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Integrable representations of root-graded Lie algebras

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ABSTRACT

In this paper we study the category of representations of a root-graded Lie algebra L which are integrable as representations of a finite-dimensional semisimple subalgebra $\mathfrak{g}$ and whose weights are bounded by some dominant weight of $\mathfrak{g}$. We link this category to the module category of an associative algebra, whose structure we determine for map algebras and $\mathfrak{sl}_n(A)$.

Our approach unifies recent work of Chari and her collaborators on map algebras, of Fourier and Savage and their collaborators on equivariant map algebras, as well as the classical work of Seligman on isotropic Lie algebras.

Contents

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Introduction

This paper unites two strands of research which up to now did not have any interaction. Historically, the first strand is the work of Seligman on representations of finite-dimensional isotropic central-simple Lie algebras L over fields $\Bbbk$ of characteristic 0 ([40,41]). While these could be studied in the spirit of Tits' approach to representation theory [42] using Galois descent, Seligman pursued "rational methods", i.e., describing the representations over $\Bbbk$ (rather than the algebraic closure of $\Bbbk$). The Lie algebra L decomposes with respect to a maximal split toral subalgebra $\mathfrak{h} \subset L$ as $L = \bigoplus_{\alpha \in \Theta \cup \{0\}} L_\alpha$ where $\Theta \subset \mathfrak{h}^*$ is an irreducible, possibly non-reduced root system and the L_α are the root spaces of L with respect to $\mathfrak{h}$. Any finite-dimensional irreducible representation of L has weights bounded by a dominant integral weight λ of Θ. Seligman introduced a unital associative algebra $\mathbb{S}^\lambda$ (see Definition 4.2) and linked the finite-dimensional irreducible representations of L and of $\mathbb{S}^\lambda$.

The second strand is essentially due to Chari and her collaborators. It concerns integrable representations of map algebras, i.e., Lie algebras $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra and A is a unital commutative associative $\mathbb{C}$-algebra. (The name "map algebra" comes from the interpretation of L as regular maps from the affine scheme $\mathrm{Spec}(A)$ to the affine variety $\mathfrak{g}$.) While originally $A = \mathbb{C}[t^{\pm 1}]$ (loop algebras [16]) or $A = \mathbb{C}[t]$ (current algebras [15]), the algebra A was soon taken to be arbitrary, see for example [18] for an earlier paper. Note that the Lie algebra $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ decomposes in a similar way as Seligman's Lie algebras: Let Δ be the root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$. Then $L = \bigoplus_{\alpha \in \Delta} L_\alpha$ where $L_\alpha = \mathfrak{g}_\alpha \otimes_{\mathbb{C}} A$ are the weight spaces of L under the canonical $\mathfrak{h}$-action. Regarding the representation theory of map algebras, a big step forward was undertaken in the paper [13] by Chari–Fourier–Khandai, which stressed a categorical point of view. Denote by $\mathcal{I}^\lambda(L, \mathfrak{g})$ the category of integrable representations of $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ whose weights with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are bounded by a dominant weight λ for $(\mathfrak{g}, \mathfrak{h})$. It was shown in [13], among many other things, that $\mathcal{I}^\lambda(L, \mathfrak{g})$ is closely related to the module category of a commutative associative $\mathbb{C}$-algebra $\mathbf{A}_\lambda$ (see Definition 3.10). Moreover, the algebra $\mathbf{A}_\lambda$ was determined in the Noetherian case. The paper also introduced global Weyl modules as the initial cyclic objects in $\mathcal{I}^\lambda(L, \mathfrak{g})$, generalizing the case $A = \mathbb{C}[t^{\pm 1}]$ of [16]. It has been the blueprint for many sequels, in which the results of [13] have been generalized to the setting of various equivariant map algebras, culminating with the recent paper [20].

The starting point of this paper is the observation that map algebras $\mathfrak{g} \otimes A$ and the equivariant map algebras studied in the sequels to [13] like for example [20], as well as the Lie algebras considered in [40,41], are examples of Lie algebras graded by finite (not necessarily reduced nor irreducible) root systems. For the sake of simplicity, in this introduction we will only consider irreducible root systems Θ. Given such a Θ we let $\Delta = \Theta$ if Θ is reduced and $\Delta = \mathrm{B}_n$ or $= \mathrm{C}_n$ if $\Theta \cong \mathrm{BC}_n$. By definition, a Lie algebra L over a field $\Bbbk$ of characteristic 0 is (Θ, Δ)-*graded* if it contains a finite-dimensional split simple subalgebra $\mathfrak{g}$ whose root system with respect to a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is Δ such that

$$L = \bigoplus\nolimits_{\alpha \in \Theta \cup \{0\}} L_\alpha \quad \text{and} \qquad L_0 = \sum\nolimits_{\alpha \in \Theta} [L_\alpha, L_{-\alpha}],$$

where the L_α are the weight spaces of L under the adjoint representation of $\mathfrak{h}$. There exists an elaborate structure theory of root-graded Lie algebras [1,2,11,12,4,33]. The reader will recognize that the two types of Lie algebras mentioned above are indeed (Θ, Δ)-graded for appropriate choices of (Θ, Δ). For example, the Lie algebras considered in [14,17] are (Θ, Δ)-graded with a non-reduced Θ. But there exist many more examples of root-graded Lie algebras (see Section 3.2 for further examples).

The main point of this paper is that root-graded Lie algebras provide a natural setting, in which one can develop a satisfactory theory of integrable representations. Indeed, we are able to define the main concepts developed for representations of map algebras on the one hand and for finite-dimensional central-simple Lie algebras on the other hand in the setting of (Θ, Δ)-graded Lie algebras $(L, \mathfrak{g})$:

- The category $\mathcal{I}(L, \mathfrak{g})$ (defined in Section 2.3) is the category of L-modules V that are integrable as $\mathfrak{g}$-modules (see Section 2.2) and thus have a weight space decomposition $V = \bigoplus_{\mu \in \mathcal{P}} V_\mu$ with respect to $\mathfrak{h}$ for $\mathcal{P}$ the set of integral weights of Δ.
- $\mathcal{I}^\lambda(L, \mathfrak{g})$ is the subcategory of $\mathcal{I}(L, \mathfrak{g})$ consisting of modules V for which all non-zero weight spaces have weights bounded above by λ in a canonical order (which is the natural partial order of $\mathcal{P}$ in the reduced case), see Section 3.4.
- Global Weyl modules $W(\lambda) \in \mathcal{I}(L, \mathfrak{g})^\lambda$ are defined by taking an appropriate quotient of $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V(\lambda)$ (3.7) and have the usual presentation (see Proposition 3.8).
- In Definition 3.10, we define the algebra $\mathbf{A}_\lambda = \mathrm{U}(L_0)/\operatorname{Ann}_{U(L_0)}(w_\lambda)$, where $w_\lambda \in W(\lambda)$ is the canonical generator of $W(\lambda)$. This extends the definition originally given by Chari and Pressley.
- In Definition 4.2, we define the algebra $\mathbb{S}^\lambda = \mathrm{U}(L_0)/J^\lambda$, where the ideal J^λ is given in terms of generators – it turns out to be the maximal quotient of $\mathrm{U}(L_0)$ which acts on the weight space V_λ for any module $V \in \mathcal{I}^\lambda(L, g)$. The algebra $\mathbb{S}^\lambda$ extends the definition of Seligman.

The main results of this paper are:

(a) Theorem 4.10: *$\mathbf{A}_\lambda$ and $\mathbb{S}^\lambda$ are equal associative algebras*, that is, $J^\lambda = \operatorname{Ann}_{U(L_0)}(w_\lambda)$. Because of this equality, this algebra will be called the Seligman–Chari–Pressley algebra.

(b) Extending results from the previously considered cases, we show that we have a restriction functor

$$\mathbf{Res}\colon \mathcal{I}^\lambda(L,\mathfrak{g}) \to \mathbb{S}^\lambda\text{-}\mathcal{M}od,$$

given by $V \mapsto V_\lambda$, and an "integrable induction" functor

$$\mathbf{Int}\colon \mathbb{S}^\lambda\text{-}\mathcal{M}od \to \mathcal{I}^\lambda(L,\mathfrak{g})$$

that satisfy $\mathbf{Res}\circ\mathbf{Int} \xrightarrow{\simeq} \mathrm{Id}_{\mathbb{S}^\lambda\text{-}\mathcal{M}od}$, see (4.7.1), and $\mathbf{Int}\circ\mathbf{Res} \Rightarrow \mathrm{Id}_{\mathcal{I}^\lambda(L,\mathfrak{g})}$, see Proposition 4.8(b).

(c) We determine the structure of $\mathbb{S}^\lambda$ in two important examples. For $L = \mathfrak{g}\otimes A$, where A is a commutative unital $\Bbbk$-algebra, and a dominant $\lambda = \sum_{i\in I} \ell_i \varpi_i$ we have

$$\mathbb{S}^\lambda \cong \mathrm{TS}^{\ell_1}(A) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathrm{TS}^{\ell_r}(A)$$

(Theorem 5.7), where $\mathrm{TS}^n(A)$ is the fixed point subalgebra of $A^{\otimes n}$ under the obvious action of the symmetric group $\mathfrak{S}_n$. This generalizes [13, Thm. 4]. For the A_{n-1}-graded Lie algebra $\mathfrak{sl}_n(A)$, $n \ge 3$ and A associative but not necessarily commutative, and a dominant weight λ with totally disconnected support, we describe $\mathbb{S}^\lambda$ in Theorem 6.5. For example, taking $\lambda = \ell\varpi_i$ with $\ell \in \mathbb{N}_+$ we have

$$\mathbb{S}^{\ell\varpi_i} \cong \begin{cases} \mathrm{TS}^\ell(A), & i = 1, \\ \mathrm{TS}^\ell(A)/\mathcal{C} & 1 < i < n-1, \\ \mathrm{TS}^\ell(A^{\mathrm{op}}) & i = n-1, \end{cases}$$

where $\mathcal{C}$ is the ideal of $\mathrm{TS}^\ell(A)$ generated by the commutator space $[\mathrm{TS}^\ell(A), \mathrm{TS}^\ell(A)]$. We point out that it may very well happen that $\mathbb{S}^\lambda = \{0\}$ in the middle case, which means that $V_\lambda = \{0\}$ for every $V \in \mathcal{I}^\lambda$.

Besides the classes of root-graded Lie algebras explicitly mentioned above, Tits–Kantor–Koecher algebras $L(J)$ of unital Jordan algebras J and their central coverings are another important class of root-graded Lie algebras — in this case $\Theta = \Delta = \mathrm{A}_1$. Integrable representations of the universal central extension $\widehat{L(J)}$ bounded by $\lambda = 2\varpi$ have been studied in the recent preprint [28] by Kashuba–Serganova. Results for $\lambda = \varpi$ are contained in the sequel [29] of [28], while integrable representations in general are investigated in on-going work of Lau and Mathieu. The cases $\lambda = \omega$ and $\lambda = 2\omega$ are closely related to associative specializations and bimodules of Jordan algebras J, which for finite-dimensional simple J have been classified in the classical paper [26] by F.D. and N. Jacobson (see also [24, Ch. II and VII]). Representations of C_n-graded Lie algebras have been investigated in [43,44].

Outlook. There are many questions left open in this paper. For instance, what is structure of the Seligman–Chari–Pressley algebra for the other types of root-graded Lie algebras? What are local Weyl modules in the setting of root-graded Lie algebras? Lie tori form a special class of root-graded Lie algebras that enter in the construction of extended affine Lie algebras, see e.g. [34–36]. It would be of interest to work out the relation between the integrable representations of Lie tori and the representations of the associated extended affine Lie algebras. Last but not least, one can also consider these questions for root-graded Lie superalgebras, for which one has a well-developed structure theory (see [3] and the references therein).

The paper is organized as follows. In Section 1 we study symmetric tensor algebras $\mathrm{TS}^\ell(A)$ for A a unital associative $\Bbbk$-algebra. Generalizing a result proven in [39, §2] for finite-dimensional A, we show in Theorem 1.7 that $\mathrm{TS}^\ell(A)$ has a universal property with respect to certain symmetric identities in the sense of [39]. In light of the explicit descriptions of $\mathbb{S}^\lambda$ given above, understanding the structure of $\mathrm{TS}^\ell(A)$ is important. We discuss this in Section 1.9. For example, $\mathrm{TS}^\ell\big(\mathrm{Mat}_d(\Bbbk)\big)$ is described by the classical Schur–Weyl duality. In Section 2 we investigate integrable representations of certain pairs $(L, \mathfrak{g})$ which in Section 3 are specialized to root-graded Lie algebras. Section 4 is devoted to the Seligman algebra $\mathbb{S}^\lambda = \mathbb{S}^\lambda(L, \mathfrak{g})$, its module category and the link to $\mathcal{I}^\lambda(L, \mathfrak{g})$. For example, when L_0 is finite-dimensional, we show that $\dim \mathbb{S}^\lambda$ is finite and indeed we give an explicit upper bound for the dimension of $\mathbb{S}^\lambda$ (see Proposition 4.15). In Sections 5 and 6 we prove our results on $\mathbb{S}^\lambda$ for map algebras and $\mathfrak{sl}_n(A)$ respectively.

Notation and conventions. Throughout the paper, $\Bbbk$ will denote a field of characteristic zero, $\mathbb{Z}$ the integers, $\mathbb{N}$ the nonnegative integers and $\mathbb{N}_+$ the positive integers. All algebras will be defined over $\Bbbk$. Unless stated otherwise we abbreviate $\otimes = \otimes_\Bbbk$. If X is a set, its cardinality will be denoted by $|X|$. We use $M \in \mathrm{Ob}\,\mathcal{I}$ to denote an object M of a category $\mathcal{I}$.

Throughout A denotes an associative unital but not necessarily commutative $\Bbbk$-algebra. Its identity element is denoted 1_A or sometimes just 1. Given another unital associative algebra B, a unital algebra homomorphism $\varphi\colon A \to B$ is a $\Bbbk$-linear map satisfying $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_i \in A$ and $\varphi(1_A) = 1_B$. As usual, $[a_1, a_2] = a_1a_2 - a_2a_1$ is the commutator in A. We denote by $\Bbbk$**-alg** the category of associative commutative unital $\Bbbk$-algebras with unital algebra homomorphisms as morphisms. We abbreviate $A \in \Bbbk\text{-}\mathbf{alg}$ for $A \in \mathrm{Ob}\,\Bbbk\text{-}\mathbf{alg}$.

Any associative algebra B becomes a Lie algebra with respect to the commutator product $[.,.]$, denoted B^-. Its derived algebra $[B, B] = \mathrm{Span}_\Bbbk\{[b_1, b_2] : b_i \in B\}$ is an ideal of the Lie algebra B^-. A linear map $A \to B$ will be called a *Lie homomorphism* if it is a homomorphism $A^- \to B^-$ of the associated Lie algebras.

The universal enveloping algebra of a Lie algebra L is denoted $\mathrm{U}(L)$. We will say that V is an L-module if there exists a Lie algebra homomorphism $\rho\colon L \to \mathfrak{gl}(V)$. However, if we need to be more precise this situation will be abbreviated by (V, ρ). Unless explicitly stated otherwise, $\mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra over $\Bbbk$. All other unexplained notation and terminology can be found in 2.1.

1. Symmetric tensor algebras

In this section we review and generalize some of Seligman's results [39] on symmetric tensor algebras. Throughout this section we fix $\ell \in \mathbb{N}_+$.

1.1. Definition *(Symmetric tensor algebras).* We denote by $A^{\otimes \ell} = A \otimes \cdots \otimes A$ the ℓth-tensor product of A. We will view $A^{\otimes \ell}$ as an associative algebra whose product is given by "coordinate-wise" multiplication, i.e., by extending linearly the assignment $(a_1 \otimes \cdots \otimes a_\ell) \cdot (b_1 \otimes \cdots \otimes b_\ell) := (a_1 b_1) \otimes \cdots \otimes (a_\ell b_\ell)$. The symmetric group $\mathfrak{S}_l$ acts by automorphisms on $A^{\otimes \ell}$ by linear extension of $\sigma \cdot (a_1 \otimes \cdots \otimes a_\ell) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(\ell)}$.

We denote by $\mathrm{TS}^\ell(A)$ the subspace of symmetric tensors in $A^{\otimes \ell}$, i.e., the invariant subspace under the $\mathfrak{S}_\ell$-action (TS stands for "tenseurs symétriques"). Since $\mathfrak{S}_\ell$ acts on $A^{\otimes \ell}$ by automorphisms, the subspace $\mathrm{TS}^\ell(A)$ is a subalgebra of $A^{\otimes \ell}$. In the following, we will always view $\mathrm{TS}^\ell(A)$ with this associative structure. We call $\mathrm{TS}^\ell(A)$ the *ℓth-symmetric tensor algebra*.

Since $\mathrm{TS}^\ell(A)$ is an associative algebra, $\mathrm{TS}^\ell(A)^-$ is a Lie algebra. For $1 \le i \le \ell$ and $a \in A$ we put

$$\begin{aligned} a^{\otimes \ell} &= a \otimes a \otimes \cdots \otimes a \in \mathrm{TS}(A) \quad (\ell \text{ factors}), \text{ and} \\ s_i(a) &= 1^{\otimes i-1} \otimes a \otimes 1^{\otimes \ell - i} \in A^{\otimes \ell}. \end{aligned}$$

Since $[s_i(a), s_j(b)] = \delta_{ij} s_i([a,b])$ the *symmetrization map*

$$\mathrm{sym}_\ell : A^- \to \mathrm{TS}^\ell(A)^-, \quad a \mapsto \textstyle\sum_{i=1}^{\ell} s_i(a) \tag{1.1.1}$$

is a homomorphism of Lie algebras.

1.2. Lemma. (a) $\mathrm{TS}^\ell(A)$ *is spanned as vector space by* $a^{\otimes \ell}$, $a \in A$.

(b) $\mathrm{TS}^\ell(A)$ *is generated as an associative algebra by* $\mathrm{sym}_\ell(A)$.

More properties of $\mathrm{TS}^\ell(A)$ are discussed in 1.9.

Proof. (a) We denote by $\mathsf{S}(A)$ the symmetric algebra of the vector space underlying A, by $\circ$ its product and by $\mathsf{S}^\ell(A)$ its ℓth graded component. Since our base field $\Bbbk$ has characteristic 0, the canonical map $\pi : A^{\otimes \ell} \to \mathsf{S}^\ell(A)$, given by $a_1 \otimes \cdots \otimes a_\ell \mapsto a_1 \circ \cdots \circ a_\ell$, becomes a linear isomorphism when restricted to $\mathrm{TS}^\ell(A)$ [6, III, §6.3 Rem.]. We have $\pi(a^{\otimes \ell}) = a \circ a \cdots \circ a$. Since $\mathsf{S}^\ell(A)$ is spanned by the powers $a \circ \cdots \circ a$ [6, III, §6.1, Rem. 3], (a) follows.

(b) Fix $a \in A$. By (a) it is enough to show that $a^{\otimes \ell}$ lies in the associative subalgebra of $\mathrm{TS}^\ell(A)$ generated by $\mathrm{sym}_\ell(A)$. Since $[s_i(a), s_j(a)] = 0$ for $1 \le i, j \le \ell$, we have a unique unital algebra homomorphism $\zeta \colon \Bbbk[x_1, \ldots, x_\ell] \to A^{\otimes \ell}$ such that $\zeta(x_i) = s_i(a)$. Recall that $\mathfrak{S}_l$ acts on the polynomial algebra $\Bbbk[x_1, \ldots, x_\ell]$ by $(\sigma \cdot p)(x_1, \ldots, x_\ell) =$

$p(x_{\sigma(1)},\dots,x_{\sigma(\ell)})$ for $\sigma\in\mathfrak{S}_l$. It follows from $\zeta(\sigma\cdot x_i)=\zeta(x_{\sigma(i)})=s_{\sigma(i)}(a)=\sigma\cdot s_i(a)=\sigma\cdot\zeta(x_i)$ that ζ is an $\mathfrak{S}_\ell$-module map. It therefore maps the symmetric polynomial algebra $k[x_1,\dots,x_\ell]^{\mathfrak{S}_l}$ to $\mathrm{TS}^\ell(A)$. Let $p_i=\sum_{j=1}^{\ell}x_j^i$ be the ith-power sum. Since $\zeta(p_i)=\mathrm{sym}_\ell(a^i)$ and the $p_1,\dots,p_\ell$ generate the algebra $\Bbbk[x_1,\dots,x_\ell]^{\mathfrak{S}_\ell}$, see e.g. [30, I.2, (2.12)], our claim follows. $\square$

We will characterize $\mathrm{TS}^\ell(A)$ by a universal property in Theorem 1.7. To do so, we will use the following technical results, the first of which is standard vector space theory.

1.3. Lemma. *Let C be a basis of a vector space Y and assume that $\mathsf{C}=\bigcup_{i=0}^{\ell}\mathsf{C}_i$ is a partition of C. Put $Y_i=\mathrm{Span}(\mathsf{C}_0\cup\dots\cup\mathsf{C}_i)$ and $Y_{-1}=\{0\}$. Furthermore, suppose that $\mathsf{B}\subseteq Y$ has a partition $\mathsf{B}=\bigcup_{i=0}^{\ell}\mathsf{B}_i$ such that the canonical images of B_i and C_i in Y_i/Y_{i-1} are identical for all i, $0\le i\le\ell$. Then $\mathsf{B}_0\cup\dots\cup\mathsf{B}_\ell$ is a basis of Y.*

1.4. Lemma. *Let $\{1_A\}\cup\mathsf{B}$ be a basis of the vector space A, and let $\le$ be a total order on B. Then*

$$\mathsf{TB}=\{1_A^{\otimes\ell}\}\cup\{\mathrm{sym}_\ell(b_1)\mathrm{sym}_\ell(b_2)\cdots\mathrm{sym}_\ell(b_j):b_i\in\mathsf{B},b_1\le b_2\le\dots\le b_j,1\le j\le\ell\}$$

is a basis of the vector space $\mathrm{TS}^\ell(A)$.

Proof. Recall the linear map $\pi\colon A^{\otimes\ell}\to\mathsf{S}^\ell(A)$, $\pi(a_1\otimes\dots\otimes a_\ell)=a_1\circ\dots\circ a_\ell$. Since its restriction to $\mathrm{TS}^\ell(A)$ is a vector space isomorphism, it suffices to show that $\pi(\mathsf{TB})$ is a basis of $\mathsf{S}^\ell(A)$. To this end, put

$$\mathsf{C}_j=\{1\circ\dots\circ1\circ b_1\circ b_2\circ\dots\circ b_j:b_1\le b_2\le\dots\le b_j\},\quad(0\le j\le\ell).$$

It is well-known that $\mathsf{C}=\bigcup_{j=0}^{\ell}\mathsf{C}_j$ is a basis of $\mathsf{S}^\ell(A)$. For $b_1\le\dots\le b_j$ we get

$$\mathrm{sym}_\ell(b_1)\cdots\mathrm{sym}_\ell(b_j)\equiv\tfrac{l!}{(l-j)!}\,1\circ\dots\circ1\circ b_1\circ\dots\circ b_j\quad\mathrm{mod}\ S_{j-1}^\ell(A):=\mathrm{Span}_{\Bbbk}(\mathsf{C}_0\cup\dots\cup\mathsf{C}_j).$$

Applying Lemma 1.3 (modulo scalars) shows that $\pi(\mathsf{TB})$ is indeed a vector space basis of $\mathsf{S}^\ell(A)$. $\square$

1.5. Seligman's symmetric identity

To a partition $p=(p_1,\dots,p_\ell)$ of ℓ, i.e., $p_i\in\mathbb{N}$ and $p_1+2p_2\cdots+\ell p_\ell=\ell$, we associate the conjugacy class $\mathcal{C}(p)\subset\mathfrak{S}_\ell$ of permutations whose cycle decomposition consists of p_i i-cycles. For example, $(\ell,0,\dots,0)$ corresponds to (the conjugacy class of) the identity $1_{\mathfrak{S}_\ell}$ and $(\ell-2,1,0,\dots,0)$ to the conjugacy class of any transposition. The map $p\mapsto\mathcal{C}(p)$ is a bijection between the partitions of ℓ and the set of conjugacy classes of $\mathfrak{S}_\ell$. Since the sign of a partition is constant on a conjugacy class, $\mathrm{sgn}\big(\mathcal{C}(p)\big)$ is well-defined as the sign of any partition in $\mathcal{C}(p)$. Following [39] we say that a linear map $\rho\colon A \to B$ into a unital associative $\Bbbk$-algebra *satisfies the ℓth-symmetric identity* if

$$\sum_{p=(p_1,\dots,p_\ell)} \operatorname{sgn}\big(\mathcal{C}(p)\big)\,\big|\big(\mathcal{C}(p)\big)\big|\,\rho(a)^{p_1}\,\rho(a^2)^{p_2}\,\cdots\rho(a^\ell)^{p_\ell}=0 \tag{1.5.1}$$

for all $a \in A$, the sum being taken over all partitions p of ℓ.

