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Polynomial Identities and Nonidentities of Split Jordan Pairs

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We show that split Jordan pairs over rings without 2-torsion can be distinguished by polynomial identities with integer coefficients. In particular, this holds for simple finite-dimensional Jordan pairs over algebraically closed fields of characteristic not 2. We also generalize results of Drensky and Racine and of Rached and Racine on polynomial identities of, respectively, Jordan algebras and Jordan triple systems.

0. INTRODUCTION

Identities are one of the key tools in Zel'manov's description of prime Jordan algebras [Z1, McZ] and Jordan triple systems [Z2, D1, D2]. This led McCrimmon to a number of questions aimed at clarifying the structure of polynomial identities of Jordan triple systems [Mc], some of which were answered by Rached and Racine: simple finite-dimensional Jordan triple systems of degree ≤ 2 over algebraically closed fields of characteristic $\neq 2$ can be separated by polynomial identities and nonidentities [RR], and the same is true for the simple exceptional Jordan triple systems [RR2]. That the isomorphism classes of simple finite-dimensional Jordan algebras over algebraically closed fields of characteristic 0 are determined by the polynomial (non)identities of the algebras had been shown before by Drensky and Racine [DR]. In this paper, we generalize these results to the setting of Jordan pairs.

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One of the polynomials used in [RR] and [RR2] is the inner Capelli polynomial IC_n, which was shown to be a nonidentity for several classes of simple Jordan triple systems ([RR, Propositions 16–19] and [RR2, Proposition 1]). These classes all have the property that the associated Jordan pair contains a connected grid of n idempotents. That Jordan pairs occur is not surprising. By its very definition, the inner Capelli polynomial is a Jordan pair polynomial rather than a Jordan triple polynomial. It is therefore more natural to work with Jordan pairs. Our first theorem (see Section 2) proves the obvious generalization of Rached's and Racine's results on the inner Capelli polynomial: over rings without 2-torsion, IC_n is not an identity for any Jordan pair containing a connected grid of n idempotents.

It is an easy consequence of this theorem that the inner Capelli polynomials can be used to distinguish between split finite-dimensional Jordan pairs of different dimensions (see Section 3 for the definition of "split"). To separate nonisomorphic Jordan pairs of the same dimension, we use a new variant of the inner Capelli polynomial and other polynomials already introduced in [DR] and [RR]. This leads to our second theorem, proven in Section 9: simple finite-dimensional Jordan pairs over algebraically closed fields of characteristic $\neq 2$ can be separated by polynomial identities. As a corollary we obtain the analogous result for Jordan algebras which generalizes the Drensky–Racine theorem [DR]. That Jordan pairs of rectangular matrices can be distinguished by polynomial identities is also proven in [I].

1. INNER CAPELLI POLYNOMIALS AND CAPELLI SEQUENCES

Unless stated otherwise, Jordan pairs will be considered over arbitrary commutative rings of scalars. We will use the notation of [L].

Let $\mathbf{X} = (X^+, X^-)$ be a pair of nonempty sets. We denote by $\mathrm{FJP}(\mathbf{X})$ the free Jordan pair over $\mathbb{Z}$ on $\mathbf{X}$, defined by the universal property that for every Jordan pair V (considered as a Jordan pair over the integers) and every map $\varphi: \mathbf{X} \to V$ there exists a unique homomorphism $\Phi: \mathrm{FJP}(\mathbf{X}) \to V$ extending φ. Such a map (φ or Φ) will be called a *substitution in V*, while elements of $\mathrm{FJP}(\mathbf{X})^\sigma$, $\sigma = \pm$, are called *Jordan polynomials*. If $f = f(\mathbf{x}, \mathbf{y}) \in \mathrm{FJP}(\mathbf{X})^\sigma$ is a Jordan polynomial in the generators $\mathbf{x} = (x_1, \dots, x_m)$, $x_i \in X^\sigma$, and $\mathbf{y} = (y_1, \dots, y_n)$, $y_j \in X^{-\sigma}$, and φ: $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{u}, \mathbf{v}) \in V^\sigma \times \cdots \times V^\sigma \times V^{-\sigma} \times \cdots \times V^{-\sigma}$ (m factors of V^σ, n factors of $V^{-\sigma}$) is a substitution in V we put $\Phi^\sigma(f) = f(\mathbf{u}, \mathbf{v}) = f(u_1, \dots, u_m, v_1, \dots, v_n)$, and say it results from f by the substitution Φ. We say that a Jordan polynomial $f \in \mathrm{FJP}(\mathbf{X})^\sigma$ *is an identity of a Jordan pair V* if $\Phi^\sigma(f) = 0$ for all substitutions in V and if f is monic in the sense that some leading monomial in f has coefficient 1.

