then extend $\lambda \in \Lambda \subset D^{0\,*}$ to a linear form of H by putting $\lambda(\mathfrak{h} \oplus D^{0\,*}) = 0$. Then H is a toral subalgebra of E with root spaces

$$E_{\xi \oplus \lambda} = \begin{cases} L_\xi^\lambda, & \xi \neq 0, \\ L_0^\lambda \oplus (D^{-\lambda})^* \oplus D^\lambda, & \xi = 0. \end{cases}$$

Observe $H = E_0$ since $\mathfrak{h} = L_0^0$ by Ex. 4.10. The symmetric bilinear form $(\cdot|\cdot)$ on E, defined by

$$\big(l_1 \oplus c_1 \oplus d_1 \mid l_2 \oplus c_2 \oplus d_2\big) = (l_1 \mid l_2)_L + c_1(d_2) + c_2(d_1)$$

where $(\cdot|\cdot)_L$ is the given bilinear form of the invariant Lie torus L, is nondegenerate and invariant. With respect to this bilinear form the set of roots of (E, H) is $R = R^0 \cup R^{\mathrm{an}}$, where

$$\begin{aligned} R^0 &= \{\lambda \in \Lambda \subset H^* : L_0^\lambda \neq 0\} \quad \text{and} \\ R^{\mathrm{an}} &= \{\xi \oplus \lambda : \xi \neq 0 \text{ and } L_\xi^\lambda \neq 0\}. \end{aligned}$$

We have now indicated that the axioms (EA1) and (EA2) of an extended affine Lie algebra holds for the pair (E, H). The verification of the remaining axioms can be easily be done by the reader, or can be looked up in [Na, Prop. 5.2.4]. This then shows part (a) of the following theorem.

Theorem 5.15 ([Ne4, Th. 6])**.** (a) *The pair (E, H) constructed above is an extended affine Lie algebra, denoted* $\mathrm{E} = \mathrm{E}(L, D, \tau)$. *Its core is $L \oplus D^{\mathrm{gr}\,*}$ and its centreless core is L.*

(b) *Conversely, let (E, H) be an extended affine Lie algebra, and let $L = E_c/Z(E_c)$ be its centreless core, which by* Cor. 4.16 *is an invariant Lie torus, say of type (S, Λ), with Λ free of finite rank.*

Then there exists a subalgebra $D \subset \mathrm{SCDer}_F(L)$ and an abelian 2-cocycle τ satisfying the conditions (5.10)–(5.12) *on (D, τ) such that $E \cong \mathrm{E}(L, D, \tau)$.*

We have defined discrete EALAs in 2.1 as a special class of EALAs over the base field $F = \mathbb{C}$. They can now be characterized as follows.

Corollary 5.16 ([Ne4, Th. 8])**.** *Let $F = \mathbb{C}$.* (a) *Let L be an invariant Lie torus of type (S, Λ) with Λ free of finite rank and let $D \subset \mathrm{SCDer}_{\mathbb{C}}(L)$ be a graded subalgebra such that the evaluation map* $\mathrm{ev} : \Lambda \to D^{0\,*}$ *is injective with discrete image. Then, for any affine 2-cocycle τ the extended affine Lie algebra $\mathrm{E}(L, D, \tau)$ is a discrete EALA. Conversely, any discrete EALA arises in this way.*

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