For example, it is easily seen that $\rho = \operatorname{sym}_2$, i.e., $\rho(a) = a\otimes 1 + 1\otimes a \in \operatorname{TS}^2(A)$, satisfies the third symmetric identity:

$$\operatorname{sym}_2(a)^3 - 3\operatorname{sym}_2(a)\operatorname{sym}_2(a^2) + 2\operatorname{sym}_2(a^3) = 0.$$

But even more is true:

1.6. Proposition. *([39, §2]) The Lie homomorphism $\operatorname{sym}_\ell\colon A \to \operatorname{TS}^\ell(A)$ satisfies the $(\ell+1)$st-symmetric identity and $\operatorname{sym}_\ell(1_A) = \ell\, 1_{\operatorname{TS}^\ell(A)}$.*

In fact, we will show in Theorem 1.7, $(\operatorname{TS}^\ell(A), \operatorname{sym}_\ell)$ is universal with respect to these two properties. Our proof is a generalization of [39, Prop. 3.1] where this result is proven for finite-dimensional A.

1.7. Theorem. *Let $\ell \in \mathbb{N}_+$. Then for every unital associative $\Bbbk$-algebra B and every Lie homomorphism $\rho\colon A \to B$ satisfying the $(\ell+1)$st-symmetric identity and $\rho(1_A) = \ell\, 1_B$ there exists a unique homomorphism $\varphi\colon \operatorname{TS}^\ell(A) \to B$ of unital associative algebras such that the diagram below is commutative.*

$$\begin{tikzcd} A \arrow[rr, "\operatorname{sym}_\ell"] \arrow[dr, "\rho"'] & & \operatorname{TS}^\ell(A) \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

Proof. Let $\mathrm{U} = \mathrm{U}(A^-)$ be the universal enveloping algebra of the Lie algebra A^-, let $\gamma\colon A^- \to U^-$ be the canonical embedding, and denote by $I \subset U$ the ideal of the associative algebra U generated by

(i) the elements $\sum_{p=(p_1,\dots,p_{\ell+1})} \operatorname{sgn}\big(\mathcal{C}(p)\big)\,\big|\big(\mathcal{C}(p)\big)\big|\,\gamma(a)^{p_1}\,\gamma(a^2)^{p_2}\,\cdots\gamma(a^{\ell+1})^{p_{\ell+1}}$ used in the definition of the $(\ell+1)$st-symmetric identity, and
(ii) $\gamma(1_A) = \ell\, 1_U$.

Let $\mathrm{can}\colon \mathrm{U} \to \mathrm{U}/I$ be the canonical quotient map. Then $\psi = \mathrm{can}\circ\gamma\colon A \to \mathrm{U}/I$ is a Lie homomorphism satisfying the $(\ell+1)$st-symmetric identity and $\psi(1_A) = \ell\, 1_{\mathrm{U}}$. Moreover, since by Proposition 1.6 the same holds for $(\operatorname{TS}^\ell(A), \operatorname{sym}_\ell)$, there exists a unique homomorphism $\eta\colon \mathrm{U}/I \to \operatorname{TS}^\ell(A)$ of unital associative algebras such that $\operatorname{sym}_\ell = \eta\circ\psi$:

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[rr, bend left, "\psi"] \arrow[dr, "\mathrm{sym}_\ell"'] & \mathrm{U} \arrow[r, "\mathrm{can}"] & \mathrm{U}/I \arrow[dl, "\eta"] \\ & \mathrm{TS}^\ell(A) & \end{tikzcd}$$

We claim that η is an isomorphism. Let $\mathsf{TB} \subset \mathrm{TS}^\ell(A)$ be the basis of Lemma 1.4. Since η is a unital algebra homomorphism, TB is the image under η of

$$\mathsf{C} = \{1_{\mathrm{U}/I}\} \cup \{\psi(b_1)\cdots\psi(b_j) : 1 \le j \le \ell, b_1 \le \cdots \le b_j\}.$$

Thus, η is an isomorphism as soon as we show that C is a spanning set of U/I. To do so, we use the canonical filtration of U determined by the generating set $\gamma(A)$ of U. Let $\mathrm{U}_{(t)}$ be the tth-term of this filtration. By the PBW Theorem, $\mathrm{U}_{(\ell+1)}/\mathrm{U}_{(\ell)} \cong \mathsf{S}^{\ell+1}(A)$. Hence, by Newton's identities (see for example [6, III, §6.1, Rem. 3]), $\mathrm{U}_{(\ell+1)}/\mathrm{U}_{(\ell)}$ is spanned by $\gamma(a)^{\ell+1} + \mathrm{U}_{(\ell)}$, $a \in A$. Since

$$\sum_{p=(p_1,\dots,p_{\ell+1})} \mathrm{sgn}\big(\mathcal{C}(p)\big)\,\big|\big(\mathcal{C}(p)\big)\big|\,\gamma(a)^{p_1}\,\gamma(a^2)^{p_2}\,\cdots\gamma(a^{\ell+1})^{p_{\ell+1}} = \gamma(a)^{\ell+1} + u_{(l)}$$

for some $u_{(\ell)} \in \mathrm{U}_{(\ell)}$ (the term $\gamma(a)^{\ell+1}$ occurs for $p = (\ell+1, 0, \dots, 0)$), it follows that every element in $\mathrm{U}_{(\ell+1)}$ is congruent to $\mathrm{U}_{(\ell)}$ modulo I. Since $\mathrm{U}_{(t)} = \mathrm{U}_{(\ell+1)}\,\mathrm{U}_{(t-\ell-1)}$ for $t > \ell$, this implies that U/I is spanned by the image of $\mathrm{U}_{(\ell)}$ under the canonical map $\mathrm{U} \to \mathrm{U}/I$.

We extend the total order $\le$ of B to a total order on $\mathsf{B}' = \{1_A\} \cup \mathsf{B}$ by $1_A \le b$ for all $b \in \mathsf{B}$. Again by the PBW Theorem (or one of its corollaries), we then know that U/I is spanned by $1_{\mathrm{U}/I}$ and elements $\psi(b_1')\cdots\psi(b_j')$, $1 \le j \le \ell$, $b_i' \in \mathsf{B}'$, $b_1' \le \cdots \le b_j'$. Finally, replacing 1_A in such an element by $1_{\mathrm{U}/I}$ using the relation (ii) above, we get that C spans U/I.

Now let $\rho\colon A \to B$ be a map as in the statement of the theorem. Using the universal property of U, it is immediate that there exists a unique unital algebra homomorphism $\varphi'\colon\ \mathrm{U}/I \to B$ such that $\rho = \varphi' \circ \psi$:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, "\rho"'] & \mathrm{U}/I \arrow[r, "\eta"] \arrow[d, "\varphi'"] & \mathrm{TS}^\ell(A) \arrow[dl, dashed, "\varphi"] \\ & B & \end{tikzcd}$$

Putting $\varphi = \varphi' \circ \eta^{-1}$ shows $\varphi \circ \mathrm{sym}_\ell = \varphi' \circ \eta^{-1} \circ \mathrm{sym}_\ell = \varphi' \circ \psi = \rho$. $\square$

A natural question arising at this point is how to check that a given map $\rho\colon A \to B$ satisfies the kth-symmetric identity for some $k \in \mathbb{N}_+$. Part of this problem is to get a good approach to the left hand side of (1.5.1). In this paper we will use the following recursion.

1.8. Recursion. ([39, §III.1, in particular (6)]) Let $\rho\colon A \to B$ be a $\Bbbk$-linear map into a unital associative $\Bbbk$-algebra B such that $[a_1, a_2] = 0 \Rightarrow [\rho(a_1), \rho(a_2)] = 0$ for all $a_i \in A$. Assume that there exists a family $(g_t)_{t\in\mathbb{N}}$ of multilinear functions $g_t\colon A^t \to B$ satisfying

(i) $g_1 = \rho$, and
(ii) if $(a_1, \dots, a_{t+1})$ is a family of commuting elements of A then

$$\begin{aligned} g_{t+1}(a_1, \dots, a_{t+1}) = &\sum_{j=1}^{t+1} \rho(a_j) g_t(a, \dots, \widehat{a_j}, \dots, a_{t+1}) \\ &- 2 \sum_{j<m} g_t(a_j a_m, a_1, \dots, \widehat{a_j}, \dots, \widehat{a_m}, \dots, a_{t+1}) \end{aligned} \tag{1.8.1}$$

where $\widehat{\ }$ indicates an omitted argument.

Then $g_t(a, a, \dots, a)$ equals the left hand side of (1.5.1):

$$g_t(a, a, \dots, a) = \sum_{p=(p_1,\dots,p_t)} \operatorname{sgn}\big(\mathcal{C}(p)\big)\, \big|\big(\mathcal{C}(p)\big)\big|\, \rho(a)^{p_1}\, \rho(a^2)^{p_2} \cdots \rho(a^t)^{p_t} \tag{1.8.2}$$

Seligman has in fact obtained a closed formula for $g_t(a_1, \dots, a_t)$ ([39, Lem. 1.1]).

In this paper we will use a two-pronged approach to verifying that a certain function satisfies the $(\ell+1)$st-symmetric identity: We will define families (g_t) of functions satisfying the recursion (1.8.1) and we will then know from the context that $g_{\ell+1} \equiv 0$ (see for example Proposition 5.4).

1.9. More properties of $\mathrm{TS}^\ell(A)$

In light of the importance of the algebra $\mathrm{TS}^\ell(A)$ for this paper, it is appropriate to establish more properties of this algebra, besides the ones given in Lemma 1.2.

(a) (*Functoriality*) It is immediate that any homomorphism $\varphi\colon A \to B$ of unital associative $\Bbbk$-algebras induces a homomorphism $\mathrm{TS}^\ell(\varphi)\colon \mathrm{TS}^\ell(A) \to \mathrm{TS}^\ell(B)$ and that the assignments $A \mapsto \mathrm{TS}^\ell(A)$ and $\varphi \mapsto \mathrm{TS}^\ell(\varphi)$ define a functor TS^ℓ on the category of unital associative $\Bbbk$-algebras. It commutes with base field extensions,

$$\mathrm{TS}^\ell(A) \otimes K \cong \mathrm{TS}^\ell(A \otimes K),$$

and moreover, is an exact functor.

(b) Let A be a finite-dimensional semisimple $\Bbbk$-algebra. Then $A^{\otimes \ell}$ is semisimple by [10, §12.7, Cor. 1] and so is $\mathrm{TS}^\ell(A)$ by [31, Cor. 6]. That $\mathrm{TS}^\ell(A)$ is semisimple in case A is finite-dimensional central-simple, is also proven in [40, IV.4].

(c) (*Schur–Weyl duality* [39, p. 467]) In particular, for $A = \mathrm{Mat}_d(\Bbbk)$, the associative algebra of $d \times d$ matrices over $\Bbbk$, we get

$$\mathrm{TS}^\ell(\mathrm{Mat}_d(\Bbbk)) \cong \big(\mathrm{End}_\Bbbk(V^{\otimes \ell})\big)^{\mathfrak{S}_\ell},$$

where the right hand side is the centralizer algebra of $\mathfrak{S}_\ell$ and is thus described by the classical Schur–Weyl duality. That is, it is a direct product of $p_d(\ell)$ matrix algebras where $p_d(\ell)$ is the number of partitions of ℓ with at most d parts, see for example [21, §9.1.1].

(d) Assume $A \in \Bbbk$-**alg** is finitely generated. Then so is $A^{\otimes \ell}$ and hence also $\mathrm{TS}^\ell(A)$, [9, I, §1.9, Th. 2]. In fact, if $\Bbbk$ is algebraically closed and X is an affine variety with coordinate ring A, then $\mathrm{TS}^\ell(A)$ is the coordinate ring of the symmetric product $X^{(\ell)}$, [22, Ex. 10.23].

As a specific example, let $A = k[x]$, the polynomial ring in the variable x. Then $\mathrm{TS}^\ell(k[x])$ is the ring of symmetric polynomials in ℓ variables, hence a polynomial ring in the elementary symmetric polynomials.

2. Categories of integrable modules for pairs $(L, \mathfrak{g})$

Let $\mathfrak{g}$ be a finite-dimensional split semisimple Lie algebra over $\Bbbk$. In this section we consider pairs $(L, \mathfrak{g})$ where

$$\begin{aligned} &L \text{ is a Lie algebra containing } \mathfrak{g} \text{ as a subalgebra such that} \\ &\quad L \text{ is an integrable module under the adjoint representation of } \mathfrak{g}, \text{ cf. } 2.2. \end{aligned} \tag{2.0.1}$$

2.1. *Notation (standard)*

Let $\mathfrak{h} \subset \mathfrak{g}$ be a splitting Cartan subalgebra. We denote by
- $\Delta \subset \mathfrak{h}^*$ the root system of $(\mathfrak{g}, \mathfrak{h})$,
- $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, the corresponding root space,
- $h_\alpha \in \mathfrak{h}$ the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying $\alpha(h_\alpha) = 2$,
- $\mathcal{Q} = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ the root lattice of $(\mathfrak{g}, \mathfrak{h})$ and
- $\mathcal{P} = \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ the (integral) weight lattice of $(\mathfrak{g}, \mathfrak{h})$.

We choose a root basis $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of Δ. Associated to Π are the following data:
- $I = \{1, \ldots, r\}$,
- $h_i := h_{\alpha_i} \in \mathfrak{h}$, $1 \le i \le r$,
- $\mathcal{Q}_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ and $\Delta_+ = \Delta \cap \mathcal{Q}_+$,
- $\mathcal{P}_+ = \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{N}, 1 \le i \le r\} \subset \mathcal{P}$,
- $\varpi_i \in \mathcal{P}_+$, $i \in I$, fundamental weights defined by $\varpi_i(h_j) = \delta_{ij}$,
- $\lambda \le \mu \iff \mu - \lambda \in \mathcal{Q}_+$ for $\lambda, \mu \in \mathcal{P}$,
- $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$.

We choose non-zero $e_i \in \mathfrak{g}_{\alpha_i}$, $1 \le i \le r$, and denote by f_i, $1 \le i \le r$, the unique element in $\mathfrak{g}_{-\alpha_i}$ satisfying $[e_i, f_i] = h_i$.

In the following we fix the data and notation introduced above, and use them without further explanation.

2.2. *Integrable $\mathfrak{g}$-modules*

It is standard that for a $\mathfrak{g}$-module V the following are equivalent:

(i) V is a sum of finite-dimensional $\mathfrak{g}$-modules;
(ii) V is a direct sum of finite-dimensional simple $\mathfrak{g}$-modules;
(iii) for each $v \in V$, the submodule $\mathrm{U}(\mathfrak{g})v$ is finite-dimensional;
(iv) for all $i \in I$, the elements e_i, f_i act locally nilpotently on V.

The implication (iv)⇒(i) follows from [27, Prop. 3.8]. A $\mathfrak{g}$-module V which satisfies the above properties is called an *integrable* $\mathfrak{g}$*-module* – it is sometimes also called a *locally finite* $\mathfrak{g}$-module. We will use the following properties of integrable $\mathfrak{g}$-modules:

(a) In a short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathfrak{g}$-modules, M is integrable if and only if M' and M'' are integrable. In this case the sequence splits in the category of $\mathfrak{g}$-modules.

(b) The tensor product and the direct sum of integrable modules are again integrable.

(c) The tensor algebra $\mathrm{T}(L)$ and its quotient $\mathrm{U}(L)$ are integrable $\mathfrak{g}$-modules with respect to the canonical extension of the adjoint action of $\mathfrak{g}$ on L to $\mathrm{T}(L)$ and $\mathrm{U}(L)$ respectively. Denoting the latter action by ρ_{U}, we have for $x \in \mathfrak{g}$ and $u \in \mathrm{U}(L)$ the formula $\rho_{\mathrm{U}}(x)(u) = xu - ux$ where the right hand side is given by the multiplication in $\mathrm{U}(L)$.

2.3. *Category $\mathcal{I}(L, \mathfrak{g})$*

We let $\mathcal{I}(L, \mathfrak{g})$ denote the category whose objects are L-modules which are integrable as $\mathfrak{g}$-modules, and whose morphisms are L-module homomorphisms. Note that in our setting L is an object of $\mathcal{I}(L, \mathfrak{g})$ with respect to the adjoint representation.

2.4. Proposition. *Let (V, ρ) be an integrable $\mathfrak{g}$-module. Then*

$$P(V, \rho) := \mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V,$$

considered as L-module under left multiplication on the left factor, is a projective object of $\mathcal{I}(L, \mathfrak{g})$.

This is a classical result that has been proven in different contexts (e.g., [13], but certainly it goes back as far as [23]). We include a proof for the sake of completeness.

Proof. We start by showing that $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V$ is an object of $\mathcal{I}(L, \mathfrak{g})$. To do so, it suffices to prove that $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V$ is an integrable $\mathfrak{g}$-module with respect to the indicated action. We know from 2.2(c) and 2.2(b) that $\mathrm{U}(L) \otimes_{\Bbbk} V$ is an integrable $\mathfrak{g}$-module with respect to the tensor product action of $\mathfrak{g}$, i.e., $x \in \mathfrak{g}$ acts on $u \otimes v \in \mathrm{U}(L) \otimes_{\Bbbk} V$ by $\rho_{\mathrm{U}}(x)(u) \otimes_{\Bbbk} v + u \otimes_{\Bbbk} \rho(x)(v) = (xu - ux) \otimes_{\Bbbk} v + u \otimes_{\Bbbk} \rho(v)$. This action descends to $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V$, and the obvious surjective linear map

$$\pi\colon \mathrm{U}(L) \otimes_{\Bbbk} V \to \mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V \quad , \quad u \otimes_{\Bbbk} v \mapsto u \otimes_{\mathrm{U}(\mathfrak{g})} v$$

is a $\mathfrak{g}$-module map with respect to the indicated $\mathfrak{g}$-actions. It then follows from 2.2(a) that $P(V, \rho)$ is indeed an integrable $\mathfrak{g}$-module.

To prove that $P(V, \rho)$ is projective, we must show that $\mathrm{Hom}_{\mathrm{U}(L)}(P(V), -)$ is an exact functor on $\mathcal{I}(L, \mathfrak{g})$. Using the standard adjunction formula we have isomorphisms of abelian groups

$$\mathrm{Hom}_{\mathrm{U}(L)}(\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V, M) \cong \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(V, \mathrm{Hom}_{\mathrm{U}(L)}(\mathrm{U}(L), M)) \cong \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(V, M)$$

for any $M \in \mathrm{Ob}\, \mathcal{I}(L, \mathfrak{g})$. Let $0 \to K \to M \to N \to 0$ be an exact sequence of objects in $\mathcal{I}(L, \mathfrak{g})$. By 2.2(a) this sequence splits in the category of $\mathfrak{g}$-modules. Hence the associated sequence

$$0 \to \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(V, K) \to \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(V, M) \to \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(V, N) \to 0$$

is also exact, in fact split-exact, which implies that $P(V, \rho)$ is projective. □

2.5. Corollary. *The category $\mathcal{I}(L, \mathfrak{g})$ has enough projectives.*

Proof. Indeed, for each $V \in \mathrm{Ob}\, \mathcal{I}(L, \mathfrak{g})$ the surjective homomorphism $\epsilon_V\colon P(V) \to V$, $u \otimes v \mapsto u \cdot v$, is an L-module map. □

2.6. Weight modules

A $\mathfrak{g}$-module M is called a *weight module* (with respect to $\mathfrak{h}$) if

$$M = \bigoplus\nolimits_{\mu \in \mathfrak{h}^*} M_\mu, \quad M_\mu = \{m \in M \;:\; h \cdot m = \mu(h)m,\ h \in \mathfrak{h}\}.$$

It is equivalent to require $M = \sum_{\mu \in \mathfrak{h}^*} M_\mu$. The *weights* of M are those μ such that $M_\mu \neq 0$.

An integrable $\mathfrak{g}$-module is necessarily a weight module with respect to any splitting Cartan subalgebra of $\mathfrak{g}$, hence in particular with respect to our chosen $\mathfrak{h}$. Furthermore, a weight module with a finite set of weights is integrable.

A *highest weight module of* $\mathfrak{g}$ of highest weight $\lambda \in \mathfrak{h}^*$ is a module V generated by a vector v_λ which satisfies $\mathfrak{n}_+ \cdot v_\lambda = 0$ and $h \cdot v_\lambda = \lambda(h) v_\lambda$ for $h \in \mathfrak{h}$. It follows that V is a weight module, the weights of V lie in $\lambda - \mathcal{Q}_+$ and $\dim_{\Bbbk} V_\mu < \infty$. Moreover, $V_\lambda = \Bbbk v_\lambda$.

For every $\lambda \in \mathcal{P}_+$ there exists a unique (up to isomorphism) highest weight module $V(\lambda)$ with highest weight λ. It is finite-dimensional and given as $V(\lambda) = \mathrm{U}(\mathfrak{g})/J(\lambda)$, where $J(\lambda)$ is the left ideal of $\mathrm{U}(\mathfrak{g})$ generated by $I(\lambda)$ and $f_i^{\lambda(h_i)+1}$. Every irreducible finite-dimensional $\mathfrak{g}$-module is isomorphic to a unique $V(\lambda)$, $\lambda \in \mathcal{P}_+$.

The following elementary lemma is a straightforward generalization of [13, Prop. 3.2].

2.7. Lemma. *Let V be a cyclic $\mathrm{U}(\mathfrak{g})$-module, i.e., $V \cong \mathrm{U}(\mathfrak{g})/\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})}(v_0)$, where v_0 is a generator of V. Then $\mathrm{U}(L)\otimes_{\mathrm{U}(\mathfrak{g})} V$ is a cyclic $\mathrm{U}(L)$-module under left multiplication with generator $1 \otimes v_0$, and $\operatorname{Ann}_{\mathrm{U}(L)}(1 \otimes v_0) = \mathrm{U}(L)\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})}(v_0)$. In particular, if $V = V(\lambda)$, then $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V(\lambda)$ is presented, as a $\mathrm{U}(L)$-module, by the relations*

$$\mathfrak{n}_+(1 \otimes v_\lambda) = 0, \quad h(1 \otimes v_\lambda) = \lambda(h)(1 \otimes v_\lambda), \quad (f_i)^{\lambda(h_i)+1}(1 \otimes v_\lambda) = 0, \quad i \in I, \quad h \in \mathfrak{h}.$$

Proof (sketch). $\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V$ is clearly cyclic with generator $1 \otimes v_0$, and it is easy to see that $\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})}(v_0) \subseteq \operatorname{Ann}_{\mathrm{U}(L)}(1 \otimes v_0)$, which is enough for one containment. For the reverse containment, use the PBW Theorem to decompose $\mathrm{U}(L)$ as a right $\mathrm{U}(\mathfrak{g})$-module. □

3. Integrable modules for root-graded Lie algebras

In this section we will specialize the pairs $(L, \mathfrak{g})$ of the previous section 2 assuming that L is a root-graded Lie algebra with grading subalgebra $\mathfrak{g}$. This will allow us to introduce and study global Weyl modules.