Warning. Our concepts differ somewhat from the ones used in [D1], [DMc2], or [McZ]. We consider free Jordan pairs only over $\mathbb{Z}$ and correspondingly Jordan polynomials and polynomial identities all have integer coefficients. We will remind the reader of this by speaking of *integral* Jordan polynomials and *integral* polynomial identities. Also, we do not require that a polynomial identity of a Jordan pair V holds in all scalar extensions of V (see, however, criterion (3.2)).

For $\mathbf{X} = (\{x_1^+, x_2^+, \ldots, x_n^+\}, \{x_1^-, x_2^-, \ldots, x_n^-\})$ and $\sigma = \pm$ we put

$$\begin{aligned} \mathrm{DDQ}_n^\sigma(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_1^{-\sigma}, x_2^{-\sigma}, \ldots, x_n^{-\sigma}) &= D(x_n^\sigma, x_n^{-\sigma}) \cdots D(x_2^\sigma, x_2^{-\sigma}) Q(x_1^\sigma) x_1^{-\sigma} \\ &= \{\cdots\{\{\{Q(x_1^\sigma) x_1^{-\sigma}, x_2^{-\sigma}, x_2^\sigma\} x_3^{-\sigma}, x_3^\sigma\} \cdots\} x_n^{-\sigma}, x_n^\sigma\}. \end{aligned}$$

The *inner Capelli polynomial* is then defined as $\mathrm{IC}_n = (\mathrm{IC}_n^+, \mathrm{IC}_n^-) \in \mathrm{FJP}(\mathbf{X})$,

$$\begin{aligned} &\mathrm{IC}_n^\sigma(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_1^{-\sigma}, x_2^{-\sigma}, \ldots, x_n^{-\sigma}) \\ &\quad = \sum_{\tau \in S_n} (-1)^\tau \mathrm{DDQ}_n^\sigma\left(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_{\tau(1)}^{-\sigma}, x_{\tau(2)}^{-\sigma}, \ldots, x_{\tau(n)}^{-\sigma}\right), \end{aligned}$$

where $(-1)^\tau$ denotes the signature of the permutation τ in the symmetric group S_n (see [RR] and [RR2]). The Jordan polynomial IC_n^σ is an alternating multilinear function in the generators $x_i^{-\sigma}$ and hence an identity of any Jordan pair for which $V^{-\sigma}$ is spanned by fewer than n elements. On the other hand, the following theorem gives a criterion for IC_n^σ not to be an identity.

To establish this result we prove the existence of a special substitution which is analogous to the concept of an Amitsur–Levitzki staircase sequence for the standard polynomial for matrix algebras. For a fixed $\sigma \in \{\pm\}$ we define an *nth order Capelli sequence in a Jordan pair V* as a pair of sequences $(\mathbf{u}; \mathbf{v}) = (u_1, \ldots, u_n; v_1, \ldots, v_n)$, $(u_i, v_i) \in V^\sigma \times V^{-\sigma}$, such that

$$\mathrm{DDQ}_n^\sigma(u_1, u_2, \ldots, u_n; v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)}) = 0$$

for every permutation $\tau \in S_n$ with $\tau \neq 1$, and thus $\mathrm{IC}_n^\sigma(\mathbf{u}; \mathbf{v}) = \mathrm{DDQ}_n^\sigma(\mathbf{u}; \mathbf{v})$. Obviously, the interest lies in those sequences with $\mathrm{IC}_n^\sigma(\mathbf{u}; \mathbf{v}) \neq 0$. But because of the situation in a Jordan pair of hermitian matrices in characteristic 2 (see the Theorem in Section 2 below), we have not included this condition as part of the definition of a Capelli sequence.