3.1. Definition *(Root-graded Lie algebras).* Let Θ be a finite root system in the sense of [5], hence not necessarily reduced or irreducible, and let $\Delta \subset \Theta$ be a subsystem. We will say that (Θ, Δ) is an *admissible pair* if for every connected component Θ_c of Θ the following hold:

(a) if Θ_c is reduced, then $\Theta_c \subset \Delta$, and
(b) if Θ_c is not reduced, say $\Theta_c = \Theta_{c,1} \cup \Theta_{c,2} \cup \Theta_{c,4}$ is of type BC_n where $\Theta_{c,i}$ is the set of roots of square length i, then
  (I) either $\Delta \cap \Theta_c = \Theta_{c,1} \cup \Theta_{c,2}$, hence is of type B_n, or
  (II) $\Delta \cap \Theta_c = \Theta_{c,2} \cup \Theta_{c,4}$ and thus is of type C_n.

In case (b) with $n = 1$ we have $\Theta_{c,2} = \emptyset$ and $\Delta \cap \Theta_c$ is of type $\mathrm{B}_1 = \mathrm{A}_1$ or $\mathrm{C}_1 = \mathrm{A}_1$. Observe that Δ is reduced and that $\Theta_c \cap \Delta$ is a connected component of Δ. We can and will view Θ as a set of weights of Δ.

Let (Θ, Δ) be an admissible pair. A (Θ, Δ)*-graded Lie algebra* is a pair $(L, \mathfrak{g})$ consisting of a Lie algebra L over $\Bbbk$ and a split semisimple subalgebra $\mathfrak{g}$ satisfying the following conditions:

(i) $\mathfrak{g}$ contains a splitting Cartan subalgebra $\mathfrak{h}$ whose root system is (isomorphic to) Δ such that

$$L = \sum_{\alpha \in \Theta \cup \{0\}} L_\alpha \tag{3.1.1}$$

where $L_\alpha = \{l \in L : [h, l] = \alpha(h)l \text{ for all } h \in \mathfrak{h}\}$ for $\alpha \in \mathfrak{h}^*$, and

(ii) $L_0 = \sum_{\alpha \in \Theta} [L_\alpha, L_{-\alpha}]$.

As splitting Cartan subalgebras of $\mathfrak{g}$ are conjugate under automorphisms of L which extend the elementary automorphisms of $\mathfrak{g}$ [8, VIII, §3.3, Cor. de la Prop. 10], condition (i) holds for one splitting Cartan subalgebra if and only if it holds for all. It is therefore not necessary to include $\mathfrak{h}$ in the notation of a (Θ, Δ)-graded Lie algebra. We will say that $(L, \mathfrak{g})$ is Δ-*graded* if $\Theta = \Delta$ is reduced and that $(L, \mathfrak{g})$ is *root-graded* if $(L, \mathfrak{g})$ is (Θ, Δ)-graded for some admissible pair (Θ, Δ). The subalgebra $\mathfrak{g}$, called the *grading subalgebra*, will usually be omitted from the notation.

Condition (3.1.1) says that L is a weight module for $\mathfrak{g}$ under the adjoint action. As its set of weights is finite, it follows that L is an integrable $\mathfrak{g}$-module. Thus $(L, \mathfrak{g})$ satisfies the assumption (2.0.1) of the previous section.

The condition (ii) serves as a normalization: If L satisfies (i) then $\big(\sum_{\alpha\in\Delta}[L_\alpha, L_{-\alpha}]\big) \oplus \big(\bigoplus_{\alpha\in\Delta} L_\alpha\big)$ is a Δ-graded ideal of L. From the definition of L_α it follows that the decomposition (3.1.1) is a grading by the weight lattice $\mathcal{P}$ of Δ, even by the root lattice if (Θ, Δ) does not have a connected component of type (bII).

Root-graded Lie algebras for irreducible Θ are considered in the papers [1,2,11,12,4,33]. But we caution the reader that in [2,11] the case $(\Theta, \Delta) = (\mathrm{BC}_n, \mathrm{D}_n)$ for $n \geq 1$ is allowed – a situation not considered here.

3.2. Examples

(a) A prime example of a root-graded Lie algebra is the Lie algebra $L = \mathfrak{g} \otimes A$ for any $A \in \Bbbk$-**alg** and $(\mathfrak{g}, \mathfrak{h})$ as in the definition above. The Lie algebra product of L is $[l_1 \otimes a_1, l_2 \otimes a_2] = [l_1, l_2] \otimes a_1 a_2$ where $l_i \in L$ and $a_i \in A$. If $\mathfrak{g} = \bigoplus_{\mu \in \Delta \cup \{0\}} \mathfrak{g}_\mu$ denotes the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, whence in particular $\mathfrak{g}_0 = \mathfrak{h}$ and $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta$, then the root spaces L_α appearing in (3.1.1) are the subspaces $L_\alpha = \mathfrak{g}_\alpha \otimes A$. We will refer to Lie algebras $\mathfrak{g} \otimes_\Bbbk A$ as *map algebras*, since they can be interpreted as regular maps from the affine scheme $\mathrm{Spec}(A)$ to the affine scheme $\mathfrak{g}$. (Sometimes they are also called *generalized current algebras*.)

(b) A second example, relevant for this paper, is the Lie algebra $L = \mathfrak{sl}_n(A)$ for $n \in \mathbb{N}_+$ and A a unital associative $\Bbbk$-algebra, see 5.1. This is a Δ-graded Lie algebra for $\Delta = \mathrm{A}_{n-1}$.

The two examples (a) and (b) above essentially describe the structure of Δ-graded Lie algebras L up to central extensions, where Δ is an irreducible reduced root system of ADE-type. More precisely, if Δ is of type D or E then L is a central extension of the example (a) and if Δ is of type A_l, $l \geq 3$, then there exists a unital associative $\Bbbk$-algebra A such that $L/\mathcal{C}(L) \cong \mathfrak{sl}_{l+1}(A)/\mathcal{C}\big(\mathfrak{sl}_{l+1}(A)\big)$ for $\mathcal{C}(\cdot)$ the center of the Lie algebra in question [4]. We note that for $\Delta = \mathrm{A}_1$ and $\Delta = \mathrm{A}_2$ more general Lie algebras than $\mathfrak{sl}_{l+1}(A)$ occur [12,4,33], see (d) for $\Delta = \mathrm{A}_1$.

(c) A finite-dimensional semisimple Lie algebra L, which is *isotropic* in the sense that L contains a non-zero split toral subalgebra, is root-graded with respect to a maximal split toral subalgebra $\mathfrak{h}$ and a suitable $\mathfrak{g}$ ([38]). We point out that in this example Θ need not be reduced.

(d) The Tits–Kantor–Koecher algebra of a unital Jordan algebra J and its central coverings are examples of root-graded Lie algebras with $\Theta = \Delta = \mathrm{A}_1$. In fact, the A_1-graded Lie algebras are precisely the central coverings of the Tits–Kantor–Koecher algebra of a unital Jordan algebra J ([12, 0.8], [33, 2.7 and 3.2(1)]).

(e) Let $\mathfrak{s}$ be a finite-dimensional simple Lie algebra over $\Bbbk$, which we assume to be algebraically closed in this example. Further, let μ be a non-trivial diagram automorphism of $\mathfrak{s}$, whence μ has order $|\mu| = 2$ or $|\mu| = 3$. Let $\zeta \in \Bbbk$ be a primitive $|\mu|$th-root of unity, put $\mathfrak{s}^{(i)} = \{x \in \mathfrak{s} : \mu(x) = \zeta^i x\}$, so that $\mathfrak{s}$ has an eigenspace decomposition

$$\mathfrak{s} = \mathfrak{s}^{(0)} \oplus \mathfrak{s}^{(1)} \quad \text{respectively} \quad \mathfrak{s} = \mathfrak{s}^{(0)} \oplus \mathfrak{s}^{(1)} \oplus \mathfrak{s}^{(2)},$$

which is a $(\mathbb{Z}/|\mu|\mathbb{Z})$-grading of $\mathfrak{s}$. In particular, $\mathfrak{s}^{(0)} =: \mathfrak{g}$ is a subalgebra of $\mathfrak{s}$ and $\mathfrak{s}^{(i)}$, $i > 0$, are $\mathfrak{g}$-modules under the adjoint action of $\mathfrak{s}^{(0)}$.

One knows (see for example [27, Prop. 7.9 and 7.10]) that $\mathfrak{g}$ is a simple subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and denote by Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. It is further known that Δ is not simply-laced and that the $\mathfrak{g}$-modules $\mathfrak{s}^{(i)}$, $i > 0$, are irreducible of highest weight equal to the highest short root of Δ or twice that root. For $\alpha \in \mathfrak{h}^*$ let $\mathfrak{s}_\alpha$ be the weight space of $\mathfrak{s}$ with respect to $\mathfrak{h}$. We then have a weight decomposition

$$\mathfrak{s} = \bigoplus\nolimits_{\alpha \in \Theta \cup \{0\}} \mathfrak{s}_\alpha,$$

for $\Theta = \Delta$ or $\Theta = \mathrm{BC}_r \supset \Delta = \mathrm{B}_r$. Thus (Θ, Δ) is an admissible pair. In fact, *the pair* $(\mathfrak{s}, \mathfrak{g})$ *is a* (Θ, Δ)*-graded Lie algebra* since the condition (ii) of 3.1 holds by simplicity of $\mathfrak{s}$. We note that every weight space $\mathfrak{s}_\alpha$ respects the eigenspace decomposition of $\mathfrak{s}$ with respect to μ: $\mathfrak{s}_\alpha = \bigoplus_{0 \le i \le |\mu|} \mathfrak{s}_\alpha^{(i)}$ with $\mathfrak{s}_\alpha^{(i)} = \mathfrak{s}^{(i)} \cap \mathfrak{s}_\alpha$.

Generalizing this example, let $A \in \Bbbk$**-alg** and assume that μ also acts on A by an algebra automorphism. Let L be the associated equivariant map algebra, i.e., the fixed point subalgebra of the Lie algebra $\mathfrak{s} \otimes_\Bbbk A$ under the diagonal action of μ ([37]). For A finitely generated, these Lie algebras are the main object of investigation in [20] (the setting in this example is assumed from section 5 on in [20]). Using obvious notation, we therefore get

$$L = \bigoplus\nolimits_{0 \le i < |\mu|} \mathfrak{s}^{(i)} \otimes A^{(-i)}.$$

We set $\mathfrak{g}$ equal to the subalgebra $\mathfrak{s}^{(0)} \otimes \Bbbk 1_A$ of L. Since $\mathfrak{h} \subset \mathfrak{g}$ acts on the factor $\mathfrak{s}$ of $\mathfrak{s} \otimes_\Bbbk A$, it follows that L has a weight space decomposition

$$L = \bigoplus\nolimits_{\alpha \in \Theta \cup \{0\}} L_\alpha, \qquad L_\alpha = \bigoplus\nolimits_{0 \le i < |\mu|} \mathfrak{s}_\alpha^{(i)} \otimes A^{(-i)}.$$

We now sketch the argument that condition (ii) of 3.1 holds, and therefore $(L, \mathfrak{g})$ *is a root-graded Lie algebra.* To this end, it is enough to verify that

$$\mathfrak{s}_0^{(i)} \otimes A^{(-i)} = \sum_{\alpha\in\Theta} \left[\mathfrak{s}_\alpha^{(0)} \otimes A^{(0)}, \mathfrak{s}_{-\alpha}^{(i)} \otimes A^{(-i)}\right] \text{ for } 0 \le i < |\mu|.$$

The latter relation will follow immediately if we can show that

$$\mathfrak{s}_0^{(i)} = \sum_{\alpha\in\Theta} \left[\mathfrak{s}_\alpha^{(0)}, \mathfrak{s}_{-\alpha}^{(i)}\right] \text{ for } 0 \le i < |\mu|. \tag{3.2.1}$$

To see why (3.2.1) is true, first note that by simplicity of $\mathfrak{s}$, and the fact that $\mathfrak{s}_0^{(0)}$ is the Cartan subalgebra of $\mathfrak{s}^{(0)}$, we have

$$\mathfrak{s}_0^{(0)} = \sum_{\alpha\in\Delta} \left[\mathfrak{s}_\alpha^{(0)}, \mathfrak{s}_{-\alpha}^{(0)}\right].$$

This immediately implies (3.2.1) for $i = 0$. Next assume that $0 < i < |\mu|$. We can also assume that $\mathfrak{s}_0^{(i)} \neq \{0\}$. Then $\mathfrak{s}^{(i)}$ is a nontrivial irreducible $\mathfrak{s}^{(0)}$-module, and therefore it is generated by a nonzero vector $v_\beta \in \mathfrak{s}_\beta^{(i)}$, for some $\beta \neq 0$. Thus every nonzero vector in $\mathfrak{s}_0^{(i)}$ is a linear combination of vectors of the form $x_{\beta_1} \cdots x_{\beta_j} v_\beta$, for some $j > 0$ and $\beta_1, \dots, \beta_j \in \Delta \cup \{0\}$. Since $\mathfrak{s}_0^{(0)}$ is the Cartan subalgebra of $\mathfrak{s}^{(0)}$, we can assume that $\beta_1 \neq 0$. Consequently, $x_{\beta_1} \cdots x_{\beta_j} v_\beta \in \left[\mathfrak{s}_{\beta_1}^{(0)}, \mathfrak{s}_{-\beta_1}^{(i)}\right]$. This completes the proof of (3.2.1).

(f) *Lie tori are examples of root-graded Lie algebras* [35,34] (these are the cores and centerless cores of extended affine Lie algebras). In particular, if $\hat{\mathfrak{g}}$ denotes an affine Kac–Moody Lie algebra then $\tilde{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ and the (twisted or untwisted) loop algebra $\tilde{\mathfrak{g}}/\mathcal{C}(\tilde{\mathfrak{g}})$ is a root-graded Lie algebra.

(g) Let L be a (Θ, Δ)-graded Lie algebra with grading subalgebra $\mathfrak{g}$ and let L' be a subalgebra of L containing $\mathfrak{g}$. Then L' has a weight space decomposition $L' = \sum_{\alpha\in\Theta\cup\{0\}} L'_\alpha$ with $L'_\alpha = L' \cap L_\alpha$. It follows that L' is root-graded as soon as condition (ii) of 3.1 is fulfilled. We give examples of this situation:

(1) The hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ of [14] is (Θ, Δ)-graded for (Θ, Δ) of type 3.1(bII). It is a subalgebra of the twisted loop algebra of type $\mathrm{A}_{2r}^{(2)}$ which is root-graded by (f). Using the notation of [14], there exists an obvious epimorphism $\pi\colon \hat{\mathfrak{h}}^* \to \mathfrak{h}^*$ with $\delta \mapsto 0$. It follows that $\pi\big(\hat{R}_{\mathrm{re}}(+) \cup \mathbb{N}\delta \cup \hat{R}_{\mathrm{re}}(-)\big) = R \cup \{0\} \cup \frac{1}{2}R_\ell \cong \mathrm{BC}_r$. The subalgebra $\mathfrak{g} \subset \mathfrak{C}\mathfrak{g} = L$ is simple of type $R = \mathrm{C}_r$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts diagonalizably on L with weights α in $\mathrm{BC}_n \cup \{0\}$, namely $L_\alpha = \bigoplus_{\pi(\beta)=\alpha} \hat{\mathfrak{g}}_\beta$.

(2) The standard current algebras considered in [14, §9] are root-graded. The details are similar to (g1). As a special case, the twisted current algebras in the sense of [17] are (Θ, Δ)-graded with (Θ, Δ) of type 3.1(bI).

The following facts about root-graded Lie algebras will be useful later.

3.3. Lemma. *Let $(L, \mathfrak{g})$ be a (Θ, Δ)-graded Lie algebra. We use the notation of* 2.1 *and* 3.1*.*

(a) *For* $\alpha \in \Delta$ *let* $(e_\alpha, h_\alpha, f_\alpha) \in \mathfrak{g}_\alpha \times \mathfrak{h} \times \mathfrak{g}_{-\alpha}$ *be an* $\mathfrak{sl}_2$*-triple. Then* $[e_\alpha, L_\beta] = L_{\alpha+\beta}$ *whenever* $\beta \in \Theta \cup \{0\}$ *satisfies* $\langle \beta, \alpha^\vee \rangle \geq -1$*. In particular,*

$$[e_\alpha, L_0] = L_\alpha \quad \textit{and} \quad [e_\alpha, L_\alpha] = L_{2\alpha}. \tag{3.3.1}$$

(b) *Let* $\mathfrak{C}$ *be the set of connected components* Θ_c *of* Θ *such that* $(\Theta_c, \Delta_c) \cong (\mathrm{BC}_n, \mathrm{C}_n)$ *for some* $n = n(c)$ *depending on* $c \in \mathfrak{C}$*. For each* $c \in \mathfrak{C}$ *let* $\beta_c \in \Pi = \{\alpha_1, \ldots, \alpha_r\}$ *be the unique root with* $\beta_c/2 \in \Theta$*. Then the* 0*-weight space* L_0 *satisfies*

$$L_0 = \textstyle\sum_{i=1}^r [L_{\alpha_i}, L_{-\alpha_i}] + \sum_{c\in\mathfrak{C}} [L_{\beta_c/2}, L_{-\beta_c/2}]. \tag{3.3.2}$$

In particular, if (Θ, Δ) *does not have a connected component of type* 3.1(bII) *then*

$$L_0 = \textstyle\sum_{i=1}^r [L_{\alpha_i}, L_{-\alpha_i}]. \tag{3.3.3}$$

(c) *If* $I \triangleleft L$ *is an ideal of* L *with* $\mathfrak{g} \subset I$*, then* $I = L$*.*

Proof. (a) The subspace $\bigoplus_{n\in\mathbb{Z}} L_{\beta+n\alpha}$ is a weight module for the adjoint action of the subalgebra $\Bbbk e_\alpha \oplus \Bbbk h_\alpha \oplus f_\alpha \cong \mathfrak{sl}_2(\Bbbk)$. The result therefore follows from standard $\mathfrak{sl}_2$-representation theory.

(b) Let $\gamma \in \Theta \cap \mathcal{Q}_+$ and assume that γ is not a simple root. It is then well-known that there exists a simple root $\alpha_i \in \Pi$ satisfying $\langle \gamma, \alpha_i^\vee \rangle > 0$ and therefore $\beta = \gamma - \alpha_i \in \Theta$.

Let $(e_i, h_i, f_i) \in \mathfrak{g}_{\alpha_i} \times \mathfrak{h} \times \mathfrak{g}_{-\alpha_i}$ be an $\mathfrak{sl}_2$-triple. Then $L_\gamma = [e_i, L_\beta]$ by (a). Hence

$$[L_\gamma, L_{-\gamma}] = [[e_i, L_\beta], L_{-\gamma}] \subset [e_i, [L_\beta, L_{-\gamma}]] + [[e_i, L_{-\gamma}], L_\beta] \subseteq [L_{\alpha_i}, L_{-\alpha_i}] + [L_\beta, L_{-\beta}]. \tag{3.3.4}$$

By induction on the height we get $[L_\gamma, L_{-\gamma}] \subset \sum_{i=1}^r [L_{\alpha_i}, L_{-\alpha_i}]$, in particular (3.3.3) follows when $\mathfrak{C} = \varnothing$. In general,

$$L_0 = \textstyle\sum_{i=1}^r [L_{\alpha_i}, L_{-\alpha_i}] + \sum_{c\in\mathfrak{C}} \sum_{\gamma\in\Theta_c\setminus\Delta_c} [L_\gamma, L_{-\gamma}].$$

Fix $c \in \mathfrak{C}$. To simplify notation, we write $\Pi \cap \Delta_c = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$ so that $\beta_c = 2\epsilon_n$. To finish the proof of (3.3.2) we need to show $[L_{\epsilon_i}, L_{-\epsilon_i}] \subset \sum_{i=1}^r [L_{\alpha_i}, L_{-\alpha_i}] + [L_{\epsilon_n}, L_{-\epsilon_n}]$. But this follows from (3.3.4) for $\gamma = \epsilon_i$, $\alpha_i = \epsilon_i - \epsilon_{i+1} \in \Pi$, $\beta = \epsilon_{i+1} = \gamma - \alpha_i$ together with a downward induction starting at n.

(c) It follows from (3.3.1) that $L_\alpha \oplus L_{2\alpha} \subset I$ for all $\alpha \in \Delta$. If there exists $\beta \in \Theta$ with $\alpha = 2\beta \in \Delta$ (case 3.1(bII)) then $[e_\alpha, L_{-\beta}] = L_\beta \subset I$ by (a), whence $L_\gamma \subset I$ for all $\gamma \in \Theta$. Now the relation $L_0 = \sum_{\alpha\in\Theta} [L_\alpha, L_{-\alpha}]$ proves $L_0 \subset I$, thus $L = I$. $\square$

3.4. The category $\mathcal{I}(L, \mathfrak{g})^\lambda$

This category has been introduced in different settings in several recent papers, e.g. [13,14,17,19,20]. We do this here for a (Θ, Δ)-graded Lie algebra $(L, \mathfrak{g})$ which covers all previous papers apart from some preliminary results in the first four sections of [20]. But note that the examples in 3.2 show that our setting applies to many more types of Lie algebras not covered in [20]. We use the notation introduced in 2.1.

Let $\Theta_+ \subset \Theta$ be a positive system, i.e., $(\Theta_+ + \Theta_+) \cap \Theta \subset \Theta_+$, $\Theta_+ \cap (-\Theta_+) = \emptyset$ and $\Theta = \Theta_+ \cup (-\Theta_+)$. We can and will assume that $\Theta_+ \cap \Delta = \Delta_+$. We denote by $\mathbb{N}[\Theta_+]$ the monoid spanned by Θ_+ in the weight lattice $\mathcal{P}$ of Δ. Then $\mathbb{N}[\Theta_+] \supset \mathcal{Q}_+$ and $\mathbb{N}[\Theta_+]/\mathcal{Q}_+$ is an abelian group of order 2^s where s is the number of irreducible components of (Θ, Δ) of type 3.1(bII). For later use we note

$$(-\Theta_+) \cap \mathbb{N}[\Theta_+] = \emptyset. \tag{3.4.1}$$

It will be convenient to put

$$\mu \preceq \lambda \quad \iff \quad \lambda - \mu \in \mathbb{N}[\Theta_+].$$

Since $\mathcal{Q}_+ \subset \mathbb{N}[\Theta_+]$ we have $\mu \le \lambda \implies \mu \preceq \lambda$. We define subalgebras

$$L_\pm := \bigoplus\nolimits_{\alpha \in \pm\Theta_+} L_\alpha$$

of L. Thus

$$L = L_- \oplus L_0 \oplus L_+ \tag{3.4.2}$$

is a decomposition of L as a direct sum of subalgebras, but we point out that this is in general not a "triangular decomposition" in the sense of [32] because for instance L_0 is not abelian in general.

For $\lambda \in \mathcal{P}_+$, let $\mathcal{I}(L, \mathfrak{g})^\lambda$ denote the full subcategory of $\mathcal{I}(L, \mathfrak{g})$ whose objects V satisfy $\mathrm{wt}(V) \subseteq \lambda - \mathbb{N}[\Theta_+]$, where $\mathrm{wt}(V)$ denotes the weights of the $\mathfrak{g}$-module V. We note some elementary properties of the category $\mathcal{I}(L, \mathfrak{g})^\lambda$.

(a) Every $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})$ has a unique decomposition $V = \bigoplus_{\mu \in \mathcal{P}_+} V_{(\mu)}$ as $\mathfrak{g}$-module where $V_{(\mu)}$ is the isotypic component of V associated to the simple $\mathfrak{g}$-module $V(\mu)$. We have $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda \Leftrightarrow V = \bigoplus_{\mu \in \mathcal{P}_+,\, \mu \preceq \lambda} V_{(\mu)}$. By standard $\mathfrak{sl}_2$-theory, if $\lambda = \sum_{i \in I} \ell_i \varpi_i$, then

$$\lambda - (\ell_i + 1)\alpha_i \notin \mathrm{wt}\, V \text{ for every } V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda \text{ and every } i \in I. \tag{3.4.3}$$

Since there are only finitely many $\mu \in \mathcal{P}_+$ with $\mu \in \lambda - \mathbb{N}[\Theta_+]$, the set of weights of any $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda$ is finite.

(b) There is a natural truncation functor $\mathbf{T}^\lambda \colon \mathcal{I}(L, \mathfrak{g}) \to \mathcal{I}(L, \mathfrak{g})^\lambda$ given on objects by

$$V \mapsto V^\lambda := V \big/ \sum\nolimits_{\mu \not\preceq \lambda} \mathrm{U}(L) \cdot V_\mu$$

and on morphisms by $\varphi \mapsto \varphi^\lambda$, where for an L-module homomorphism $\varphi \colon V \to N$ the map φ^λ is the induced quotient map keeping in mind that $\varphi(\sum_{\mu \not\preceq \lambda} \mathrm{U}(L) \cdot V_\mu) \subset \sum_{\mu \not\preceq \lambda} \mathrm{U}(L) \cdot N_\mu$. An object $V \in \mathcal{I}(L, \mathfrak{g})$ lies in $\mathcal{I}(L, \mathfrak{g})^\lambda$ if and only if $V = V^\lambda$.

The functor $\mathbf{T}^\lambda$ is exact and preserves projective objects. In particular, in view of Proposition 2.4 we have (cf. [13, 3.2, Cor. 1])

$$P(V,\rho)^\lambda \textit{ is projective for any } (V,\rho) \in \mathcal{I}(L,\mathfrak{g}). \tag{3.4.4}$$

(c) If $\dim V_\lambda = 1$ and $V = \mathrm{U}(L)V_\lambda$, then V has a unique maximal L-submodule and hence a unique irreducible quotient.

3.5. Remark. For $V \in \mathcal{I}(L,\mathfrak{g})^\lambda$ denote by $\mathrm{wt}(V)$ the set of weights of V. Then $\mathrm{wt}(V) \subset \mathrm{wt}(V(\lambda))$ whenever one of the following holds,

(i) $\mathbb{N}[\Theta_+] = \mathcal{Q}_+$, i.e., only the cases 3.1(a) and 3.1(bI) occur,
(ii) $(\Theta,\Delta) \cong (\mathrm{BC}_r, \mathrm{C}_r)$, $\lambda = \sum_{i\in I} \ell_i \varpi_i$ with $\ell_r > 0$, where ϖ_r is the fundamental weight corresponding to the unique long root in Π.

3.6. Examples

(a) ($\lambda = 0$) By 3.4(a), an object V in $\mathcal{I}(L,\mathfrak{g})^0$ has the property that $\mathfrak{g}\cdot V = 0$. It then follows from Lemma 3.3(c) that $\mathrm{Ann}_L(V) = L$, i.e., $L \cdot V = 0$.

(b) ([41, I, §2, Prop. 1.1] for Example 3.2(c)) *For every finite-dimensional irreducible L-module V there exists a unique $\lambda \in \mathcal{P}_+$ such that $V \in \mathrm{Ob}\,\mathcal{I}(L,\mathfrak{g})^\lambda$.* Indeed, such a V is an integrable $\mathfrak{g}$-module by 2.2 with finitely many weights. Since $L_\alpha \cdot V_\mu \subset V_{\alpha+\mu}$ there exists a weight $\lambda \in \mathcal{P}_+$ of V such that $L_\alpha \cdot V_\lambda = 0$ for all $\alpha \in \Theta_+$. One checks that $V' = \mathrm{Span}\{l_1 \cdots l_k \cdot v : l_i \in L_{-\beta_i}, \beta_i \in \Theta_+, v \in V_\lambda\}$ is an L-submodule, whence $V' = V$. By construction the weights μ of V' lie in $\lambda - \mathbb{N}[\Theta_+]$. This also shows uniqueness of λ.

(c) If Θ is irreducible and θ the highest root with respect to Θ_+, we have $L \in \mathcal{I}(L,\mathfrak{g})^\theta$.

3.7. Global Weyl modules

For $\lambda \in \mathcal{P}_+$ the *global Weyl module (of highest weight λ)* is defined as

$$W(\lambda) := P(V(\lambda))^\lambda = \big(\mathrm{U}(L) \otimes_{\mathrm{U}(\mathfrak{g})} V(\lambda)\big)^\lambda \in \mathrm{Ob}\,\mathcal{I}(L,\mathfrak{g})^\lambda.$$

By Lemma 2.7 we see that $W(\lambda)$ is a cyclic $\mathrm{U}(L)$-module, generated by the image of $1 \otimes v_\lambda \in \mathrm{U}(L) \otimes_{U(\mathfrak{g})} V(\lambda)$ in $W(\lambda)$, which we will denote by w_λ, and from (3.4.4) we know that $W(\lambda)$ is projective in $\mathcal{I}(L,\mathfrak{g})^\lambda$.

The terminology "global Weyl module" first appeared in this form in [13] in the context $L = \mathfrak{g} \otimes A$ for $A \in \Bbbk$**-alg**, but these modules were first defined in [16]. The name is justified in view of their universal property 3.8(b) and the analogous universal property of Weyl modules in the theory of algebraic groups [25, II, 2.13(b)]. For more background information on the significance of global Weyl modules and the choice of nomenclature we refer the reader to the papers by Chari and her collaborators that are listed in our bibliography.

3.8. Proposition *(Characterization of global Weyl modules, [13, Prop. 4]).* (a) *The global Weyl module $W(\lambda)$ is generated by w_λ with defining relations*

$$L_+ \cdot w_\lambda = 0, \qquad (h - \lambda(h)1) \cdot w_\lambda = 0 \quad (h \in \mathfrak{h}), \qquad f_i^{\lambda(h_i)+1} \cdot w_\lambda = 0, \quad i \in I. \tag{3.8.1}$$

In other words, $\mathrm{Ann}_{\mathrm{U}(L)}(w_\lambda)$ *is the left ideal of* $\mathrm{U}(L)$ *generated by the set*

$$L_+ \cup \left\{ f_i^{\lambda(h_i)+1} \ : \ i \in I \right\} \cup \{ h - \lambda(h) \ : \ h \in \mathfrak{h} \} . \tag{3.8.2}$$

(b) *For every cyclic module $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda$ with generator $v_\lambda \in V_\lambda$ there exists a unique surjective L-module homomorphism $W(\lambda) \to V$ sending w_λ to v_λ.*

Proof. The proof is essentially the same as in [13, Prop. 4]. We give a sketch for the convenience of the reader.

(I) Let $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda$ be a cyclic module as in (b). Then $L_+ \cdot V_\lambda \subset \sum_{\mu \not\leq \lambda} V_\mu$ by (3.4.1), which forces $L_+ \cdot v_\lambda = 0$. Obviously, $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$. Finally, (3.4.3) implies $f_i^{\lambda(h_i)+1} \cdot v_\lambda = 0$.

(II) Proof of (a): We have already noted that $W(\lambda)$ is a cyclic module. It thus follows from (I) that $(W(\lambda), w_\lambda)$ satisfies the relations (3.8.1). Let W be the cyclic L-module generated by a vector w with the given relations. Since the relations hold in $W(\lambda)$, we get that $W(\lambda)$ is a quotient of W. Since $\mathfrak{n}_+ \subseteq L_+$, it follows from the presentation given in Lemma 2.7 that W is a quotient of $P(V(\lambda))$. The relations imply that W is a weight module, and from $L_+ \cdot w = 0$, we see that the weights of W lie in $\lambda - \mathbb{N}[\Theta_+]$. Hence, W is a quotient of $W(\lambda) = P(V(\lambda)) / \sum_{\mu \notin \lambda - \mathbb{N}[\Theta_+]} \mathrm{U}(L) \cdot P(V(\lambda))_\mu$, so that $W \cong W(\lambda)$.

(III) Proof of (b): This follows from (I) and (a). □

In the remainder of this section we relate the representation theory of L with that of a certain associative algebra. We start with a generalization of [13, §3.4] using essentially the same proof.

3.9. Lemma. *The formula*

$$(u \cdot w_\lambda) \cdot a := ua \cdot w_\lambda, \quad u \in \mathrm{U}(L), \quad a \in \mathrm{U}(L_0),$$

endows $W(\lambda)$ with the structure of a $(\mathrm{U}(L), \mathrm{U}(L_0))$-bimodule.

Proof. Let $J = \mathrm{Ann}_{\mathrm{U}(L)}(w_\lambda)$, the left ideal of $\mathrm{U}(L)$ described in (3.8.2). Then the given action is well-defined if and only if $JL_0 \subseteq J$. To see that this holds, it suffices to show that $ga \in J$ for g one of the generators of J described in Proposition 3.8 and $a \in L_0$.

(i) To see that this is true for $g = \ell_+ \in L_+$, note that $\ell_+ a = [\ell_+, a] + a\ell_+ \in L_+ + L_0 L_+ \subseteq J$, since $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, and thus $[L_0, L_+] \subseteq L_+$.

(ii) $(h - \lambda(h)1)a = a(h - \lambda(h)1) \in J$ since $[L_0, \mathfrak{h}] = 0$.

(iii) In $W(\lambda)$ we have $\mathfrak{n}_+ a \cdot w_\lambda = [\mathfrak{n}_+, a] \cdot w_\lambda + a\mathfrak{n}_+ \cdot w_\lambda = 0$, because $\mathfrak{n}_+$ and $[\mathfrak{n}_+, a]$ lie in L_+. Since $(h - \lambda(h)1)a \cdot w_\lambda = 0$ for $h \in \mathfrak{h}$, either $a \cdot w_\lambda = 0$, or $a \cdot w_\lambda$ is a primitive vector for the $\mathfrak{g}$-action on $W(\lambda)$. In the latter case, since $W(\lambda)$ is a completely reducible $\mathfrak{g}$-module, we have $U(\mathfrak{g})(a \cdot w_\lambda) \cong V(\lambda)$. It follows that for $1 \le i \le n$ we have $f_i^{\lambda(\alpha_i^\vee)+1} a \cdot w_\lambda = 0$, so that $f_i^{\lambda(\alpha_i^\vee)+1} a \in J$.

This shows that the given action is well-defined. Since it clearly commutes with the left action of $\mathrm{U}(L)$, the proof is complete. □

3.10. The Chari–Pressley algebra $\mathbf{A}_\lambda$

We have

$$\begin{aligned}\mathrm{Ann}_{\mathrm{U}(L_0)}(w_\lambda) &= \{x \in \mathrm{U}(L_0) \ : \ w_\lambda \cdot x = 0\} \\ &= \{x \in \mathrm{U}(L_0) \ : \ W(\lambda) \cdot x = 0\} = \mathrm{Ann}_{\mathrm{U}(L_0)}(W(\lambda)),\end{aligned}$$

a two-sided ideal of $\mathrm{U}(L_0)$. Hence

$$\mathbf{A}_\lambda := \mathrm{U}(L_0) / \mathrm{Ann}_{\mathrm{U}(L_0)}(w_\lambda)$$

is an associative $\Bbbk$-algebra, called the Chari–Pressley algebra for now, but see Theorem 4.10.

Let $\mathbf{A}_\lambda$-mod denote the category of left $\mathbf{A}_\lambda$-modules.

3.11. Lemma. *([13, Lem. 4]) For $\lambda \in \mathcal{P}_+$ and $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda$ we have $\mathrm{Ann}_{\mathrm{U}(L_0)}(w_\lambda) \cdot V_\lambda = 0$. In particular, V_λ is a left $\mathbf{A}_\lambda$-module.*

Proof. Let $v \in V_\lambda$. Then v satisfies the relations of Proposition 3.8, so there is a homomorphism of L-modules $\pi : W(\lambda) \to \mathrm{U}(L) \cdot v$ which maps $w_\lambda \mapsto v$. In particular, if $u \in \mathrm{Ann}_{\mathrm{U}(L_0)}(w_\lambda)$ then $0 = \pi(u \cdot w_\lambda) = u \cdot v$. □

4. Admissible modules, Seligman algebras

In this section we continue with the assumptions of Section 3: L is a (Θ, Δ)-graded Lie algebra with grading subalgebra $\mathfrak{g}$ and splitting Cartan subalgebra $\mathfrak{h}$.

In this section we study in more detail the highest weight space V_λ of modules $V \in \mathrm{Ob}\,\mathcal{I}(L, \mathfrak{g})^\lambda$. Our main goal is to relate the module categories $\mathcal{I}(L, \mathfrak{g})^\lambda$ and $\mathbf{A}_\lambda$-mod to the work of Seligman ([40,41]) on rational modules of simple Lie algebras.

4.1. The Harish-Chandra homomorphism

We know from 2.2(c) that $\mathrm{U}(L)$ is an integrable $\mathfrak{g}$-module with respect to the action ρ_{U}, whence in particular a weight module: $\mathrm{U}(L) = \bigoplus_{\mu \in \mathcal{Q}} \mathrm{U}(L)_\mu$ with weight spaces $\mathrm{U}(L)_\mu$.

Clearly $\mathrm{U}(L)_0$ is a subalgebra of $\mathrm{U}(L)$ containing the center of $\mathrm{U}(L)$. By a standard consequence of the PBW Theorem we have

(a) $K = \big(L_- \,\mathrm{U}(L)\big) \cap \mathrm{U}(L)_0 = \big(\mathrm{U}(L)L_+\big) \cap \mathrm{U}(L)_0$ is a two-sided ideal of $\mathrm{U}(L)_0$.

(b) $\mathrm{U}(L)_0 = \mathrm{U}(L_0) \oplus K$ and the associated projection $\pi_0 : \mathrm{U}(L)_0 \to \mathrm{U}(L_0)$ with $\mathrm{Ker}(\pi_0) = K$ is a homomorphism of algebras. Following the standard terminology for the case $L = \mathfrak{g}$, we call π_0 the *Harish-Chandra homomorphism.*

4.2. Definition *(Admissible modules, Seligman algebra).* Fix $\lambda = \sum_{i\in I} \ell_i\omega_i \in \mathcal{P}_+$.

(a) We say an L_0-module $M \neq 0$ has *weight* λ if $h \cdot m = \lambda(h)m$ for all $h \in \mathfrak{h}$ and $m \in M$.

(b) An L_0-module M of weight λ is called λ-*admissible* if for all $i \in I$ and for any pair of sequences

$$\begin{aligned} &(x_1, \dots, x_{m_i}), \quad x_j \in L_{\beta_j}, \beta_j \in \Theta_+, \\ &(y_1, \dots, y_{\ell_i+1}), \quad y_j \in L_{-\alpha_i}, \quad \textstyle\sum_{j=1}^{m_i} \beta_j = (\ell_i + 1)\alpha_i, \end{aligned} \tag{4.2.1}$$

we have

$$\pi_0(x_1x_2 \cdots x_{m_i}y_1y_2 \cdots y_{\ell_i+1}) \cdot M = 0.$$

An L_0-module will be called *admissible* if it is λ-admissible for some $\lambda \in \mathcal{P}_+$. If $\Theta = \Delta$ is reduced the condition $\sum_{j=1}^{m_i} \beta_j = (\ell_i + 1)\alpha_i$ is equivalent to $m_i = \ell_i + 1$ and all $\beta_j = \alpha_i$. If Θ is not reduced and $s\alpha_i \in \Theta$ with $s = 2$ or $s = \frac{1}{2}$, then $\beta_j \in \{\alpha_i, s\alpha_i\}$ follows, but not necessarily $\beta_j = \alpha_i$.

(c) Let J^λ denote the two-sided ideal of the associative algebra $\mathrm{U}(L_0)$ generated by $\bigcup_{i\in I} G_i$ where G_i, $i \in I$, is the set composed of

(i) $\pi_0(x_1x_2 \cdots x_{m_i}y_1y_2 \cdots y_{\ell_i+1})$ for any pair of sequences $(x_1, \dots x_{m_i})$ and $(y_1, \dots, y_{\ell_i+1})$ as in (4.2.1)

and

(ii) $h_i - \ell_i\mathbb{1}_{\mathrm{U}}$ for $\mathbb{1}_{\mathrm{U}}$ the identity in $\mathrm{U}(L_0)$.

(d) We call

$$\mathbb{S}^\lambda(L, \mathfrak{g}) = \mathrm{U}(L_0)/J^\lambda,$$

often abbreviated by $\mathbb{S}^\lambda$, the *Seligman algebra* since it seems to have appeared for the first time in [40], but see Theorem 4.10. By construction, it is a unital associative $\mathbb{k}$-algebra. Denoting by $u \mapsto \mathrm{can}(u)$ the canonical map $\mathrm{U}(L_0) \to \mathbb{S}^\lambda$, a λ-admissible module M becomes an $\mathbb{S}^\lambda$-module under the action $\mathrm{can}(u) \cdot m = u \cdot m$. In this way we obtain an *isomorphism between the category of λ-admissible L_0-modules whose morphisms are L_0-module maps and the category $\mathbb{S}^\lambda$-$\mathcal{M}od$ of left $\mathbb{S}^\lambda$-modules.* In the future we will take this isomorphism as an identification.

The motivation for studying admissible L_0-modules comes from the close connection between the categories $\mathcal{I}(L,\mathfrak{g})^\lambda$ and $\mathbb{S}(L,\mathfrak{g})^\lambda$-$\mathcal{M}od$ which we investigate now. We note that any weight space of a weight module of L is invariant under the action of L_0.

4.3. Proposition *(Restriction). Let $\lambda \in \mathcal{P}_+$ and let $V \in \mathrm{Ob}\,\mathcal{I}(L,\mathfrak{g})^\lambda$.*

(a) *The weight space V_λ is a λ-admissible L_0-module.*
(b) *If V is irreducible as an L-module, then V_λ is irreducible as an L_0-module.*
(c) *If V is a cyclic L-module with generator $v_\lambda \in V_\lambda$, then V_λ is a cyclic L_0-module.*

Proof. (a) Assume $\lambda = \sum_{i\in I} \ell_i \varpi_i$. It follows from (3.4.3) that for any $i \in I$ and any sequence $(y_1, \dots, y_{\ell_i+1})$ with $y_j \in L_{-\alpha_i}$ we have $y_1 \cdot y_2 \cdots y_{\ell_i+1} \cdot V_\lambda \in V_{\lambda-(\ell_i+1)\alpha_i} = 0$. Now let $(x_1, \dots, x_{m_i})$ be a sequence as in (4.2.1) and consider the monomial $\mathbf{m} := x_1 x_2 \cdots x_{m_i} y_1 y_2 \cdots y_{\ell_i+1} \in \mathrm{U}(L)_0$. By what we just observed, $\mathbf{m} \cdot V_\lambda = 0$. On the other hand, converting $\mathbf{m}$ to the PBW order given by the decomposition $L = L_- \oplus L_0 \oplus L_+$ of (3.4.2) yields the equation $\mathbf{m} = \pi_0(\mathbf{m}) + \mathbf{m}'$, $\mathbf{m}' \in \mathrm{U}(L)L_+$. Since $L_+ \cdot V_\lambda = 0$ by hypothesis, this shows that $\pi_0(\mathbf{m}) \cdot V_\lambda = 0$, which proves (a).

(b) To see that V_λ is irreducible as an L_0-module, suppose $U \subseteq V_\lambda$ is a non-zero submodule. By the irreducibility of V and the PBW Theorem, we have $V = \mathrm{U}(L) \cdot U = \mathrm{U}(L_-) \cdot U$. It follows that $V_\lambda = U$, which proves the second statement.

(c) Applying again the PBW Theorem we have $V = \mathrm{U}(L)v_\lambda = \mathrm{U}(L_-)_+ \mathrm{U}(L_0)v_\lambda$ with $\mathrm{U}(L_0)v_\lambda \subset V_\lambda$ and $\mathrm{U}(L_-)_+ V_\lambda \subset \bigoplus_{\mu \preceq \lambda, \mu\neq\lambda} V_\mu$ for $\mathrm{U}(L_-)_+$ the augmentation ideal of $\mathrm{U}(L_-)$. It follows that $\mathrm{U}(L_0)v_\lambda = V_\lambda$ (and $\mathrm{U}(L_-)_+ V_\lambda = \bigoplus_{\mu \preceq \lambda, \mu\neq\lambda} V_\mu$). $\square$

The next result concerns the opposite direction: Associating to an admissible L_0-module an integrable L-module.

4.4. Proposition *(Integrable induction). Fix $\lambda = \sum_{i\in I} \ell_i \varpi_i \in \mathcal{P}_+$.*

(a) *Let M be a λ-admissible L_0-module. We give M the structure of an $(L_0 \oplus L_+)$-module by requiring $L_+ \cdot M = 0$ and define the induced L-module*

$$\widetilde{M} := \mathrm{U}(L) \otimes_{\mathrm{U}(L_0 \oplus L_+)} M \qquad \text{and its submodule}$$
$$\widetilde{N} := \textstyle\sum_{i\in I} \sum_{m\in M} \mathrm{U}(L) \cdot (f_i^{\ell_i+1} \otimes m).$$

Then the L-module $\mathbf{Int}(M) := \widetilde{M}/\widetilde{N} \in \mathrm{Ob}\,\mathcal{I}(L,\mathfrak{g})^\lambda$. It has the following properties.

(i) *The map $\xi_M \colon M \to \mathbf{Int}(M)_\lambda$, $m \mapsto 1 \otimes_{\mathrm{U}(L_0\oplus L_+)} m + \widetilde{N}$, is an isomorphism of L_0-modules.*

(ii) *If M is a cyclic L_0-module, then $\mathbf{Int}(M)$ is a cyclic L-module.*

(b) *For an L_0-module map $f\colon M_1 \to M_2$ of λ-admissible L_0-modules the map $\widetilde{f} = \mathrm{Id}_{\mathrm{U}(L)} \otimes f \colon \widetilde{M_1} \to \widetilde{M_2}$ satisfies $\widetilde{f}(\widetilde{N_1}) \subset \widetilde{N_2}$ and hence induces an L-module map $\mathbf{Int}(f)\colon \mathbf{Int}(M_1) \to \mathbf{Int}(M_2)$. It has the following properties.*

(i) *The diagram below commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\xi_M"'] & N \arrow[d, "\xi_N"] \\ \mathbf{Int}(M)_\lambda \arrow[r, "\mathbf{Int}(f)_\lambda"'] & \mathbf{Int}(N)_\lambda \end{tikzcd} \tag{4.4.1}$$

(ii) *If f is surjective, then so is* $\mathbf{Int}(f)$.

Proof. (a) Since $\mathrm{U}(L)$ is a weight module by 2.2(c) and 2.6, it follows that $\widetilde{M}$ is a weight module too. By the PBW Theorem, $\widetilde{M} = \mathrm{U}(L_-) \otimes_{\mathbb{C}} M$ as vector spaces, whence all weights of $\widetilde{M}$ are contained in $\lambda - \mathbb{N}[\Theta_+]$. The same then holds for the quotient $\mathbf{Int}(M)$. To see that this L-module is integrable, observe that by construction all e_i and f_i act locally nilpotently on the generating set $(1 \otimes M) + \widetilde{N}$ of the L-module $\mathbf{Int}(M)$. Since $\mathrm{ad}_L\, e_i$ and $\mathrm{ad}_L\, f_i$ are also locally nilpotent (in fact nilpotent), [27, Lem. 3.4(b)] says that e_i and f_i act locally nilpotently on $\mathbf{Int}(M)$, whence $\mathbf{Int}(M) \in \mathcal{I}(L, \mathfrak{g})^\lambda$ by 2.2.