We give an example of a Capelli sequence in the Jordan pair I_{pq}, $p \le q$, of rectangular $p \times q$ matrices over a base ring k with Jordan pair product given by $Q_x y = xy^{\mathrm{T}}x$. Let E_{ij} denote the usual rectangular matrix units: the (ij)-entry of E_{ij} is 1 while all other entries of E_{ij} are 0. The pairs (u_j, v_j) of an nth order Capelli sequence, $n = pq$, and $\sigma = +$ are

$$
\begin{aligned}
&(E_{11}, E_{11}), (E_{21}, E_{21}), \dots, (E_{p-1,1}, E_{p-1,1}), (E_{p1}, E_{p1}),\\
&(E_{1q}, E_{1q}), (E_{1,q-1}, E_{1,q-1}), \dots, (E_{13}, E_{13}), (E_{22}, E_{12}),\\
&(E_{2q}, E_{2q}), (E_{2,q-1}, E_{2,q-1}), \dots, (E_{23}, E_{23}), (E_{32}, E_{22}),\\
&(E_{3q}, E_{3q}), (E_{3,q-1}, E_{3,q-1}), \dots, (E_{33}, E_{33}), (E_{42}, E_{32}),\\
&\dots, \dots, \quad \dots \quad \dots, \dots\\
&(E_{p-1,q}, E_{p-1,q}), (E_{p-1,q-1}, E_{p-1,q-1}), \dots,\\
&\qquad (E_{p-1,3}, E_{p-1,3}), (E_{p2}, E_{p-1,2}),\\
&(E_{pq}, E_{pq}), (E_{p,q-1}, E_{p,q-1}), \dots, (E_{p3}, E_{p3}), (E_{p2}, E_{p,2}).
\end{aligned}
$$

Note the "wrinkle" in the choice of the u_js in the last entry of the second-to-last row: the u_js miss E_{12} but repeat E_{p2}. For this Capelli sequence one finds $\mathrm{IC}_n^+(\mathbf{u}, \mathbf{v}) = E_{p1}$. The reader may prove this now or specialize the proof of the following Theorem, Section 2, which establishes a more general result.

This generalization arises from the observation that the Jordan pair I_{pq} is a Jordan pair containing a finite connected standard grid, namely, the rectangular grid $\mathscr{R}(p, q) = \{(E_{ij}, E_{ij});\ 1 \le i \le p,\ 1 \le j \le q\}$, and that the Capelli sequence above comprises $\pm$-parts of idempotents in this grid (most but not all (u_j, v_j) are idempotents in $\mathscr{R}(p, q)$). We claim that Capelli sequences always exist in Jordan pairs containing grids. In particular, we will give an inductive construction of Capelli sequences which works for all Jordan pairs containing orthocollinear connected standard grids and which produces the sequence above when applied to the Jordan pair I_{pq} and the rectangular grid $\mathscr{R}(p, q)$. Regarding grids and standard grids in Jordan pairs the reader is referred to [N2] and [N3]. Some of the properties of standard grids are reviewed in the proof of the Theorem, Section 2, but we recall here that a connected standard grid is either an orthocollinear grid (any two idempotents in the grid are either equal, orthogonal, or collinear) or an odd quadratic form grid or a hermitian grid, see also Section 3 below.