For the proof of (ai), we first consider a fixed $i \in I$ and $m \in M$, and set $N = N(i,m) = \mathrm{U}(L) \cdot (f^{\ell_i+1} \otimes m)$. As a submodule of a weight module (or since $\mathrm{U}(L)$ is a weight module), N is a weight module with weights contained in $\lambda - \mathbb{N}[\Theta_+]$ since this is so for $\widetilde{M}$. The crucial point now is to show that the weights of N are strictly below λ. Suppose, to the contrary, that $N_\lambda \neq 0$. That is, there is some

$$u = \sum_{j=1}^{p} u_j^- u_j^0 u_j^+ \in \mathrm{U}(L), \qquad u_j^\pm \in \mathrm{U}(L_\pm), \quad u_j^0 \in \mathrm{U}(L_0),$$

with $0 \neq u f_i^{\ell_i+1} \otimes m \in N(i,m)_\lambda$, where we may assume without loss of generality that $0 \neq u_j^+ f_i^{\ell_i+1} \otimes m \in N(i,m)$ for each $1 \leq j \leq p$ and that the elements $u_j^\pm$ are "monomials" in $\mathrm{U}(L)$ in the following sense

$$u_j^+ = \prod_{s=1}^{r_j} b_{\eta_{s,j}}, \quad b_{\eta_{s,j}} \in L_{\eta_{s,j}}, \quad \eta_{s,j} \in \Theta_+, \qquad \text{and}$$

$$u_j^- = \prod_{t=1}^{q_j} c_{\nu_{t,j}}, \quad c_{\nu_{t,j}} \in L_{\nu_{t,j}}, \quad \nu_{t,j} \in -\Theta_+.$$

Since $u f_i^{\ell_i+1} \otimes m \in N(i,m)_\lambda$ and $m \in \widetilde{M}_\lambda$, we have

$$\sum_{t=1}^{q_j} \nu_{t,j} + \sum_{s=1}^{r_j} \eta_{s,j} = (\ell_i + 1)\alpha_i, \qquad 1 \leq j \leq p.$$

If $\nu_{t,j} \neq 0$ for some pair (t,j), then $\sum_{s=1}^{r_j} \eta_{s,j} = (\ell_i+1)\alpha_i + \mu_j$ for some $\mu_j \in \mathbb{N}[\Theta_+] \backslash \{0\}$, which implies that $u_j^+ f_i^{\ell_i+1} \otimes m = 0$. Thus, we have in fact

$$u = \sum_{j=1}^{p} u_j^0 u_j^+ \quad \text{and} \quad \sum_{s=1}^{r_j} \eta_{s,j} = (\ell_i + 1)\alpha_i, \qquad 1 \le j \le p.$$

It then follows that each $\eta_{s,j}$ is a positive multiple of α_i. Thus

$$\begin{aligned} uf_i^{\ell_i+1} \otimes m &= \sum_{j=1}^{p} u_j^0 u_j^+ f_i^{\ell_i+1} \otimes m = \sum_{j=1}^{p} 1 \otimes u_j^0 \pi_0(u_j^+ f_i^{\ell_i+1}) \cdot m \\ &= \sum_{j=1}^{p} 1 \otimes u_j^0 \pi_0(b_{\eta_{1,j}} b_{\eta_{2,j}} \cdots b_{\eta_{r,j}} f_i^{\ell_i+1}) \cdot m. \end{aligned}$$

By the λ-admissibility of M, each term on the right is zero, letting us conclude that $N_\lambda = 0$.

Since $\widetilde{N} = \sum_{i\in I} \sum_{m\in M} N(i,m)$ is a sum of weight modules, it now follows that the weights of $\widetilde{N}$ are all strictly below λ. Hence the composition of L_0-module maps

$$M \ni m \mapsto \widetilde{m} = 1 \otimes_{\mathrm{U}(L_0\oplus L_+)} m \mapsto \widetilde{m} + \widetilde{N} \in (\widetilde{M}/\widetilde{N})_\lambda$$

is bijective. This proves (ai). For (aii) suppose $M = \mathrm{U}(L_0)m_0$. Then $\widetilde{M}$ is cyclic as L-module: $M = \mathrm{U}(L) \otimes_{\mathrm{U}(L_0\oplus L_+)} \mathrm{U}(L_0)m_0 = \mathrm{U}(L) \otimes \Bbbk m_0$. This implies (aii). (b) is straightforward from the definitions, observing that f is an $L_0 \oplus L_+$-module homomorphism. □

4.5. The left-regular representation

Let $\lambda \in \mathcal{P}_+$. Then the map $L_0 \to \mathrm{End}_\Bbbk(\mathrm{U}(L_0))$ given by $u \mapsto \mathrm{L}_u$, where $\mathrm{L}_u(x) = ux$ for every $x \in \mathrm{U}(L_0)$, is a representation of the Lie algebra L_0 on $\mathrm{U}(L_0)$. The ideal J^λ is L_0-invariant, and it is clear from the definition that the quotient L_0-module $\mathbb{S}^\lambda = \mathrm{U}(L_0)/J^\lambda$ is λ-admissible, or equivalently, an $\mathbb{S}^\lambda$-module. We will call the latter $\mathbb{S}^\lambda$-module the *left-regular* $\mathbb{S}^\lambda$*-module*, and denote it by ${}_{\mathrm{reg}}\mathbb{S}^\lambda$.

The next lemma shows that the ideal J^λ defined in 4.2(c) contains elements that are obtained by a relaxation of the definition of the generators.

4.6. Lemma. *Let $\lambda = \sum_{i\in I} \ell_i \varpi_i \in \mathcal{P}_+$, and fix $i \in I$. Let $(x_1, \dots, x_{m_i})$ and $(y_1, \dots, y_{n_i})$ be a pair of sequences satisfying*

$$x_j \in L_{\beta_j},\ \beta_j \in \Theta_+,\ 1 \le j \le m_i, \quad y_j \in L_{-\gamma_j},\ \gamma_j \in \Theta_+,\ 1 \le j \le n_i, \quad \textit{and}$$
$$\sum_{j=1}^{m_i} \beta_j = \sum_{j=1}^{n_i} \gamma_j = r_i\alpha_i,$$

where $r_i > \ell_i$. Then $\pi_0(x_1 \cdots x_{m_i} y_1 \cdots y_{n_i}) \in J^\lambda$.

Proof. By the universal property of $\mathrm{U}(L_0)$, the L_0-action on ${}_{\mathrm{reg}}\mathbb{S}^\lambda$ extends to a representation of the associative algebra $\mathrm{U}(L_0)$ on ${}_{\mathrm{reg}}\mathbb{S}^\lambda$. The annihilator of this representation is J^λ. Therefore it is enough to prove that $\pi_0(x_1 \cdots x_{m_i} y_1 \cdots y_{n_i})({}_{\mathrm{reg}}\mathbb{S}^\lambda) = 0$. By Proposition 4.4(ai), it is enough to verify that $x_1 \cdots x_{m_i} y_1 \cdots y_{n_i}$ acts on $\mathbf{Int}(M)$ trivially. But indeed already $y_1 \cdots y_{n_i}$ acts on $\mathbf{Int}(M)$ trivially. The latter statement follows from an $\mathfrak{sl}_2$-theory argument similar to the one given for (3.4.3). □

4.7. Functors **Res** *and* **Int**

Let $\lambda \in \mathcal{P}_+$. We abbreviate $\mathcal{I}(L, \mathfrak{g})^\lambda = \mathcal{I}^\lambda$ and $\mathbb{S}(L, \mathfrak{g})^\lambda = \mathbb{S}^\lambda$.

For any morphism $\varphi\colon V \to V'$ in $\mathcal{I}^\lambda$ the restriction $\varphi_\lambda\colon V_\lambda \to V'_\lambda$ is an L_0-module homomorphism, whence, by Proposition 4.3, a homomorphism of $\mathbb{S}^\lambda$-modules. The assignments $V \mapsto V_\lambda$ and $\varphi \mapsto \varphi_\lambda$ define a restriction functor

$$\mathbf{Res}^\lambda_{(L,\mathfrak{g})}\colon \mathcal{I}^\lambda \to \mathbb{S}^\lambda\text{-}\mathcal{M}od,$$

which is exact. Using Proposition 4.4, we have a functor

$$\mathbf{Int}^\lambda_{(L,\mathfrak{g})}\colon \mathbb{S}^\lambda\text{-}\mathcal{M}od \to \mathcal{I}^\lambda$$

in the other direction, defined by $M \mapsto \mathbf{Int}(M)$ and $f \mapsto \mathbf{Int}(f)$. Unless there is a danger of confusion we will abbreviate

$$\mathbf{Res} = \mathbf{Res}^\lambda_{(L,\mathfrak{g})} \quad \text{and} \quad \mathbf{Int} = \mathbf{Int}^\lambda_{(L,\mathfrak{g})}.$$

A re-interpretation of Proposition 4.4(bi) is that the family of isomorphisms (ξ_M) is an isomorphism of functors

$$\mathrm{Id}_{\mathbb{S}^\lambda\text{-}\mathcal{M}od} \xrightarrow{\cong} \mathbf{Res} \circ \mathbf{Int}. \tag{4.7.1}$$

We will deal with the composition of **Res** and **Int** in the other order in the following result. It is in spirit closely related to [13, Prop. 5 and Cor. 2] dealing with the special case of map algebras, although the functor **W** used in [13] is not the same as our functor **Int**.

4.8. Proposition *(The functor* **Int**$\circ$**Res***). We fix $\lambda \in \mathcal{P}_+$ and use the abbreviations of 4.7.*

(a) *For $V \in \mathcal{I}^\lambda$ the structure map $\mathrm{U}(L) \otimes_{\Bbbk} V \to V$ induces an L-module map $\eta_V\colon \mathbf{Int}(V_\lambda) \to V$ with image $\mathrm{U}(L) \cdot V_\lambda$.*
(b) *The collection of maps (η_V) is a natural transformation* $\mathbf{Int} \circ \mathbf{Res} \Rightarrow \mathrm{Id}_{\mathcal{I}^\lambda}$.
(c) **Res** *is a right adjoint of* **Int**.
(d) *The functor* **Int** *maps projective objects to projective objects.*

Proof. (a) The structure map $\mathrm{U}(L) \times V_\lambda \to V$, $(u, v_\lambda) \mapsto u \cdot v_\lambda$ is $\mathrm{U}(L_0 \oplus L_+)$-balanced, whence gives rise to an $\mathrm{U}(L)$-linear map $\mathrm{U}(L) \otimes_{\mathrm{U}(L_0 \oplus L_+)} V_\lambda = \widetilde{V_\lambda} \to V$ with $\mathrm{U}(L)$ acting on the left factor of $\widetilde{V_\lambda}$. Because of (3.4.3), this map annihilates the submodule $\widetilde{N}$ of Proposition 4.4(a). The quotient map is the map η_V of our claim.

Regarding (b) we need to show that for any morphism $f \colon V \to V'$ in $\mathcal{I}^\lambda$ the diagram

$$\begin{tikzcd} \mathbf{Int}(V_\lambda) \arrow[r, "\eta_V"] \arrow[d, "\mathbf{Int}(f_\lambda)"'] & V \arrow[d, "f"] \\ \mathbf{Int}(V'_\lambda) \arrow[r, "\eta_{V'}"'] & V' \end{tikzcd}$$

commutes, which is immediate from the definitions.

(c) We need to prove that for all $M \in \mathbb{S}^\lambda\text{-}\mathcal{M}od$ and $V \in \mathcal{I}^\lambda$ there exists an isomorphism of abelian groups

$$\tau_{M,V} \colon \operatorname{Hom}_{\mathcal{I}^\lambda}\big(\mathbf{Int}(M), V\big) \to \operatorname{Hom}_{\mathbb{S}^\lambda}\big(M, \mathbf{Res}(V)\big)$$

which is natural in M and V. There is no other reasonable choice but to define $\tau_{M,V}(\varphi) = \varphi_\lambda \circ \xi_M = \mathbf{Res}(\varphi) \circ \xi_M$ for $\varphi \in \operatorname{Hom}_{\mathcal{I}^\lambda}\big(\mathbf{Int}(M), V\big)$. This map is clearly additive in φ. It is injective since $\mathbf{Int}(M)$ is generated by $\xi_M(M) \cong M$ as L-module. It is surjective since $\tau_{M,V}\big(\eta_V \circ \mathbf{Int}(f)\big) = f$ for any $f \in \operatorname{Hom}_{\mathbb{S}^\lambda}\big(M, \mathbf{Res}(V)\big)$. Indeed, the commutative diagram (4.4.1) specializes to

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\xi_M"'] & V_\lambda \arrow[d, "\xi_{V_\lambda}"] \\ \mathbf{Int}(M)_\lambda \arrow[r, "(\mathbf{Res} \circ \mathbf{Int})(f)"'] & \big(\mathbf{Int}(V_\lambda)\big)_\lambda \end{tikzcd}$$

so that $\tau\big(\eta_V \circ \mathbf{Int}(f)\big) = \mathbf{Res}\big(\eta_V \circ \mathbf{Int}(f)\big) \circ \xi_M = \mathbf{Res}(\eta_V) \circ (\mathbf{Res} \circ \mathbf{Int})(f) \circ \xi_M = \mathbf{Res}(\eta_V) \circ \xi_{V_\lambda} \circ f$. Our claim then follows from $\mathbf{Res}(\eta_V) \circ \xi_{V_\lambda} = \mathrm{Id}_{V_\lambda}$ which easily follows from the definitions. We leave it to the reader to check that $\tau_{M,V}$ is natural in M and V.

(d) is a standard property of adjoint functors. □

4.9. Corollary *(The Weyl module $W(\lambda)$). The map $\eta_{W(\lambda)} \colon \mathbf{Int}(W(\lambda)_\lambda) \to W(\lambda)$ of Proposition 4.8(a) is an isomorphism of L-modules. Moreover, let ${}_{\mathrm{reg}}\mathbb{S}^\lambda$ be the left-regular $\mathbb{S}^\lambda$-module and $f \colon {}_{\mathrm{reg}}\mathbb{S}^\lambda \to W(\lambda)_\lambda$ be the $\mathbb{S}^\lambda$-module map sending $1_{\mathbb{S}^\lambda}$ to the generator w_λ of the cyclic L_0-module $W(\lambda)_\lambda$. Then the map $\mathbf{Int}(f) \colon \mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda) \to \mathbf{Int}(W(\lambda)_\lambda)$ is an isomorphsim of L-modules. In sum,*

$$\mathbf{Int}(W(\lambda)_\lambda) \cong W(\lambda) \cong \mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda)$$

as L-modules.

Proof. By Proposition 4.3(c), $W(\lambda)_\lambda$ is a cyclic L_0-module, whence $\mathbf{Int}(W(\lambda)_\lambda)$ is a cyclic L-module in $\mathcal{I}^\lambda$ by Proposition 4.4(aii). The map $\eta_{W(\lambda)}$ sends the generator $\mathbf{Int}(w_\lambda)$ onto the canonical generator w_λ of $W(\lambda)$. It then follows from the universal property of $W(\lambda)$, cf. Proposition 3.8(b), that $\eta_{W(\lambda)}$ is an isomorphism.

Since $W(\lambda)_\lambda$ is a cyclic $\mathbb{S}^\lambda$-module we have a surjective $\mathbb{S}^\lambda$-module map $f\colon {}_{\mathrm{reg}}\mathbb{S}^\lambda \to W(\lambda)_\lambda$ and then by Proposition 4.4(bii) a surjective L-module map $\mathbf{Int}(f)\colon \mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda) \to \mathbf{Int}(W(\lambda)_\lambda)$. Furthermore, by Proposition 4.4(aii) the module $\mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda)$ is cyclic, and therefore the canonically induced map $W(\lambda) \to \mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda)$ is a surjection. It then follows as in the first part that $\mathbf{Int}(f)$ is an isomorphism. □

4.10. Theorem. *For any $\lambda \in \mathcal{P}_+$ we have $\mathbb{S}^\lambda = \mathbf{A}_\lambda$ as associative $\Bbbk$-algebras.*

Because of this result, we will refer to $\mathbf{A}_\lambda$ and $\mathbb{S}^\lambda$ as the *Seligman–Chari–Pressley algebra* in the future. For simpler notation we will denote it by $\mathbb{S}^\lambda$.

Proof. Recall that $\mathbb{S}^\lambda = \mathrm{U}(L_0)/J^\lambda$ and $\mathbf{A}_\lambda = \mathrm{U}(L_0)/\operatorname{Ann}_{\mathrm{U}(L_0)}(w_\lambda)$. From Proposition 4.3(a) it follows that $J^\lambda \subseteq \operatorname{Ann}_{\mathrm{U}(L_0)}(w_\lambda)$. To complete the proof, we need to show the reverse inclusion. Consider the L_0-module map

$$f : \mathbb{S}^\lambda \to W(\lambda)_\lambda \ , \ u \mapsto u \cdot w_\lambda.$$

By Proposition 4.4(b) and Corollary 4.9, f gives rise to an isomorphism of L-modules

$$\mathbf{Int}(f) : \mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda) \to \mathbf{Int}(W(\lambda)_\lambda) \cong W(\lambda).$$

Write $\lambda = \sum_{i=1}^r \ell_i \varpi_i$ and fix $u_\circ \in \operatorname{Ann}_{\mathrm{U}(L_0)}(w_\lambda)$. Our goal is to prove that $u_\circ \in J^\lambda$. Let $\overline{u}_\circ$ be the projection of $u_\circ$ in $\mathbb{S}^\lambda$. First note that $\mathbf{Int}(f)(1\otimes\overline{u}_\circ) = 1\otimes(u_\circ\cdot w_\lambda) = 0$, and since $\mathbf{Int}(f)$ is an isomorphism, we should have $1 \otimes \overline{u}_\circ = 0$ as an element of $\mathbf{Int}({}_{\mathrm{reg}}\mathbb{S}^\lambda)$. It follows that

$$1 \otimes \overline{u}_\circ \in \sum_{i=1}^r \sum_{s\in\mathbb{S}^\lambda} \mathrm{U}(L) \cdot (f_i^{\ell_i+1} \otimes s).$$

Fix a basis of L consisting of $\mathfrak{h}$-root vectors. The basis of L yields a PBW basis of $\mathrm{U}(L)$. We can now write

$$1 \otimes \overline{u}_\circ = \sum_{i=1}^r \sum_{j=1}^\infty q_{i,j} f_i^{\ell_i+1} \otimes s_{i,j}, \tag{4.10.1}$$

where every $q_{i,j}$ is a monomial in the PBW basis of $\mathrm{U}(L)$, and all but finitely many of the $s_{i,j} \in \mathbb{S}^\lambda$ are zero. Next fix i, j and write

$$q_{i,j} = y_1 \cdots y_a h_1 \cdots h_b x_1 \cdots x_c,$$

where $y_s \in L_{-\beta_s}$ for $1 \le s \le a$, $h_s \in L_0$ for $1 \le s \le b$, and $x_s \in L_{\gamma_s}$ for $1 \le s \le c$, with $\beta_1, \dots, \beta_a, \gamma_1, \dots, \gamma_c \in \Theta_+$. The equality of $\mathfrak{h}$-weights of both sides of (4.10.1) implies that

$$\gamma_1 + \cdots + \gamma_c = \beta_1 + \cdots + \beta_a + (\ell_i + 1)\alpha_i.$$

If $a \ge 1$, then the $\mathfrak{h}$-weight of $x_1 \cdots x_c f_i^{\ell_i+1}$ is in $\mathbb{N}[\Theta_+] \setminus \{0\}$, from which it follows that $x_1 \cdots x_c f_i^{\ell_i+1} \in \mathrm{U}(L)L_+$, and therefore $q_{i,j} f_i^{\ell_i+1} \otimes s_{i,j} = 0$. We can therefore assume $a = 0$. Then $x_1 \cdots x_c f_i^{\ell_i+1} \in \mathrm{U}(L)_0$ and

$$x_1 \cdots x_c f_i^{\ell_i+1} = \pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) + u_+ \quad \text{for some} \quad u_+ \in \mathrm{U}(L)L_+,$$

so that

$$\begin{aligned} q_{i,j} f_i^{\ell_i+1} \otimes s_{i,j} &= \Big(h_1 \cdots h_b \pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) + h_1 \cdots h_b u_+\Big) \otimes s_{i,j} \\ &= h_1 \cdots h_b \pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) \otimes s_{i,j} = 1 \otimes h_1 \cdots h_b \pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) s_{i,j}. \end{aligned}$$

Note that $h_1 \cdots h_b \pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) s_{i,j} = 0$ in $\mathbb{S}^\lambda$ since already $\pi_0(x_1 \cdots x_c f_i^{\ell_i+1}) = 0$ in $\mathbb{S}^\lambda$. Therefore the above arguments show that $1 \otimes \overline{u}_\circ = 0$ as an element of $\mathrm{U}(L) \otimes_{\mathrm{U}(L_0 \oplus L_+)} \mathbb{S}^\lambda$. Since $\mathrm{U}(L)$ is a free $\mathrm{U}(L_0 \oplus L_+)$-module, it follows that $\overline{u}_\circ = 0$ in $\mathbb{S}^\lambda$, so that $u_\circ \in J^\lambda$. $\square$

In the remainder of this section we will derive results concerning the structure of $\mathbb{S}^\lambda$.

4.11. *Example* $\mathrm{U}(L_0)_+ \subseteq J^\lambda$

Let $\mathrm{U}(L_0)_+$ be the augmentation ideal of $\mathrm{U}(L_0)$. Since $\mathrm{U}(L_0)_+$ is a maximal ideal there are exactly the cases (a) and (b) below whenever $\mathrm{U}(L_0)_+ \subseteq J^\lambda$:

(a) $\mathrm{U}(L_0)_+ = J^\lambda$: We have

$$\mathrm{U}(L_0)_+ = J^\lambda \quad \iff \quad \lambda = 0,$$

and in this case (obviously) $\mathbb{S}^\lambda(L, \mathfrak{g}) = \mathbb{k}1$ is 1-dimensional.

Indeed, if $\mathrm{U}(L_0)_+ = J^\lambda$, then $h - \lambda(h)\mathbb{1}_{\mathrm{U}} \in J^\lambda$ and $h \in J^\lambda$ forces $\lambda = 0$. Conversely, assume $\lambda = 0$. Then $\pi_0(L_{\alpha_i} L_{-\alpha_i}) = [L_{\alpha_i}, L_{-\alpha_i}] \subset J^0$ by definition of the ideal J^λ in 4.2(c). Using the notation of Lemma 3.3(b), we also have $\pi_0(L_{\beta_c/2} L_{\beta_c/2} L_{-\beta_c}) = [L_{\beta_c/2}, [L_{\beta_c/2}, L_{\beta_c}] = [L_{\beta_c/2}, L_{\beta_c/2}] \subset J^0$. It then follows from (3.3.2) and that J^0 is generated by L_0, i.e., $J^0 = \mathrm{U}(L_0)_+$.

We point out that $\dim \mathbb{S}^\lambda = 1$ does not imply $\lambda = 0$. For example, as already shown in [40, p. 59], for $L = \mathfrak{sl}_n(A)$, A central-simple, $\lambda = \ell\varpi_i$ for $1 < i < n-1$, we always have $\dim \mathbb{S}^\lambda \le 1$ and $\dim = 1 \iff d|\ell$ where d is the degree of A. We will see in 6.6 and 6.8 that examples with $\dim \mathbb{S}^\lambda \le 1$ arise naturally.

(b) $\mathrm{U}(L_0) = J^\lambda$, i.e., $\mathbb{S}^\lambda(L, \mathfrak{g}) = \{0\}$: Then $\lambda \neq 0$, and the only λ-admissible L_0-module is $\{0\}$. Hence any module V in $\mathcal{I}^\lambda(L, \mathfrak{g})$ has $V_\lambda = \{0\}$.

4.12. The subalgebras $\mathbb{S}_i^\lambda < \mathbb{S}^\lambda$

One approach to the analysis of the structure of $\mathbb{S}^\lambda(L, \mathfrak{g}) = \mathbb{S}^\lambda$ is through certain subalgebras $\mathbb{S}_i^\lambda$, which we now define using the notation of 2.1 and 4.2. For $i \in I$ let

$$\begin{aligned} L_i &= \Big(\textstyle\sum_{\beta \in \mathbb{R}\alpha_i \cap \Theta} L_\beta \Big) \bigoplus \textstyle\sum_{\beta \in \mathbb{R}_+\alpha_i \cap \Theta} [L_\beta, L_{-\beta}], \\ \mathfrak{g}_i &= \mathfrak{g}_{\alpha_i} \oplus \Bbbk h_i \oplus \mathfrak{g}_{-\alpha_i} \simeq \mathfrak{sl}_2(\Bbbk). \end{aligned} \tag{4.12.1}$$

Then L_i is a subalgebra of L and $(L_i, \mathfrak{g}_i)$ is either an A_1-graded or a BC_1-graded Lie algebra with 0-part $L_{i0} = L_i \cap L_0$. It is immediate from the Jacobi identity that L_{i0} is an ideal of L_0. Let ξ_i be the restriction of the fundamental weight ϖ_i to $\Bbbk h_i$, the fundamental weight of $\mathfrak{g}_i$. Under the canonical injection $\iota_i \colon \mathrm{U}(L_{i0}) \hookrightarrow \mathrm{U}(L_0)$ the ideal $J^{\ell_i \xi_i}$ is mapped to J^λ, giving rise to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J^{\ell_i\xi_i} \arrow[r] \arrow[d, hook] & \mathrm{U}(L_{i0}) \arrow[r, "\mathrm{can}"] \arrow[d, hook, "\iota_i"] & \mathbb{S}^{\ell_i\xi_i}(L_i, \mathfrak{g}_i) \arrow[r] \arrow[d, dashed, "\phi_i"] & 0 \\ 0 \arrow[r] & J^\lambda \arrow[r] & \mathrm{U}(L_0) \arrow[r, "\mathrm{can}"] & \mathbb{S}^\lambda(L, \mathfrak{g}) \arrow[r] & 0 \end{tikzcd}$$

where $\mathbb{S}^{\ell_i\xi_i}(L_i, \mathfrak{g}_i)$ is defined for $(L_i, \mathfrak{g}_i)$ as in Definition 4.2, the maps “can” are the canonical quotient maps, and ϕ_i is the unital algebra homomorphism induced by ι_i. We denote by

$$\mathbb{S}_i^\lambda = \phi_i \left(\mathbb{S}^{\ell_i\xi_i}(L_i, \mathfrak{g}_i) \right) < \mathbb{S}^\lambda$$

the image of ϕ_i. Since $L_0 = \sum_{i \in I} L_{i0}$ by (3.3.2), the algebra $\mathbb{S}^\lambda$ is generated as associative algebra by the subalgebras $\mathbb{S}_i^\lambda$, $i \in I$. Note $\mathbb{S}_i^\lambda = \Bbbk 1\!\!1_{\mathrm{U}}$ if $\ell_i = 0$ since then $J^{0\xi_i}$ is generated by L_{i0} and thus equals the augmentation ideal of $\mathrm{U}(L_{i0})$.

Using the subalgebras $\mathbb{S}_i^\lambda$ to determine the structure of $\mathbb{S}^\lambda$ leads to two problems: First, $\mathbb{S}_i^\lambda$ need not be isomorphic to $\mathbb{S}^{\ell_i\xi_i}(L_i, \mathfrak{g}_i)$ and, second, the interplay between the subalgebras $\mathbb{S}_i^\lambda$ of $\mathbb{S}^\lambda$ seems to be complicated in general. But it can be understood in the following situation.

4.13. Proposition. *Let L be a Δ-graded Lie algebra and let $M_0 \lhd L_0$ be an ideal of L_0 satisfying*

(i) *$M_0 \subset J^\lambda$, viewed in $\mathrm{U}(L_0)$,*
(ii) *$M_0 = \sum_{i \in I} M_0 \cap L_{i0}$, and*
(iii) *denoting by $\bar{\ }$ the canonical quotient map $L_0 \to L_0/M_0$ we have $L_0/M_0 = \boxplus_{i \in I} \overline{L_{i0}}$, a direct sum of ideals $\overline{L_{i0}}$ of L_0/M_0.*

Then multiplication induces a unital algebra isomorphism between the tensor product algebra of the $\mathbb{S}_i^\lambda$, $i \in I$, and $\mathbb{S}^\lambda$:

$$\bigotimes_{i\in I} \mathbb{S}_i^\lambda \xrightarrow{\simeq} \mathbb{S}^\lambda.$$

Proof. By [7, §2.7, Cor. 6], multiplying the factors is an isomorphism of vector spaces $\bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}}) \to \mathrm{U}(\overline{L_0})$. But since the factors $\mathrm{U}(\overline{L_{i0}})$ commute in $\mathrm{U}(\overline{L_0})$, this map is in fact a unital algebra isomorphism. To simplify the notation, we will view it as an identification in the following.

Let $\mathrm{id}(M_0) \lhd \mathrm{U}(L_0)$ be the ideal of $\mathrm{U}(L_0)$ generated by M_0. By [7, §2.3, Prop. 3] this is the kernel of the quotient map $\bar{p}\colon\ \mathrm{U}(L_0) \to \bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})$. Observe that $\bar{p}$ maps $\mathrm{U}(L_{j0}) \subset \mathrm{U}(L_0)$ onto $\mathrm{U}(\overline{L_{j0}}) \subset \bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})$. We have the following commutative diagram, defining the unital surjective algebra homomorphism φ :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{id}(M_0) \arrow[r] \arrow[d, hook] & \mathrm{U}(L_0) \arrow[r, "\bar{p}"] \arrow[d, equal] & \bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}}) \arrow[r] \arrow[d, dashed, "\varphi"] & 0 \\
0 \arrow[r] & J^\lambda \arrow[r] & \mathrm{U}(L_0) \arrow[r, "\mathrm{can}"] & \mathbb{S}^\lambda(L, \mathfrak{g}) \arrow[r] & 0
\end{tikzcd}$$

The kernel of φ is $\bar{p}(J^\lambda)$, the ideal of $\bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})$ generated by $\bigcup_{i\in I} \bar{p}(G_i)$. From the algebra structure of $\bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})$ it follows that $\bar{p}(J^\lambda) = \sum_{i\in I} \mathrm{U}(\overline{L_{10}}) \otimes \cdots \mathrm{U}(\overline{L_{i-1,0}}) \otimes X_i \otimes \mathrm{U}(\overline{L_{i+1,0}}) \otimes \cdots \mathrm{U}(\overline{L_{r0}})$ where X_i is the ideal of $\mathrm{U}(\overline{L_{i0}})$ generated by $\bar{p}(G_i)$. Hence,

$$\mathbb{S}^\lambda \simeq \big(\bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})\big)\big/\operatorname{Ker}(\varphi) \simeq \bigotimes_{i\in I} \mathrm{U}(\overline{L_{i0}})/X_i$$

But $\mathbb{S}_i^\lambda = \mathrm{U}(L_{i0})/(J^\lambda \cap \mathrm{U}(L_{i0})) = \bar{p}(\mathrm{U}(L_{i0}))/\bar{p}(\mathrm{U}(L_{i0}) \cap J^\lambda) = \mathrm{U}(\overline{L_{i0}})/X_i$. $\square$

4.14. Examples

(a) Let $(L, \mathfrak{g}) = (\mathfrak{g} \otimes A, \mathfrak{g})$ be a map algebra as in 3.2(a). Then $L_0 = \mathfrak{h} \otimes A$ is abelian and a direct sum of the ideals $\mathfrak{g}_i \otimes A$. Thus, Proposition 4.13 applies with $M_0 = \{0\}$. Observe that in this case $\bar{p}$ is an isomorphism and $J^{\ell_i \xi_i} = X^i$ in the notation of the proof above. It follows that the maps ϕ_i of 4.12 are isomorphisms. We will use this in the proof of Theorem 5.7.

(b) Suppose $\operatorname{supp}\lambda = \{i \in I : \ell_i > 0\}$ is *totally disconnected* in the sense that $i \in \operatorname{supp}\lambda$ implies that both $i-1 \notin \operatorname{supp}\lambda$ and $i+1 \notin \operatorname{supp}\lambda$. Then $M_0 = \sum_{k\notin \operatorname{supp}\lambda} L_{k0}$ is an ideal of L_0 satisfying the conditions (i)–(iii) of Proposition 4.13.

We also note the following structural result which generalizes [40, Prop. I.10] proven there for finite-dimensional semisimple Lie algebras as in Example 3.2(c).

4.15. Proposition. *Let $(L, \mathfrak{g}, \mathfrak{h})$ be a (Θ, Δ)-graded Lie algebra, and let $\lambda = \sum_{i\in I} \ell_i \varpi_i \in \mathcal{P}_+$. We set $\ell_{\max} := \max\{\ell_i : i \in I\}$. If $\dim L_0 < \infty$, then*

$$\dim \mathbb{S}^\lambda(L, \mathfrak{g}) \le (2\ell_{\max} + 1)^{\dim L_0 - \dim \mathfrak{h}}.$$

If there are no connected components Θ_c *of* Θ *satisfying* $(\Theta_c, \Delta_c) \cong (\mathrm{BC}_n, \mathrm{C}_n)$*, then the stronger inequality* $\dim \mathbb{S}^\lambda(L, \mathfrak{g}) \le (\ell_{\max} + 1)^{\dim L_0 - \dim \mathfrak{h}}$ *holds.*

Proof. We can essentially follow Seligman's proof in [40]. But we give a sketch of the argument for the convenience of the reader.

(I) For $\alpha \in \Theta_+$, $x \in L_\alpha, y \in L_{-\alpha}$ and $j \in \mathbb{N}_+$ there exist $a_{jk} \in \mathbb{N}_+$ such that

$$x^j y^j \equiv \textstyle\sum_{k=0}^{j} a_{jk} y^k [x, y]^{j-k} x^k \mod \textstyle\sum_{k=0}^{j-1} L_{-\alpha}^k \,\mathrm{U}(L_0)_{(j-k-1)} L_\alpha^k.$$

This is proven in [40, Lemma I.2] and [13, 4.5, Lem. 5] for Examples 3.2(c) and 3.2(a) respectively. Both proofs work in our setting. In particular, for $\alpha = \alpha_i$ and $j = \ell_i + 1$ we get, after application of π_0, that

$$\pi_0(x^{\ell_i+1} y^{\ell_i+1}) \equiv a_{\ell_i+1,0}\, [x, y]^{\ell_i+1} \mod \mathrm{U}(L_0)_{(\ell_i)}. \tag{4.15.1}$$

The left hand side lies in J^λ, hence $[x, y]^{\ell_i+1} \in J^\lambda$. If $\alpha = \frac{1}{2}\alpha_i$ (this occurs when Θ has connected components of the form $(\Theta_c, \Delta_c) \cong (\mathrm{BC}_n, \mathrm{C}_n)$), then setting $j = 2\ell_i + 1$ and using Lemma 4.6 implies that $[x, y]^{2\ell_i+1} \in J^\lambda$. The algebra L_0 has a basis B that consists of elements of the form $[x, y]$ with $x \in L_{\alpha_i}$ and $y \in L_{-\alpha_i}$, or $x \in L_{\frac{1}{2}\alpha_i}$ and $y \in L_{-\frac{1}{2}\alpha_i}$ (where in the latter case α_i is a long simple root of a connected component Δ_c of type C_n). Furthermore, using the Chevalley basis of $\mathfrak{g}$ we can assume that B contains a basis of $\mathfrak{h} \subset \mathfrak{g}$. By the PBW Theorem, the relation (ii) of 4.2(c), and the observations made above, $\mathrm{U}(L_0)/J^\lambda$ is spanned by monomials in the elements of $\mathsf{B}\backslash\mathfrak{h}$ in which none of the exponents exceed $2\ell_i$. It follows that $\dim \mathbb{S}^\lambda(L, \mathfrak{g}) \le (2\ell_{\max} + 1)^{\dim L_0 - \dim \mathfrak{h}}$. From the proof it is clear that when there are no connected components of $(\mathrm{BC}_n, \mathrm{C}_n)$ type, the stronger inequality holds. □

5. The Seligman–Chari–Pressley algebra of map algebras

We continue with the notation of the previous sections: $\Bbbk$ is a field of characteristic 0 and $\mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra over $\Bbbk$. The ultimate goal of this section is to determine $\mathbb{S}(L, \mathfrak{g})^\bullet$ for the root-graded Lie $\Bbbk$-algebra $L = \mathfrak{g} \otimes A$, $A \in \Bbbk$-**alg**, of 3.2(a).

We will start this section by defining and describing the structure for $\mathfrak{sl}_n(A)$ for A unital associative (5.1–5.2). We will then specialize to $n = 2$ with A as before, and at the end consider $L = \mathfrak{g} \otimes A$ for $A \in \Bbbk$-**alg**. Proceeding in this way will allow us to use some formulas needed for $\mathfrak{g} \otimes A$ also in the case $\mathfrak{sl}_n(A)$ of §6.

Thus, unless stated otherwise, in this section $n \in \mathbb{N}$, $n \ge 2$, and A is a unital associative, but not necessarily commutative $\Bbbk$-algebra.

5.1. The Lie algebra $\mathfrak{sl}_n(A)$

By definition $\mathfrak{sl}_n(A)$ is the derived algebra of the Lie algebra $\mathrm{Mat}_n(A)^- = \mathfrak{gl}_n(A)$ of $n \times n$ matrices over A:

$$L = \mathfrak{sl}_n(A) = [\mathfrak{gl}_n(A),\, \mathfrak{gl}_n(A)]$$

(see (5.2.3) for another description of $\mathfrak{sl}_n(A)$). For general A the structure map $\Bbbk \to \Bbbk \cdot 1_A$ gives rise to an injective homomorphism of $\mathfrak{sl}_n(\Bbbk) = [\mathfrak{gl}_n(\Bbbk), \mathfrak{gl}_n(\Bbbk)]$ into $\mathfrak{sl}_n(A)$. Its image is the subalgebra of $\mathfrak{sl}_n(A)$ consisting of the trace-less matrices over $\Bbbk$. In the future we will take this homomorphism as an identification:

$$\mathfrak{g} = \mathfrak{sl}_n(\Bbbk) \subset L.$$

We claim that L is a Δ-graded Lie algebra for the root system $\Delta = \mathrm{A}_{n-1}$ in the sense of 3.1. Indeed, $\mathfrak{g}$ is split simple and contains

$$\mathfrak{h} = \{h \in \mathfrak{sl}_n(\Bbbk) : h \text{ diagonal}\}$$

as a splitting Cartan subalgebra. Denoting by E_{ij} the usual matrix units in $\mathrm{Mat}_n(A)$ and abbreviating $E_{ij}(a) = aE_{ij}$ for $a \in A$, the Lie algebra $\mathfrak{sl}_n(A)$ has a weight space decomposition

$$L = L_0 \oplus \textstyle\bigoplus_{i \neq j} L_{\epsilon_i - \epsilon_j}, \quad L_{\epsilon_i - \epsilon_j} = \{E_{ij}(a) : a \in A\} =: E_{ij}(A)$$

where L_0 consists of the diagonal matrices in $\mathfrak{sl}_n(A)$ and $L_{\epsilon_i - \epsilon_j}$ is the weight space for the root $\epsilon_i - \epsilon_j \in \Delta$ realized in the standard way: ϵ_i is the projection onto the ith-component of $h \in \mathfrak{h}$. Hence 3.1(i) holds. In our setting condition 3.1(ii) says $L_0 \subseteq \sum_{i \neq j} [E_{ij}(A), E_{ji}(A)]$. Since $\mathfrak{gl}_n(A) = \mathfrak{gl}_n(A)_0 \oplus \big(\bigoplus_{i \neq j} E_{ij}(A)\big)$ is a grading by the root lattice of Δ, it follows from the definition of $\mathfrak{sl}_n(A)$ that we need to show $[\mathfrak{gl}_n(A)_0, \mathfrak{gl}_n(A)_0] \subset \sum_{i \neq j} [E_{ij}(A), E_{ji}(A)]$, or equivalently

$$[a,b]E_{ii} \in \sum_{i \neq j} [E_{ij}(A), E_{ji}(A)]. \tag{5.1.1}$$

To do so we will use the matrices

$$\begin{aligned} e_i(a) &:= E_{i,i+1}(a), \quad f_i(a) := E_{i+1,i}(a), \\ H_i(a,b) &:= [e_i(a), f_i(b)] = abE_{ii} - baE_{i+1,i+1} \in L_0, \\ h_i(a) &= H_i(a, 1_A) = H_i(1_A, a) = a(E_{ii} - E_{i+1,i+1}) \end{aligned} \tag{5.1.2}$$

defined for $a, b \in A$ and $1 \leq i < n$. It is straightforward to verify

$$[a,b]E_{ii} = \begin{cases} H_i(a,b) - h_i(ba), & (1 \le i < n), \\ H_{i-1}(a,b) - h_{i-1}(ab) & (1 < i \le n), \end{cases} \tag{5.1.3}$$

which clearly implies (5.1.1).

We will use the standard root basis of the root system Δ, for which the simple roots are the $\alpha_i = \epsilon_i - \epsilon_{i+1}$, cf. 2.1. Hence the subalgebra L_+ (resp. L_-) is the subalgebra of strictly upper (resp. lower) triangular matrices in $\mathfrak{sl}_n(A)$.

5.2. Structure of $L_0 = \mathfrak{sl}_n(A)_0$

Since L_0 is A_{n-1} graded, we know that

$$L_0 = \textstyle\sum_{i\neq j}[E_{ij}(A), E_{ji}(A)] = \mathrm{Span}_{\Bbbk}\{H_i(a,b) : 1 \le i < n\}. \tag{5.2.1}$$

The second equality follows from the general formula (3.3.3), which in our setting can of course be easily verified:

$$abE_{ii} - baE_{jj} = \big(\textstyle\sum_{k=i}^{j-2} h_k(ab)\big) - H_{j-1}(a,b) \qquad (1 \le i < j-1 < n).$$

Observe that (5.2.1) implies that every $x = \sum_{i=1}^n x_i E_{ii} \in L_0$ has trace $\mathrm{tr}(x) = \sum_{i=1}^n x_i \in [A,A]$. Conversely, since any diagonal $x = \sum_i x_i E_{ii} \in \mathfrak{gl}_n(A)$ can be written in the form

$$\begin{aligned} x &= \mathrm{tr}(x)E_{11} + \textstyle\sum_{j=2}^n x_j(E_{jj} - E_{11}) = \mathrm{tr}(x)E_{11} + \sum_{j=2}^n [x_jE_{j1}, E_{1j}] \\ &= \mathrm{tr}(x)E_{11} - \textstyle\sum_{j=2}^n \big(h_1(x_j) + \cdots + h_{j-1}(x_j)\big) \end{aligned} \tag{5.2.2}$$

it follows from (5.1.3) that any diagonal $x \in \mathfrak{gl}_n(A)$ with $\mathrm{tr}(x) \in [A,A]$ lies in L_0. Thus

$$\mathfrak{sl}_n(A) = \{x \in \mathfrak{gl}_n(A) : \mathrm{tr}(x) \in [A,A]\}. \tag{5.2.3}$$

This formula justifies the notation $\mathfrak{sl}_n$. Indeed, if A is commutative then $\mathfrak{sl}_n(A)$ consists of the trace-less matrices in $\mathfrak{gl}_n(A)$ and is in fact the base ring extension of $\mathfrak{sl}_n(\Bbbk)$ by A, i.e., $\mathfrak{sl}_n(A) \cong \mathfrak{sl}_n(\Bbbk) \otimes_{\Bbbk} A$.

For later use we will establish some more formulas for L_0. We put

$$H_j(A,A) = \mathrm{Span}\{H_j(a,b) : a,b \in A\} \quad \text{and} \quad h_j(A) = \{h_j(a) : a \in A\}.$$

Obviously

$$L_0' := \{l \in L_0 : \mathrm{tr}(l) = 0\} = \textstyle\bigoplus_{j=1}^{n-1} h_j(A). \tag{5.2.4}$$

Moreover, (5.2.2) implies that $L_0 = [A,A]E_{11} \oplus L_0' = H_1(A,A) + L_0'$ where the second equality follows from (5.1.3). By symmetry we thus have for any j, $1 \le j < n$,

$$L_0 = [A,A]E_{jj} \oplus L_0' = H_j(A,A) + L_0'. \tag{5.2.5}$$

For $a, b, c \in A$ we abbreviate

$$\{a\ b\ c\} = abc + cba,$$

the Jordan triple product of A. It enters in the description of the algebra L:

$$\begin{aligned}
[H_i(a,b),\ H_i(c,d)] &= H_i(\{a\ b\ c\}, d) - H_i(c, \{b\ a\ d\}), \\
[H_i(a,b),\ e_i(c)] &= e_i(\{a\ b\ c\}) \quad \text{and} \\
[H_i(a,b),\ f_i(c)] &= -f_i(\{b\ a\ c\}).
\end{aligned} \tag{5.2.6}$$

A special case of (5.2.6) is

$$[h_i(a), h_i(b)] = [a,b](E_{ii} + E_{i+1,i+1}). \tag{5.2.7}$$

5.3. Identities in $\mathrm{U}(L)$ *and* $\mathrm{U}(L_0)$

In this subsection we will establish several identities involving products with factors $e_i(a), f_i(b)$ for $a, b \in A$ and a fixed i, see (5.1.2) for the definition of e_i and f_i. For better readability we put

$$e(a) = e_i(a), \quad H(a,b) = H_i(a,b), \quad h(a) = h_i(a) \quad \text{and} \quad f(a) = f_i(a) \tag{5.3.1}$$

for $a, b \in A$.

Applying the basic commutation rules $[e(a), f(b)] = H(a,b)$ and $[e(a), H(b,c)] = e(-\{a\,c\,b\})$ in $\mathrm{U}(L)$ repeatedly to the product $e(a_1)\cdots e(a_t) \in \mathrm{U}(L)$, $t \in \mathbb{N}_+$, yields

$$\begin{aligned}
[e(a_1)\cdots e(a_t), f(b)] &= \sum_{j=1}^{t} e(a_1)\cdots e(a_{j-1}) H(a_j, b) e(a_{j+1})\cdots e(a_t) \\
[e(a_1)\cdots e(a_t), H(b,c)] &= \sum_{m=1}^{t} e(a_1)\cdots e(a_{m-1}) e(-\{a_m\ c\ b\}) e(a_{m+1})\cdots e(a_t) \\
&= -\sum_{m=1}^{t} e(\{a_m\ c\ b\}) e(a_1)\cdots \widehat{e(a_m)} \cdots e(a_t),
\end{aligned}$$

where in the last equality we used $[e(A), e(A)] = 0$ in $\mathrm{U}(L)$ and where as usual $\widehat{x}$ indicates that x has been omitted. Combining these two formulas we get for $t \in \mathbb{N}_+$

$$\begin{aligned}
[e(a_1)\cdots e(a_t), f(b)] = &\sum_{j=1}^{t} H(a_j, b) e(a_1)\cdots \widehat{e(a_j)} \cdots e(a_t) \\
&- \sum_{1 \le j < m \le t} e(\{a_j\ b\ a_m\}) e(a_1)\cdots \widehat{e(a_j)} \cdots \widehat{e(a_m)} \cdots e(a_t).
\end{aligned}$$

Applying the Harish-Chandra homomorphism $\pi_0\colon \mathrm{U}(L)^0 \to \mathrm{U}(L_0)$ defined in 4.1 and keeping in mind that $\pi_0\big(f(b_1)\cdots f(b_t)e(a_1)\cdots e(a_t)\big) = 0$ we now obtain, again for $t \in \mathbb{N}_+$,

$$\begin{aligned}\pi_0\big(e(a_1)\cdots e(a_t)f(b_1)\cdots f(b_t)\big) &= \\ &= \sum_{j=1}^{t} H(a_j,b_1)\pi_0\big(e(a_1)\cdots \widehat{e(a_j)}\cdots e(a_t)f(b_2)\cdots f(b_t)\big) \qquad (5.3.2)\\ &\quad - \sum_{1\le j<m\le t} \pi_0\big(e(\{a_j\ b_1\ a_m\})e(a_1)\cdots \widehat{e(a_j)}\cdots \widehat{e(a_m)}\cdots e(a_t)f(b_2)\cdots f(b_t)\big),\end{aligned}$$

cf. [40, p. 62]. Denoting by $\mathrm{U}(L_0)_{(t)}$ the tth-term of the standard filtration of $\mathrm{U}(L_0)$ we now claim for any $t \in \mathbb{N}_+$

$$\pi_0\big(e(a_1)\cdots e(a_t)f(b_1)\cdots f(b_t)\big) \equiv \sum_{\sigma\in\mathfrak{S}_t} H(a_1,b_{\sigma(1)})\cdots H(a_t,b_{\sigma(t)}) \mod \mathrm{U}(L_0)_{(t-1)}. \qquad (5.3.3)$$

This is obvious for $t = 1$, and follows for $t \ge 2$ by induction. Indeed, assuming (5.3.3) for $t-1$ the second term on the right hand side of (5.3.2) lies in $\mathrm{U}(L_0)_{(t-1)}$ whence

$$\begin{aligned}&\pi_0\big(e(a_1)\cdots e(a_t)f(b_1)\cdots f(b_t)\big)\\ &\quad\equiv \sum_{j=1}^{t} H(a_j,b_1)\pi_0\big(e(a_1)\cdots \widehat{e(a_j)}\cdots e(a_t)f(b_2)\cdots f(b_t)\big)\\ &\quad\equiv \sum_{j=1}^{t} H(a_j,b_1)\\ &\qquad\times\Big(\sum_{\sigma\in\mathfrak{S}_t,\sigma(j)=1} H(a_1,b_{\sigma(1)})\cdots \widehat{H(a_j,b_1)}\cdots H(a_t,b_{\sigma(t)})\Big) \mod \mathrm{U}(L_0)_{(t-1)}.\end{aligned}$$

Using the PBW Theorem we can move $H(a_j,b_1)$ to the jth place modulo terms in $\mathrm{U}(L_0)_{(t-1)}$, finishing the proof of (5.3.3). In particular,

$$\pi_0\big(e(a_1)\cdots e(a_t)f(b)^t\big) \equiv t!\,H(a_1,b)\cdots H(a_t,b) \mod \mathrm{U}(L_0)_{(t-1)}. \qquad (5.3.4)$$

5.4. Proposition. *We fix i, $1 \le i < n$, and assume that $\eta\colon \mathrm{U}(L_0) \to B$ is a unital algebra homomorphism Let $\rho\colon A \to B$ be the $\Bbbk$-linear map defined by $\rho(a) = \eta\big(h_i(a)\big)$.*

(a) *The sequence $(g_t)_{t\in\mathbb{N}_+}$ of functions,*

$$g_t\colon A^t \to B, \qquad g_t(a_1,\ldots,a_t) = (\eta\circ\pi_0)\left(e_i(a_1)\cdots e_i(a_t)f_i(1_A)^t\right) \qquad (5.4.1)$$

satisfies the recursion 1.8 with respect to ρ.