2. EXISTENCE OF CAPELLI SEQUENCES

THEOREM. *Suppose V contains a finite connected standard grid $\mathcal{G}$ of size $|\mathcal{G}| = n$. Then there exists a Capelli sequence $(\mathbf{u}; \mathbf{v})$ in V such that*

(a) $\mathbf{u} = (u_1, \dots, u_n) \subset \mathcal{G}^\sigma := \{g^\sigma;\ g = (g^+, g^-) \in \mathcal{G}\}$,

(b) $\mathbf{v} = (v_1, \dots, v_n) \subset \mathcal{G}^{-\sigma}$ *is an enumeration of* $\mathcal{G}^{-\sigma}$,

(c) *if $\mathcal{G}$ is not a hermitian grid then* $\mathrm{IC}_n^\sigma(\mathbf{u}; \mathbf{v}) = \pm e^\sigma$ *for some* $e \in \mathcal{G}$, *while in the case of a hermitian grid* $\mathcal{G} = \{h_{ij};\ 1 \le i \le j \le r\}$ *of rank* $r \ge 2$ *we have* $\mathrm{IC}_n^\sigma(\mathbf{u}; \mathbf{v})^\sigma = 2^{r-2} h_{1r}^\sigma$.

Proof. We will first consider orthocollinear grids and construct the sequences $\mathbf{u}$ and $\mathbf{v}$ for all such grids at once. Thus, let $\mathcal{G}$ be a connected orthocollinear standard grid. We will need the following known facts about $\mathcal{G}$:

Fact 1. For any $g, h \in \mathcal{G}$ the product $Q(g^\sigma)h^{-\sigma}$ is zero unless $g = h$, in which case $Q(g^\sigma)h^{-\sigma} = g^\sigma$.

Fact 2. If $g_1, g_2 \in \mathcal{G}$ are collinear ($g_1 \in V_1(g_2)$ and $g_2 \in V_1(g_1)$, denoted $g_1 \top g_2$) then for any $g \in \mathcal{G}$ the product $\{g_1^\sigma g^{-\sigma} g_2^\sigma\}$ is either zero or $g = g_i$, $i = 1, 2$, in which case, respectively, $\{g_1^\sigma g_1^{-\sigma} g_2^\sigma\} = g_2^\sigma$ or $\{g_1^\sigma g_2^{-\sigma} g_2^\sigma\} = g_1^\sigma$.

Fact 3. If $g_1, g_2 \in \mathcal{G}$ are two orthogonal idempotents ($g_1 \perp g_2$) then for any $g \in \mathcal{G}$ the product $\{g_1^\sigma g^{-\sigma} g_2^\sigma\}$ either vanishes or (g_1, g, g_2) is a *hook*, i.e., $g_1 \top g \top g_2 \perp g_1$. In the latter case, there exists $\epsilon \in \{\pm 1\}$ and $e \in \mathcal{G}$ such that $(g_1, g, g_2, \epsilon e)$ is a quadrangle of idempotents. In particular, $g \perp e \top g_i$, $i = 1, 2$, and $\{g_1^\sigma g^{-\sigma} g_2^\sigma\} = \epsilon e^\sigma$.

For $\mathcal{F} \subset \mathcal{G}$ and $g \in \mathcal{G}$ we put $\mathcal{F}_i(g) = \{f \in \mathcal{F};\ f \in V_i(g)\}$. We will use subfamilies $\mathcal{F}$ of $\mathcal{G}$ and elements $e \in \mathcal{G} \setminus \mathcal{F}$ which are *hooked up* to $\mathcal{F}$: for any $f \in \mathcal{F}$ with $f \perp e$ there exists $\hat{f} \in \mathcal{F}$ such that e is hooked to f via $\hat{f}$, i.e., $(e, \hat{f}, f)$ is a hook. An example of such a configuration is:

$$\begin{gathered} \text{Let } \mathcal{F} \subset \mathcal{G} \text{ be a connected subgrid of } \mathcal{G}. \\ \text{Then any } e \in \mathcal{G} \setminus \mathcal{F} \text{ with } \mathcal{F}_1(e) \neq \emptyset \text{ is hooked up to } \mathcal{F}. \end{gathered} \tag{1}$$