(b) *Let $\ell \in \mathbb{N}_+$, and assume that ρ is a Lie homomorphism. Then ρ satisfies the ℓth-symmetric identity if and only if*

$$(\eta \circ \pi_0)\left(e_i(a_1)\cdots e_i(a_\ell) f_i(b_1)\cdots f_i(b_\ell)\right) = 0 \tag{5.4.2}$$

for arbitrary $a_j, b_j \in A$.

Proof. (a) We will employ the notation (5.3.1). First note that the condition 1.8(i) holds by definition: $g_1(a) = (\eta \circ \pi_0)\big(e(a)f(1_A)\big) = \eta\big(h(a)\big) = \rho(a)$. To verify the condition 1.8(ii) we suppose that $(a_1, \ldots, a_{t+1})$ is a family of commuting elements of A. Then, using (5.3.2) with $b_i = 1_A$ we get

$$\begin{aligned} g_{t+1}(a_1, \ldots, a_{t+1}) &= \sum_{j=1}^{t+1} \rho(a_j) g_t(a_1, \ldots, \widehat{a_j}, \ldots a_{t+1}) \\ &\quad - 2 \sum_{1 \le j < m \le t} g_t(a_j a_m, a_1, \ldots, \widehat{a_j}, \ldots, \widehat{a_m}, \ldots, a_{t+1}) \end{aligned}$$

since $\{a_j\ 1_A\ a_m\} = a_j a_m + a_m a_j = 2 a_j a_m$.

(b) We start by noting that

$$[\eta(h(a)), \eta(h(b))] = \eta(h([a,b])) = \eta([a,b]E_{i,i} - [a,b]E_{i+1,i+1})$$

since ρ is a Lie homomorphism. But η is also a homomorphism of associative algebras, so that the left hand side of the last equation is equal to $\eta\big([a,b](E_{i,i} + E_{i+1,i+1})\big)$. It follows that $\eta([a,b]E_{i+1,i+1}) = 0$, and the latter relation can be written as

$$\eta(H(a,b)) = \eta(h(ab)). \tag{5.4.3}$$

Next we show for $t \in \mathbb{N}_+$ and $a_k, b \in A$ that

$$(\eta \circ \pi_0)\left(e(a_1)\cdots e(a_t) f(b)^t\right) = (\eta \circ \pi_0)\left(e(a_1 b)\cdots e(a_t b) f(1_A)^t\right). \tag{5.4.4}$$

Since $\pi_0\big(e(a)f(b)\big) = H(a,b)$, our assertion holds for $t = 1$. Let now $t > 1$. Then, by (5.4.3), (5.3.2) and induction,

$$\begin{aligned} (\eta \circ \pi_0)\big(e(a_1)\cdots e(a_t) f(b)^t\big) &= \sum_{j=1}^{t} \eta\big(H(a_j, b)\big)\, (\eta \circ \pi_0)\big(e(a_1)\cdots \widehat{e(a_j)} \cdots e(a_t) f(b)^{t-1}\big) \\ &\quad - \sum_{1 \le j < m \le t} (\eta \circ \pi_0)\big(e(\{a_j\ b\ a_m\}) e(a_1)\cdots \widehat{e(a_j)} \cdots \widehat{e(a_m)} \cdots e(a_t) f(b)^{t-1}\big) \\ &= \sum_{j=1}^{t} \eta\big(h(a_j b)\big)\, (\eta \circ \pi_0)\big(e(a_1 b)\cdots \widehat{e(a_j b)} \cdots e(a_t b) f(1_A)^{t-1}\big) \\ &\quad - \sum_{1 \le j < m \le t} (\eta \circ \pi_0)\big(e(\{a_j\ b\ a_m\} b) e(a_1 b)\cdots \widehat{e(a_j b)} \cdots \widehat{e(a_m b)} \cdots e(a_t b) f(1_A)^{t-1}\big) \end{aligned}$$

The same calculation applied to the right hand side of (5.4.4) yields

$$
\begin{aligned}
&(\eta \circ \pi_0)\big(e(a_1 b)\cdots e(a_t b) f(1_A)^t\big) \\
&\quad = \sum_{j=1}^{t} \eta\big(h(a_j b)\big)\, (\eta \circ \pi_0)\big(e(a_1 b)\cdots \widehat{e(a_j b)} \cdots e(a_t b) f(1_A)^{t-1}\big) \\
&\quad - \sum_{1 \le j < m \le t} (\eta \circ \pi_0)\big(e(\{a_j b,\ 1_A,\ a_m b\}) e(a_1 b)\cdots \widehat{e(a_j b)} \cdots \widehat{e(a_m b)} \cdots e(a_t b) f(1_A)^{t-1}\big).
\end{aligned}
$$

The equality (5.4.4) now follows by comparing the two equations and using $\{a_j\ b\ a_m\}b = a_j b a_m b + a_m b a_j b = \{a_j b,\ 1_A,\ a_m b\}$.

Since $[f(A), f(A)] = 0$ in L, the function $A^\ell \to \mathrm{U}(L_0)$, $(b_1, \dots, b_\ell) \mapsto e(a_1)\cdots e(a_\ell) f(b_1)\cdots f(b_\ell)$, is symmetric. Hence (5.4.2) is equivalent to $(\eta \circ \pi_0)\big(e(a_1)\cdots e(a_\ell) f(b)^\ell\big) = 0$ for arbitrary $a_i, b \in A$. Using (5.4.4), this is in turn equivalent to

$$
g_\ell(a_1, \dots, a_\ell) = (\eta \circ \pi_0)\left(e(a_1)\cdots e(a_\ell) f(1_A)^\ell\right) = 0. \tag{5.4.5}
$$

Since the functions g_ℓ are symmetric, (5.4.2) is also equivalent to $g_\ell(a, \dots, a) = 0$. But by (a) and (1.8.2) this holds if and only if ρ satisfies the ℓth-symmetric identity. $\square$

5.5. Corollary. *Assume $\lambda = \sum_{j=1}^{n-1} \ell_j \varpi_j \in \mathcal{P}_+$ satisfies $\ell_{i+1} = 0$ for some i, $1 \le i < n-1$. Denote by* $\mathrm{can}\colon \mathrm{U}(L_0) \to \mathbb{S}^\lambda(L, \mathfrak{g}) =: \mathbb{S}^\lambda$ *the canonical epimorphism and define $\rho_i\colon A \to \mathbb{S}^\lambda$, $\rho_i(a) = \mathrm{can}\left(h_i(a)\right)$.*

Then ρ_i satisfies the $(\ell_i + 1)$st-symmetric identity and $\rho(1_A) = \ell_i 1_{\mathbb{S}^\lambda}$. Hence there exists a unique homomorphism of unital associative algebras

$$
\varphi_i\colon \mathrm{TS}^{\ell_i}(A) \to \mathbb{S}^\lambda, \quad \varphi_i\big(\mathrm{sym}_{\ell_i}(a)\big) = \mathrm{can}\left(h_i(a)\right).
$$

Proof. From 4.2(c) with i replaced by $i+1$ we obtain $\pi_0\big(e_{i+1}(a) f_{i+1}(b)\big) = H_{i+1}(a, b) \in J^\lambda$, whence $\mathrm{can}\left(H_i(a,b) - h_i(ab)\right) = \mathrm{can}([a,b]E_{i+1,i+1}) = 0$ since $[a,b]E_{i+1,i+1} \in H_{i+1}(A, A)$ by (5.1.3). Thus $\eta = \mathrm{can}$ satisfies the assumptions of Proposition 5.4. The map $\rho\colon A \to \mathbb{S}^\lambda$ defined there is our map ρ_i. Again by 4.2(c) the function g_{ℓ_i+1} of (5.4.1) vanishes, whence by (1.8.2) the function ρ_i satisfies the (ℓ_i+1)st-symmetric identity. We also have $\rho_i(1_A) = \mathrm{can}\left(h_i(1_A)\right) = \mathrm{can}\left(\lambda(h_i)\mathbb{1}_{\mathrm{U}}\right) = \ell_i 1_{\mathbb{S}^\lambda}$. The last part of our claim is an application of Theorem 1.7. $\square$

5.6. Lemma. *Let $L = \mathfrak{sl}_2(A)$ for $A \in \Bbbk$-**alg**, and let $\lambda = \ell\varpi_1 \in \mathcal{P}_+$ for some $\ell \in \mathbb{N}$. We further suppose that B' is a basis of A containing 1_A and that $\le$ is a total order on $\mathsf{B} = \mathsf{B}' \setminus \{1_A\}$. Then*

$$
\{h_1(b_1)\cdots h_1(b_k) + J^\lambda : b_i \in \mathsf{B}, b_1 \le b_2 \cdots \le b_k, k \in \mathbb{N}, k \le \ell, \}
$$

is a spanning set of the vector space $\mathbb{S}^\lambda(L, \mathfrak{g})$.

Proof. Since A is commutative, we have $L_0 = h_1(A)$ by (5.2.5), whence L_0 is commutative too. It then follows from the PBW Theorem that $\mathbb{S}(L,\mathfrak{g})^\lambda$ is spanned by

$$h_1(b'_1)\cdots h_1(b'_k) + J^\lambda, \quad b'_i \in \mathsf{B}', b'_1 \le b'_2 \le \cdots \le b'_k, \quad k \in \mathbb{N}, \tag{5.6.1}$$

with $k = 0$ being the identity element $\mathbb{1}_{\mathrm{U}}$. In such an expression we can eliminate a factor $b'_i = 1_A$ using the relation $h_1(1_A) - \ell\mathbb{1}_{\mathrm{U}} \in J^\lambda$. Thus we can assume that all $b'_i = b_i \in \mathsf{B}$. We can then bound the length k of the products using (5.3.4):

$$\tfrac{1}{(\ell+1)!}\pi_0\big(e_1(b_1)\cdots e_1(b_{\ell+1})f_1(1)^{\ell+1}\big) \equiv h_1(b_1)\cdots h_1(b_{\ell+1}) \mod \mathrm{U}(L_0)_{(\ell)}.$$

Since the left hand side lies in J^λ, any expression (5.6.1) with $k > \ell$ can be iteratively reduced to a linear combination of elements (5.6.1) with $k \le \ell$ (observe that one can achieve an increasing order of the b_i's using commutativity of L_0). $\square$

We now have enough preparation to identify $\mathbb{S}^\lambda(L,\mathfrak{g})$ when $L = \mathfrak{g} \otimes A$, defined in Example 3.2(a). We use the notation 2.1.

5.7. Theorem. *Let $L = \mathfrak{g} \otimes A$ for some $A \in \mathbb{k}$-**alg** and let $\lambda = \sum_{i=1}^r \ell_i\varpi_i \in \mathcal{P}_+$. Then there exists an isomorphism*

$$\mathbb{S}^\lambda(L,\mathfrak{g}) \to \mathrm{TS}^\lambda(A) := \mathrm{TS}^{\ell_1}(A) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \mathrm{TS}^{\ell_r}(A)$$

of unital associative $\mathbb{k}$-algebras induced by

$$h_i(a) \mapsto 1 \otimes \cdots \otimes 1 \otimes \mathrm{sym}_{\ell_i}(a) \otimes 1 \otimes \cdots \otimes 1$$

with $\mathrm{sym}_{\ell_i}(a)$ put in the ith factor. The algebra structure of $\mathrm{TS}^\lambda(A)$ is that of the tensor product of the associative commutative algebras $\mathrm{TS}^{\ell_i}(A)$.

For $\mathbb{k}$ an algebraically closed field and A a finitely generated $\mathbb{k}$-algebra with trivial Jacobson radical the isomorphism $\mathbb{S}^\lambda(L,\mathfrak{g}) \cong \mathrm{TS}^\lambda(A)$ was shown in [13, Thm. 4] with a different proof. The proof of [13] does not generalize to our setting.

Proof. For $1 \le i \le r$ define $\mathfrak{g}_i \simeq \mathfrak{sl}_2(\mathbb{k})$ as the subalgebra generated by e_i, f_i and put $L_i = \mathfrak{g}_i \otimes A = (\mathbb{k}e_i \otimes A) \oplus (\mathbb{k}h_i \oplus A) \oplus (\mathbb{k}f_i \otimes A)$. We have seen in Proposition 4.13 and Example 4.14(a) that multiplication induces a unital algebra homomorphism $\bigotimes_{i\in I} \mathbb{S}^{\ell_i\xi_i}(L_i,\mathfrak{g}_i) \xrightarrow{\sim} \mathbb{S}^\lambda(L,\mathfrak{g})$. It is therefore sufficient to deal with the rank-1-case, i.e., $L = \mathfrak{sl}_2(A)$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$, and show $\mathbb{S}^\lambda := \mathbb{S}^\lambda(L,\mathfrak{g}) \cong \mathrm{TS}^\ell(A)$ for $\lambda = \ell\omega$, the isomorphism being induced by $h(a) \mapsto \mathrm{sym}_\ell(a)$.

Since A is commutative, hence so is $L_0 = h(A)$. By (1.1.1), we have a homomorphism $L_0 \to \mathrm{TS}^\ell(A)^-$, $h(a) \mapsto \mathrm{sym}_\ell(a)$, whence a homomorphism

$$\eta\colon\ \mathrm{U}(L_0) \to \mathrm{TS}^\ell(A), \quad h(a) \mapsto \mathrm{sym}_\ell(a)$$

of unital associative algebras. We claim that η annihilates the ideal $J^\lambda \lhd \mathrm{U}(L_0)$. To prove this, recall from 4.2(c) that J^λ is generated by elements of type

(i) $\pi_0\big(e(a_1)\dots e(a_{\ell+1})f(b_1)\cdots f(b_{\ell+1})\big)$, $a_i, b_i \in A$, and

(ii) $h(1_A) - \ell\mathbb{1}_{\mathrm{U}}$.

It is clear that $\eta\big(h(1_A) - \ell\mathbb{1}_{\mathrm{U}}\big) = 0$. We need to work more to deal with elements of type (i). First, note that we can apply Proposition 5.4 with $B = \mathrm{TS}^\ell(A)$ because A is commutative. The map ρ of Proposition 5.4 is $\rho = \mathrm{sym}_\ell$ and hence satisfies the $(\ell+1)$st symmetric identity by Proposition 1.6. Thus, by Proposition 5.4(b) with ℓ replaced by $\ell+1$, η also annihilates elements of type (i). Hence η descends to a homomorphism

$$\eta\colon\ \mathrm{U}(L_0)/J^\lambda = \mathbb{S}^\lambda \to \mathrm{TS}^\lambda(A), \quad u + J^\lambda \mapsto \eta(u)$$

of unital associative algebras. By Lemma 5.6 and Lemma 1.4, η maps a spanning set of $\mathbb{S}^\lambda(A)$ to a vector space basis of $\mathrm{TS}^\ell(A)$, whence is an isomorphism. $\square$

6. $L = \mathfrak{sl}_n(A)$, $n \geq 3$, and A associative

In this section we consider the root-graded Lie algebra

$$L = \mathfrak{sl}_n(A),\ \mathfrak{g} = \mathfrak{sl}_n(\mathbb{k}); \quad n \geq 3,$$

as defined in 5.1. We point out that A is a unital associative $\mathbb{k}$-algebra which need not be commutative. Our goal in this section is to describe $\mathbb{S}^\lambda = \mathbb{S}^\lambda(L, \mathfrak{g})$ for special weights $\lambda \in \mathcal{P}_+$. We do this for λ totally disconnected in Theorem 6.5. Furthermore, in Proposition 6.11 we show that for $(L, \mathfrak{g}) = (\mathfrak{sl}_4(A), \mathfrak{sl}_4(k))$ and $\lambda = \varpi_1 + \varpi_2$, we have $\mathbb{S}^\lambda \cong A$ for $A = \mathrm{Mat}_2(\mathbb{k})$, whereas $\mathbb{S}^\lambda(A) = \{0\}$ for $A = \mathrm{Mat}_{2n}(\mathbb{k})$ with $n > 1$. In particular, unlike the case when A is commutative, for $\lambda = \varpi_1 + \varpi_2$ the algebra $\mathbb{S}^\lambda$ is not necessarily isomorphic to $A \otimes A$.

6.1. The isomorphism θ

Let $d \in \mathrm{Mat}_n(\mathbb{k})$ be the matrix with 1's on the second diagonal and 0's elsewhere. Note that $d^2 = \sum_{i=1}^n E_{i,i}$. The map

$$\theta = \theta_A\colon \mathfrak{sl}_n(A) \to \mathfrak{sl}_n(A^{\mathrm{op}}), \quad x \mapsto -dx^t d$$

is an isomorphism of Lie algebras with $\theta_A^{-1} = \theta_{A^{\mathrm{op}}}$. To simplify notation in the following we will view $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$ as a subalgebra of both $\mathfrak{sl}_n(A)$ and $\mathfrak{sl}_n(A^{\mathrm{op}})$, and do the same for $\mathfrak{h}$. Since $\theta(aE_{ij}) = -aE_{n+1-j,n+1-i}$ we have $\theta(\mathfrak{g}) = \mathfrak{g}$ and $\theta(\mathfrak{h}) = \mathfrak{h}$. We denote by $\theta^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$ the transpose of $\theta|_{\mathfrak{h}} = (\theta|_{\mathfrak{h}})^{-1}$. Then $\theta(\mathfrak{sl}_n(A)_\alpha) = \mathfrak{sl}_n(A^{\mathrm{op}})_{\theta^*(\alpha)}$ holds for all $\alpha \in \Delta \cup \{0\}$, and θ^* induces the non-trivial automorphism of the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$. Hence

$$\theta^*(\varpi_i) = \varpi_{n-i} \tag{6.1.1}$$

Assigning to a representation $\rho\colon \mathfrak{sl}_n(A) \to \mathfrak{gl}(V)$ the representation $\rho \circ \theta_{A^{\rm op}} : \mathfrak{sl}_n(A^{\rm op}) \to \mathfrak{gl}(V)$ gives rise to an isomorphism between the representation categories of $\mathfrak{sl}_n(A)$ and $\mathfrak{sl}_n(A^{\rm op})$. It preserves integrable representations and because of (6.1.1) induces an isomorphism

$$\mathcal{I}(\mathfrak{sl}_n(A), \mathfrak{g})^\lambda \xrightarrow{\sim} \mathcal{I}(\mathfrak{sl}_n(A^{\rm op}), \mathfrak{g})^{\theta^*(\lambda)}. \tag{6.1.2}$$

The functors **Res** and **Int** of 4.7 together with the isomorphism (6.1.2) induce isomorphisms of the module categories $\mathbb{S}^\lambda$-$\mathcal{M}od$ and $\mathbb{S}^{\theta^*(\lambda)}$-$\mathcal{M}od$, that is,

$$\begin{tikzcd}
\mathcal{I}(\mathfrak{sl}_n(A), \mathfrak{g})^\lambda \arrow[r, "\simeq"] & \mathcal{I}(\mathfrak{sl}_n(A^{\rm op}), \mathfrak{g})^{\theta^*(\lambda)} \arrow[d, "\mathbf{Res}"] \\
\mathbb{S}^\lambda\text{-}\mathcal{M}od \arrow[u, "\mathbf{Int}"] \arrow[r, dashed, "\sim"] & \mathbb{S}^{\theta^*(\lambda)}\text{-}\mathcal{M}od
\end{tikzcd}$$

whence the isomorphisms

$$\mathbb{S}^\lambda(\mathfrak{sl}_n(A), \mathfrak{g}) \xrightarrow{\sim} \mathbb{S}^{\theta^*(\lambda)}(\mathfrak{sl}_n(A^{\rm op}), \mathfrak{g}) \tag{6.1.3}$$

of the associative algebras. The isomorphism (6.1.3) can of course also be derived directly from the definitions. Indeed, the canonical extension of θ to an isomorphism $\mathrm{U}(\theta)\colon \mathrm{U}(\mathfrak{sl}_n(A)) \to \mathrm{U}(\mathfrak{sl}_n(A^{\rm op}))$ mapping $\mathrm{U}(\mathfrak{sl}_n(A)_0)$ to $\mathrm{U}(\mathfrak{sl}_n(A^{\rm op})_0)$, is equivariant with respect to the Harish-Chandra homomorphisms, mapping the ideal $J^\lambda \lhd \mathrm{U}(\mathfrak{sl}_n(A)_0)$ of 4.2(c) to the ideal $J^{\theta^*(\lambda)} \lhd \mathrm{U}(\mathfrak{sl}_n(A^{\rm op}))$ and thus descends to the isomorphism (6.1.3).

Recall the subalgebras $L_i = L_i(A)$ of $L = \mathfrak{sl}_n(A)$ and $\mathbb{S}_i^\lambda$ of $\mathbb{S}^\lambda$ defined in 4.12. In our setting, for $i \in I = \{1, \dots, n-1\}$ we have $L_i(A) = e_i(A) \oplus H_i(A, A) \oplus f_i(A)$. It is an A_1-graded subalgebra of L with grading subalgebra $\mathfrak{g}_i = e_i(1_A) \oplus h_i(1_A) \oplus f_i(1_A)$ and 0-part $L_{i0} = H_i(A, A)$. Restricting the isomorphism θ to $L_i(A)$ shows

$$(L_i(A), \mathfrak{g}_i) \simeq (L_{n-i}(A^{\rm op}), \mathfrak{g}_{n-i}). \tag{6.1.4}$$

As before we let $\mathrm{can}\colon \mathrm{U}(L_0) \to \mathbb{S}^\lambda$ be the canonical algebra homomorphism. Then

$$\mathbb{S}_i^\lambda = \mathbb{S}_i^\lambda(A) = \langle \mathrm{can}\,\big(H_i(A, A)\big)\rangle \tag{6.1.5}$$

is the unital subalgebra of $\mathbb{S}^\lambda$ generated by $\mathrm{can}\,\big(H_i(A, A)\big)$. The isomorphism (6.1.4) induces an isomorphism

$$\mathbb{S}_i^\lambda(A) \simeq \mathbb{S}_{n-i}^{\theta^*(\lambda)}(A^{\rm op})$$

for all $i \in I$.