Indeed, let $f \in \mathcal{F}$ with $f \perp e$. By assumption we know that there exists $\hat{f} \in \mathcal{F}$ with $e \top \hat{f}$. If $\hat{f} \top f$ we are done: e is hooked to f via $\hat{f} \in \mathcal{F}$. Otherwise $\hat{f} \perp f$, and by Fact 3 applied to the connected grid $\mathcal{F}$ there exist $f_1, f_2 \in \mathcal{F}$ such that $(\hat{f}, f_1, f, \pm f_2) \subset \mathcal{F}$ is a quadrangle of idempotents. By orthocollinearity of $\mathcal{G}$ we have $f_1 \in V_i(e) \cap V_1(f)$ for $i = 1$ or 0. If $i = 1$ then e is hooked up to f via $f_1 \in \mathcal{F}$. If $i = 0$ then e is hooked up to f via

$f_2 \in \mathscr{F}$: by the Peirce multiplication rules

$$f_2 = \pm\left(\left\{\hat{f}^+ f_1^- f^+\right\}, \left\{\hat{f}^- f_1^+ f^-\right\}\right) \in V_1(e) \cap V_1(f).$$

We also need:

$$\text{If } \mathscr{F} \subset \mathscr{G} \text{ is a connected subgrid then so is } \mathscr{F}_0(g) \text{ for any } g \in \mathscr{G}. \quad (2)$$

That $\mathscr{F}_0(g)$ is again a standard grid is immediate from the definitions. So it remains to prove connectivity, i.e., any orthogonal pair $(f_1, f_3) \subset \mathscr{F}_0(g)$ imbeds in a hook in $\mathscr{F}_0(g)$. But since $\mathscr{F}$ is connected there exists an orthogonal pair $(f_2, f_4) \subset \mathscr{F}$ such that $(f_1, f_2, f_3, \pm f_4)$ is a quadrangle, and a quadrangle is orthogonal to g as soon as two opposite corners are: since $f_1 + f_3 \approx f_2 + f_4$ we have $g \in V_0(f_1 + f_3) = V_0(f_2 + f_4)$, and thus $f_2, f_4 \in \mathscr{F}_0(g)$.

We are now ready to construct a Capelli sequence with properties (a)–(c) for an orthocollinear grid $\mathscr{G}$. If $n = 1$ we are trivially done: $\mathscr{G} = \{e_1\}$ and $(u; v) = (e_1^\sigma; e_1^{-\sigma})$ is a Capelli sequence for IC_n^σ. So in the following let $n > 1$. The enumeration (v_i) of $\mathscr{G}^{-\sigma}$ and the elements $u_i \in \mathscr{G}^\sigma$ comprising the Capelli sequence will be constructed inductively. The ith induction step $(i \geq 1)$ will use the following data:

(i) a subset $\mathscr{G}^i \subset \mathscr{G}$ which is a connected subgrid for $i \geq 2$,

(ii) $e_i \in \mathscr{G} \setminus \mathscr{G}^i$ which is hooked up to $\mathscr{G}^i$,

(iii) a choice of $(u_1, \dots, u_{mi})$, $u_j \in \mathscr{G}^\sigma$, and an enumeration $(v_1, \dots, v_{m_i})$ of $(\mathscr{G} \setminus \mathscr{G}^i)^{-\sigma}$ such that, regardless of the choice of the following $u_{m_i+1}, \dots, u_n$ and the enumeration $(v_{m_i+1}, \dots, v_n)$ of $(\mathscr{G}^i)^{-\sigma}$, we have

$$\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v})) = 0 \qquad \text{unless } \tau(j) = j \text{ for } 1 \leq j \leq m_i,$$

where $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v})) := \mathrm{DDQ}_n^\sigma(u_1, u_2, \dots, u_n; v_{\tau(1)}, v_{\tau(2)}, \dots, v_{\tau(n)})$, and

$$\left\{\{\dots\{Q(u_1)v_1, v_2, u_2\} \cdots \}v_{m_i}, u_{m_i}\right\} = \pm e_i^\sigma.$$

Thus, for $\tau \in \mathrm{S}_n$ with $\tau(j) = j$ for $1 \leq j \leq m_i$ we have

$$\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v})) = \pm\left\{\left\{\dots\left\{e_i^\sigma, v_{\tau(m_i+1)}^{-\sigma}, u_{m_i+1}^\sigma\right\}\dots\right\} v_{\tau(n)}^{-\sigma} u_n^\sigma\right\}.$$

In the ith induction step we will construct the data (i)–(iii) for $i + 1$ such that $\mathscr{G}^{i+1}$ is a proper subset of $\mathscr{G}^i$. After a finite number of steps this process stops, producing a Capelli sequence with properties (a)–(c). The reader may want to keep in mind the example above, which arises from the general construction in p steps by taking $e_i = (E_{i1}, E_{i1})$, $1 \leq i \leq p$, and $\mathscr{G}^i = \{(E_{lj}, E_{lj};\ i \leq l \leq p,\ 2 \leq j \leq q\}$ for $i \geq 2$.

Beginning of Induction ($i = 1$). We start by choosing arbitrarily some $e_1 \in \mathscr{G}$. By Fact 3, e_1 is hooked up to $\mathscr{G}^1 := \mathscr{G} \setminus \{e_1\}$. We let $(u_1, v_1) = (e_1^{\sigma}, e_1^{-\sigma})$. Then $Q(u_1)v_1 = e_1^{\sigma}$ and, by Fact 1, for any enumeration $(v_2, \dots, v_n)$ of the remaining $g^{-\sigma}$ in $\mathscr{G}^{1,\sigma}$ the term $Q(u_1)v_{\tau(1)}$ vanishes if $\tau(1) \neq 1$. Thus (i)–(iii) hold for $i = 1$ with $m_1 = 1$.

Induction Step. We suppose that we are given the data described in (i)–(iii) for some $i \geq 1$. We will distinguish two cases A and B depending on whether or not all idempotents of $\mathscr{G}^i$ are collinear to e_i.

Case A. Not all idempotents of $\mathscr{G}^i$ are collinear to e_i. Let $h \in \mathscr{G}^i$ with $h \perp e_i$. Since e_i is hooked up to $\mathscr{G}^i$, there exists $f \in \mathscr{G}^i$ such that (e_i, f, h) is a hook. By Fact 3, it can be completed to a quadrangle: there exists $e_{i+1} \in \mathscr{G}$ such that $(e_i, f, h, \pm e_{i+1})$ is a quadrangle of idempotents. We put $\mathscr{G}^{i+1} = \mathscr{G}_0^i(e_i)$. This is a proper subset of $\mathscr{G}^i$ since $f \in \mathscr{G}^i \setminus \mathscr{G}^{i+1}$. It is also a connected subgrid of $\mathscr{G}$: for $i = 1$ we have $\mathscr{G}^2 = \mathscr{G}_0(e_1)$ so that connectivity follows from (2); for $i \geq 2$ we know by induction that $\mathscr{G}^i$ is connected and hence again by (2) that $\mathscr{G}^{i+1}$ is connected. Thus (i) holds for $i + 1$. We also have (ii). Indeed, $e_i \top e_{i+1}$ implies that $e_{i+1} \notin \mathscr{G}^{i+1}$ and since $h \in \mathscr{G}_0^i(e_i) \cap \mathscr{G}_1(e_{i+1}) = (\mathscr{G}^{i+1})_1(e_{i+1})$ it follows from (1) that e_{i+1} is hooked up to $\mathscr{G}^{i+1}$. We now enumerate $\mathscr{G}_1^i(e_i) = (g_{m_i+1}, \dots, g_l = f)$, $l = m_{i+1}$, and choose

$$(u_{m_i+1}, \dots, u_{l-1}, u_l) = \left(g_{m_i+1}^{\sigma}, \dots, g_{l-1}^{\sigma}, h^{\sigma}\right),$$

$$(v_{m_i+1}, \dots, v_{l-1}, v_l) = \left(g_{m_i+1}^{-\sigma}, \dots, g_{l-1}^{-\sigma}, g_l^{-\sigma} = f^{-\sigma}\right).$$