6.2. The structure of $L_0 = \mathfrak{sl}_n(A)_0$ again

We fix $k \in I = \{1, \dots, n-1\}$ and recall the decomposition (5.2.5):

$$L_0 = [A, A]E_{kk} \oplus \big(\oplus_{i \in I} h_i(A) \big). \tag{6.2.1}$$

Observe for $a, b \in A$ and $i \in I$

$$\begin{aligned} aE_{ii} &= \begin{cases} h_i(a) + \cdots + h_{k-1}(a) + aE_{kk}, & i < k. \\ aE_{kk} - \big(h_k(a) + \cdots + h_{i-1}(a)\big), & i > k. \end{cases} \\ H_i(a,b) &= abE_{ii} - baE_{i+1,i+1} \\ &= \begin{cases} h_i(ab) + h_{i+1}([a,b]) + \cdots + h_{k-1}([a,b]) + [a,b]E_{kk}, & i+1 < k, \\ h_i(ab) + [a,b]E_{kk}, & i+1 = k, \\ [a,b]E_{kk} + h_k(ba), & i = k, \\ [a,b]E_{kk} + h_k([b,a]) + \cdots + h_{i-1}([b,a]) + h_i(ba), & i > k. \end{cases} \end{aligned} \tag{6.2.2}$$

We use this to describe the Lie algebra structure of L_0 in terms of the decomposition (6.2.1):

$$[A, A]E_{kk} \lhd L_0; \tag{6.2.3}$$

$$\begin{aligned} [h_i(a), h_{i+1}(b)] &= [b,a]E_{i+1,i+1} \\ &= \begin{cases} h_{i+1}([b,a]) + \cdots + h_{k-1}([b,a]) + [b,a]E_{kk}, & i+1 < k, \\ h_k([a,b]) + \cdots + h_i([a,b]) + [b,a]E_{kk}, & i+1 > k; \end{cases} \end{aligned} \tag{6.2.4}$$

$$\begin{aligned} [h_i(a), h_i(b)] &= [a,b](E_{ii} + E_{i+1,i+1}) \\ &= \begin{cases} h_i([a,b]) + 2h_{i+1}([a,b]) + \cdots 2h_{k-1}([a,b]) + 2[a,b]E_{kk}, & i+1 < k, \\ h_i([a,b]) + 2[a,b]E_{kk}, & i+1 = k, \\ 2[a,b]E_{kk} - h_k([a,b], & i = k, \\ 2[a,b]E_{kk} - 2\big(h_k([a,b]) + \cdots + h_{i-1}([a,b]) - h_i([a,b]), & i > k; \end{cases} \end{aligned} \tag{6.2.5}$$

$$[h_i(a), h_j(b) = 0 \quad \text{if } |i - j| > 1. \tag{6.2.6}$$

6.3. Lemma. *As in 6.2 we fix $k \in I$. Let $\rho_i \colon A \to B$, $i \in I$, be linear maps into a unital associative $\Bbbk$-algebra B satisfying*

(I) $[\rho_i(A), \rho_j(A)] = 0$ *if $i \neq j$;*

(II) $\rho_i([A, A]) = 0$ *if $1 < i < n-1$ and also for $i = 1$ in case $k = 1$;*

(III) ρ_i *is a Lie homomorphism for $1 \leq i < k$, and a Lie anti-homomorphism for $k \leq i \leq n-1$.*

Then the map

$$\eta_{L_0} \colon L_0 \to B, \quad cE_{kk} + \sum_{i\in I} h_i(a_i) \mapsto \sum_{i\in I} \rho_i(a_i)$$

is a Lie homomorphism.

Proof. This is immediate from the multiplication rules (6.2.3)–(6.2.6). □

6.4. Lemma. *Let $\lambda = \sum_{i\in I} \ell_i \varpi_i \in \mathcal{P}_+$ be totally disconnected, cf.* 4.14(b). *Then the subalgebras $\mathbb{S}_i^\lambda$ of $\mathbb{S}^\lambda$, cf.* (6.1.5), *have the following properties.*

(a) $\mathbb{S}_i^\lambda$ *is generated by* $\operatorname{can}\big(h_i(A)\big)$ *as associative algebra.*
(b) *If $\ell_i = 0$, then $\mathbb{S}_i^\lambda = \Bbbk 1_{\mathbb{S}^\lambda}$.*
(c) *If $1 < i < n-1$, then $\mathbb{S}_i^\lambda$ is commutative and $h_i([A, A]) = 0$.*
(d) *$[\mathbb{S}_i^\lambda(A), \mathbb{S}_j^\lambda(A)] = 0$ for $i \neq j$.*

Proof. Recall from 4.2(d) that $H_k(A, A) \subset J^\lambda$ whenever $\ell_k = 0$, which implies (b). The identities (5.1.3),

$$H_j(a, b) - h_j(ba) = [a, b]E_{jj} = H_{j-1}(a, b) - h_{j-1}(ab), \qquad j \in I, j > 1,$$

imply that $\operatorname{can}(H_i(A, A)) = \operatorname{can}(h_i(A))$ since the image under can of the left or right side of the equation vanishes. This proves (a). Moreover, for $1 < i < n-1$ with $\ell_i > 0$ the equation also implies that $[h_i(a), h_i(b)] = [a, b](E_{ii} + E_{i+1,i+1}) \in H_{i-1}(A, A) + H_{i+1}(A, A) \subset J^\lambda$, whence the first part of (c). The second follows in the same way from $\operatorname{can}\big(h_i([a, b])\big) = \operatorname{can}\big([a, b](E_{ii} - E_{i+1,i+1}\big)$. Finally, for the proof of (d) we can assume that $|i - j| > 1$, in which case already $[H_i(A, A), H_j(A, A)] = 0$ in $\mathrm{U}(L_0)$. □

We are now in a position to prove the main result of this section. For simpler notation we put $\mathrm{TS}^0(A) = \Bbbk 1$ and let $\mathrm{sym}_0 \colon A \to \mathrm{TS}^0(A)$ be the zero map.

6.5. Theorem. *Let $\lambda = \sum_{i\in I} \ell_i \varpi_i \in \mathcal{P}_+$ be totally disconnected. For $1 < i < n-1$ denote by $\mathcal{C}_i$ the ideal of $\mathrm{TS}^{\ell_i(A)}$ generated by the commutator space $[\mathrm{TS}^{\ell_i}(A), \mathrm{TS}^{\ell_i}(A)]$ and define*

$$B_i = \begin{cases} \mathrm{TS}^{\ell_1}(A), & i = 1 \\ \mathrm{TS}^{\ell_i}(A)/\mathcal{C}_i, & 1 < i < n-1, \\ \mathrm{TS}^{\ell_{n-1}}(A^{\mathrm{op}}) & i = n-1. \end{cases}$$

Then

$$\mathbb{S}^\lambda\big(\mathfrak{sl}_n(A)\big) \cong \bigotimes_{i\in I} B_i$$

as unital associative $\Bbbk$-algebras, where the algebra structure of $\bigotimes_{i\in I} B_i$ is that of the tensor product algebra.

Proof. We will construct unital algebra homomorphisms in both directions and show that they are inverses of each other. For easier notation we put $\mathbb{S}^\lambda = \mathbb{S}^\lambda(\mathfrak{sl}_n(A))$ and $\mathcal{C}_i = \{0\} \subset B_i$ for $i = 1$ and $i = n-1$.

(a) *The homomorphism* $\bar\eta\colon \mathbb{S}^\lambda \to B = \bigotimes_{i\in I} B_i$. We fix $k \in I$ with $\ell_k = 0$, and for $i \in I$ define

$$\rho_i\colon A \to B, \quad \rho_i(a) = 1_{B_1} \otimes \cdots \otimes 1_{B_{i-1}} \otimes \big(\mathrm{sym}_{\ell_i}(a) + \mathcal{C}_i\big) \otimes 1_{B_{i+1}} \otimes \cdots \otimes 1_{B_{n-1}}.$$

We claim that the map

$$\eta_{L_0}\colon L_0 \to B, \quad cE_{kk} + \textstyle\sum_{i\in I} h_i(a) \mapsto \sum_{i\in I} \rho_i(a_i)$$

of Lemma 6.3 is a Lie homomorphism. To show this, we verify the conditions (I)–(III) of Lemma 6.3.

Identifying B_i with the obvious unital subalgebra of B, we have $\rho_i(A) \subset B_i$, so that (I) holds because of $[B_i, B_j] = 0$ for $i \neq j$. Recall from (1.1.1) that $a \mapsto \mathrm{sym}_\ell(a)$ is a Lie homomorphism, whence $\rho_i([A,A]) = [\mathrm{sym}_{\ell_i}(A), \sum_{\ell_i}(A)] + \mathcal{C}_i = \mathcal{C}_i$ for $1 < i < n-1$, and (II) follows in that case. Since $\rho_1 = 0$ in case $k = 1$, we have now proven (II). The condition (III) is clear from the proof of (II) if $1 \le i \le k$, and also for $k \le i < n-1$ since then $\rho_i([a,b]) = 0 = [\rho_i(b), \rho_i(a)]$. Finally, (III) for $i = n-1$ follows from the fact that $\mathrm{sym}_{\ell_{n-1}}\colon A^{\mathrm{op}} \to \mathrm{TS}^{\ell_{n-1}}(A^{\mathrm{op}}) = \mathrm{TS}^{\ell_{n-1}}(A)^{\mathrm{op}}$ is a Lie homomorphism.

Having proven that η_{L_0} is a Lie homomorphism, we next claim that the unique extension of η_{L_0} to a unital associative algebra homomorphism $\eta\colon \mathrm{U}(L_0) \to B$ annihilates the ideal J^λ defining $\mathbb{S}^\lambda$. Thus, specializing 4.2(d) to our setting, we need to show that η vanishes on elements of the following types:

(i) $H_i(a,b) = \pi_0\big(e_i(a)f_i(b)\big)$ for $i \in I$ with $\ell_i = 0$;
(ii) $\pi_0\big(e_i(a_1)\cdots e_i(a_{\ell_i+1})f_i(b_1)\cdots f_i(b_{\ell_i+1})\big)$ for $a_r, b_s \in A$ and $\ell_i > 0$;
(iii) $h_i(1_A) - \ell_i \mathbb{1}_{\mathrm{U}}$ for $i \in I$ with $\ell_i > 0$.

Re (i): We have $\eta\big(H_i(a,b)\big) = \eta_{L_0}\big(H_i(a,b)\big)$ which can be calculated using the formulas in (6.2.2). For example, if $i+1 < k$ then $\eta_{L_0}\big(H_i(a,b)\big) = \rho_i(ab) + \rho_{i+1}([a,b]) + \cdots + \rho_{k-1}([a,b]) = 0$ using $\rho_i = 0$ because $\ell_i = 0$ and $\rho_j([a,b]) = 0$ for $1 < j < n-1$ as shown in the proof of (II) above. The proof in the other cases is similar.

Re (ii): By Proposition 1.6 the map $A \to \mathrm{TS}^{\ell_i}(A)$ satisfies the (ℓ_i+1)st-symmetric identity, whence so does ρ_i for $1 \le i < n-1$. Since $\rho_i(a) = \eta\big(h_i(a)\big)$, Proposition 5.4 applies and yields our claim. We can argue similarly in case $i = n-1$: The map $\mathrm{sym}_{\ell_{n-1}}\colon A^{\mathrm{op}} \to \mathrm{TS}^{\ell_{n-1}}(A^{\mathrm{op}}) = B_{n-1}$ satisfies the $(\ell_{n-1}+1)$st-symmetric identity. Since powers in A and A^{op} coincide, it follows from Definition 1.5 that $\rho_{n-1}\colon A \to B_{n-1}$ satisfies the same symmetric identity. We can then conclude as before by applying Proposition 5.4.

Re (iii): These elements are annihilated because $\rho_i(1_A)) = \ell_i 1_{B_i}$ and $\eta(1_{\mathrm{U}(L_0)} = 1_B$.

We now know that $\eta(J^\lambda) = 0$. The induced algebra homomorphism $\bar\eta\colon \mathbb{S}^\lambda \to B$ is the map we were looking for. Observe that

$$\bar{\eta}\big(\operatorname{can}\big(h_i(a)\big)\big) = \operatorname{sym}_{\ell_i}(a) + \mathcal{C}_i \qquad (i \in I). \tag{6.5.1}$$

(b) *The homomorphism* $\bar{\varphi}\colon B \to \mathbb{S}^\lambda$. We will use Lemma 6.4 to construct this map. First we claim that there exists a well-defined unital algebra homomorphism

$$\varphi_i\colon B_i \to \mathbb{S}^\lambda, \qquad \varphi_i(\operatorname{sym}_{\ell_i}(a) + \mathcal{C}_i) = \operatorname{can}\big(h_i(a)\big) \in \mathbb{S}_i^\lambda.$$

This is obvious in case $\ell_i = 0$ since then $B_i = \Bbbk 1_B$ (after identification) and also $\operatorname{can}\big(h_i(a)\big) = 0$. Let then $\ell_i > 0$. In case $1 \le i < n-1$ the existence of φ_i follows from Corollary 5.5, namely directly in case $i = 1$ and from commutativity of $\mathbb{S}_i^\lambda$ in case $1 < i < n-1$. It remains to consider $i = n-1$. Here $\theta^*(\lambda) = \ell_{n-1}\varpi_1 + 0\varpi_2 + \cdots$, whence Corollary 5.5 for A^{op} yields the existence of a unital algebra homomorphism

$$\varphi_{n-1}^{\mathrm{op}}\colon \mathrm{TS}^{\ell_{n-1}}(A^{\mathrm{op}}) \to \mathbb{S}^{\theta^*(\lambda)}\big(\mathfrak{sl}_n(A^{\mathrm{op}})\big), \quad \operatorname{sym}_{\ell_{n-1}}(a) \mapsto \operatorname{can}\big(h_1(a)\big).$$

Following this map with the isomorphism $\mathbb{S}_1^{\theta^*(\lambda)}\big(\mathfrak{sl}_n A^{\mathrm{op}})\big) \cong \mathbb{S}_i^\lambda(A)$ established in (6.1.3) provides a map $\varphi_{n-1}\colon B_{n-1} \to \mathbb{S}^\lambda$ as claimed.

We combine the maps φ_i, $i \in I$, to get a multilinear map $\varphi\colon \prod_{i\in I} B_i \to \mathbb{S}^\lambda$, $(b_i)_{i\in I} \mapsto \prod_{i\in I}\varphi_i(b_i)$, and denote by $\bar{\varphi}\colon \bigotimes_{i\in I} B_i \to \mathbb{S}^\lambda$ the unique linear map defined by $\bar{\varphi}(b_1 \otimes \cdots \otimes b_{n-1}) \mapsto \varphi_1(b_1)\cdots\varphi_{n-1}(b_{n-1})$. It is immediate that $\bar{\varphi}$ is a unital algebra homomorphism.

By construction

$$\bar{\varphi}(\operatorname{sym}_{\ell_i}(a) + \mathcal{C}_i) = \operatorname{can}\big(h_i(a)\big) \qquad (i \in I). \tag{6.5.2}$$

(c) $\bar{\eta}$ *and* $\bar{\varphi}$ *are inverses of each other.* The associative algebra B is generated by $B_i \subset B$, $i \in I$, which, as shown in Lemma 1.2(b), are in turn generated by $\operatorname{sym}_{\ell_i}(A)$. Similarly, since $L_0 = \sum_{i\in I} H_i(A,A)$, the associative algebra $\mathbb{S}^\lambda$ is generated by $\operatorname{can}\big(H_i(A,A)\big) = \operatorname{can}(h_i(A)$, $i \in I$. Because of (6.5.1) and (6.5.2), $\bar{\eta}\circ\bar{\varphi}$ and $\bar{\varphi}\circ\bar{\eta}$ are the identity map on generating sets of B and $\mathbb{S}^\lambda$ respectively, proving that they are inverses of each other. □

6.6. *Example* $\mathbb{S}^\lambda = \{0\}$

Let $\lambda = \sum_i \ell_i\varpi_i \in \mathcal{P}_+$ be totally disconnected with $\ell_1 = 0 = \ell_{n-1}$. Then $\mathbb{S}^\lambda$ is commutative by Theorem 6.5 (or Lemma 6.4(c)). If $A = (\Bbbk 1_A + [A,A]) \oplus X$ for some complementary space X, then Lemma 6.4(c) shows that $\mathbb{S}_i^\lambda$ is generated by $\operatorname{can}\big(h_i(X)\big)$ as unital algebra. Moreover, if $1_A \in [A,A]$ then $\mathbb{S}^\lambda = \{0\}$ since all $\mathbb{S}_i^\lambda$ vanish in that case. We note that there exist natural algebras A with $1 \in [A,A]$, for example the Weyl algebra. See 4.11(b) for a discussion of the consequences of $\mathbb{S}^\lambda = \{0\}$.

We will use the next lemma to construct representations.

6.7. Lemma. *Let* $\rho\colon A \to B$ *be a linear map into a unital associative* $\Bbbk$*-algebra* B *satisfying*

$$\rho([a,b]) = 0 = [\rho(a), \rho(b)] \tag{6.7.1}$$

for $a, b \in A$. For i with $1 < i < n-1$ define $\eta\colon L_0 \to B$ by

$$[a,b]E_{11} \oplus \big(\textstyle\bigoplus_{j=1}^{n-1} h_j(a_j)\big) \mapsto \rho(a_i). \tag{6.7.2}$$

Then $\eta\colon L_0 \to B^-$ is a Lie homomorphism. The unique extension of η to a unital associative algebra homomorphism $\mathrm{U}(L_0) \to B$ has the following properties:

(i) *$\eta\big(h_i(a)\big) = \rho(a)$ and $\eta\big(H_i(a,b)\big) = \eta\big(h_i(ab)\big)$ for all $a, b \in A$;*
(ii) *η annihilates the ideal J^λ if and only if $\rho(1_A) = \ell 1_B$ and ρ satisfies the $(\ell+1)$st-symmetric identity.*

Part (i) is the special case $k = 1$ of Lemma 6.3. The proof of part (ii) is similar to part (a) in the proof of Theorem 6.5 and will be left to the reader.

6.8. *Example $A = \mathcal{Z}(A) \oplus [A, A]$*

We denote by $\mathcal{Z}(A)$ the center of A and suppose that $A = \mathcal{Z}(A) \oplus [A, A]$. Some examples for which this is the case are:

(i) A is commutative;
(ii) A is finite-dimensional central-simple over $\Bbbk$;
(iii) A is an Azumaya over some $R \in \Bbbk$-**alg**;
(iv) A is a quantum torus over $\Bbbk$.

We denote by $\tau\colon A \to \mathcal{Z}(A)$ the projection with kernel $[A, A]$, and by $L_z \in \mathrm{End}_\Bbbk\big(\mathcal{Z}(A)\big)$ the left multiplication by $z \in \mathcal{Z}(A)$. Let $\lambda = \ell\varpi_i$ for some $1 < i < n-1$ and $\ell \in \mathbb{N}_+$. The map

$$\rho\colon A \to \mathrm{End}_\Bbbk \mathcal{Z}(A), \quad a \mapsto \ell\, L_{\tau(a)}$$

satisfies (6.7.1).

The homomorphism $\eta\colon \mathrm{U}(L_0) \to \mathrm{End}_\Bbbk\big(\mathcal{Z}(A)\big)$, defined in (6.7.2), descends to a unital homomorphism $\mathbb{S}^\lambda \to \mathrm{End}_\Bbbk\big(\mathcal{Z}(A)\big)$ and thus *defines a λ-admissible L_0-module if and only if ρ, equivalently $\ell\tau$, satisfies the $(\ell+1)$st-symmetric identity.*

In particular, assume $\mathcal{Z}(A) = \Bbbk 1_A$. Then Lemma 6.4(c) shows $\dim \mathbb{S}^\lambda \le 1$. We claim in this case

$$\dim \mathbb{S}^\lambda = 1 \quad \iff \quad \ell\tau \text{ satisfies the } (\ell+1)\text{st-symmetric identity.} \tag{6.8.1}$$

Indeed, if $\mathbb{S}^\lambda = \Bbbk 1_{\mathbb{S}^\lambda} \ne \{0\}$, the map $\rho_i\colon A \to \mathbb{S}^\lambda$ of Corollary 5.5 is here given by $\rho_i(a) = \ell\tau(a) 1_{\mathbb{S}^\lambda}$, and hence can be identified with $\ell\tau\colon A \to \Bbbk$. We then know from

Corollary 5.5 and $1_{\mathbb{S}^\lambda} \neq 0$ that $\ell\tau$ satisfies the $(\ell+1)$st-symmetric identity. Conversely, if this is the case, we have seen above that then $\mathbb{S}^\lambda$ has a 1-dimensional irreducible representation. Since the zero algebra does not have such a representation we must have $\dim \mathbb{S}^\lambda = 1$.

The assumption $A = \Bbbk 1_A \oplus [A, A]$ holds in case (ii) above. For those algebras, (6.8.1) is shown in [40, V.2]. We note that Seligman has also shown ([40, Cor. III.V]) that (6.8.1) holds if and only if ℓ is a multiple of the degree of A.

6.9. Remark. Let $\mathcal{C}_A = \mathrm{id}([A, A])$ be the ideal of A generated by $[A, A]$. By exactness of the functor TS^ℓ, 1.9(a), the epimorphism $c\colon A \to A/\mathcal{C}_A$ leads to an epimorphism $\mathrm{TS}^\ell(c)\colon \mathrm{TS}^\ell(A) \twoheadrightarrow \mathrm{TS}^\ell(A/\mathcal{C}_A)$ which by commutativity of $\mathrm{TS}^\ell(A/\mathcal{C}_A)$ factors through an epimorphism

$$\bar{c}\colon \mathrm{TS}^\ell(A)/\mathcal{C} \twoheadrightarrow \mathrm{TS}^\ell(A/\mathcal{C}_A). \tag{6.9.1}$$

The map $\bar{c}$ is in general not an isomorphism. For example, if A is finite-dimensional central-simple over $\Bbbk$ and non-commutative, i.e., $\dim A > 1$, we have $A/\mathcal{C}_A = \{0\}$, whence also $\mathrm{TS}^\ell(A/\mathcal{C}_A) = \{0\}$. But by Example 6.8 we know $\dim \mathrm{TS}^\ell(A)/\mathcal{C} = \dim \mathbb{S}^{\ell\varpi_i} = 1$ if (and only if) ℓ is a multiple of the degree of A.

6.10. $\mathbb{S}^\lambda$ for $\lambda = \varpi_1 + \varpi_2$ and $L = \mathfrak{sl}_4(A)$

Recall that a *generalized quaternion algebra* is a $\Bbbk$-algebra with generators $\mathbf{i}, \mathbf{j}$ which satisfy the relations

$$\mathbf{i}^2 = a,\ \mathbf{j}^2 = b,\ \mathbf{ij} = -\mathbf{ji} \text{ where } a, b \in \Bbbk^\times.$$

Setting $\mathbf{k} = \mathbf{ij}$, we obtain a basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for A with relations $\mathbf{k}^2 = -ab$, $\mathbf{ki} = -a\mathbf{j}$, and $\mathbf{jk} = -b\mathbf{i}$. The following proposition provides an example that indicates that beyond the case of totally disconnected λ, the structure of $\mathbb{S}^\lambda$ might depend on A as well as on λ. The proof of this proposition is given in the longer version of the paper which is available on the arXiv.

6.11. Proposition. *Assume that $L = \mathfrak{sl}_4(A)$, and set $\lambda = \varpi_1 + \varpi_2$. If A is a generalized quaternion algebra over $\Bbbk$, then the map*

$$A \to \mathbb{S}^\lambda\ ,\ a \mapsto h_2(a) + J^\lambda$$

is an isomorphism of associative algebras. Furthermore, if $A = \mathrm{Mat}_{2n}(\Bbbk)$ for an integer $n \geq 2$, then $\mathbb{S}^\lambda = \{0\}$.

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