Note the choice of (u_l, v_l)! By construction, all $u_j \in \mathscr{G}$, $1 \leq j \leq l$, and $(v_1, \dots, v_l)$ is an enumeration of $(\mathscr{G} \setminus \mathscr{G}^{i+1})^{-\sigma}$. To show the remaining parts of (iii) we suppose that the sequence $\{u_1, \dots, u_l; v_1, \dots, v_l)$ has been completed to a sequence $(\mathbf{u}; \mathbf{v})$ satisfying (a) and (b) of the theorem, and we let $\tau \in S_n$. We can assume $\tau(j) = j$ for $1 \leq j \leq m_i$. For $m_i + 1 < l = m_{i+1}$ we have $g_{m_i+1} \top e_i$ and hence, by Fact 2, $\{e_i^{\sigma}, v_{\tau(m_i+1)}, u_{m_i+1}\} = 0$ unless $\tau(m_i + 1) = m_i + 1$, in which case $\{e_i^{\sigma}, v_{m_i+1}, u_{m_i+1}\} = e_i$. Analogously, for any $j < l$ we have

$$\left\{\left\{ \dots \left\{e_i^{\sigma}, v_{\tau(m_i+1)}, u_{m_i+1}\right\} \dots \right\} v_{\tau(j)}, u_j\right\} = 0$$

unless $\tau(k) = k$ for $m_i \leq k \leq j$, and in this case the product equals e_i. Finally, we consider the product $\{e_i^{\sigma}, v_{\tau(l)}, u_l\} = \{e_i^{\sigma}, v_{\tau(l)}, h^{\sigma}\}$. We can assume that $\tau(j) = j$ for $1 \leq j < l$. Then $v_{\tau(l)} = g^{-\sigma}$ for some $g \in \{f\} \cup \mathscr{G}_0^i(e_i)$. Hence, by Fact 3, this product vanishes unless $g = f$, i.e., $\tau(l) = l$, and in this case we obtain $\{\{ \dots \{e_i^{\sigma}, v_l, h^{\sigma}\} \dots \}\} = \pm e_{i+1}$. This finishes the induction step in Case A.

Case B. All idempotents in $\mathscr{G}^n$ are collinear to e_i. In this case the induction stops: we enumerate $\mathscr{G}^i = (g_{m_i+1}, \dots, g_n)$ arbitrarily and put $(u_j, v_j) = (g_j^\sigma, g_j^{-\sigma})$ for $m_i < j \le n$. Then $(\mathbf{u}; \mathbf{v}) = (u_1, \dots, u_n; v_1, \dots, v_n)$ satisfies (a) and (b) of the theorem, and we claim that it is also a Capelli sequence with property (c). Indeed, it is enough to consider $\tau \in \mathrm{S}_n$ with $\tau(j) = j$ for $1 \le j \le m_i$. Since then $v_{\tau(m_i+1)} \ne e_i^\sigma$ while $g_{m_i+1} \top e_i$, Fact 2 shows that $\{e_i, v_{\tau(m_i+1)}, u_{m_i+1}\} = 0$ unless $\tau(m_i + 1) = m_i + 1$, and in this case $\{e_i^\sigma, v_{m_i+1}, u_{m_i+1}\} = e_i^\sigma$, so

$$\mathrm{DDQ}_n^\sigma(\mathbf{u}; \tau(\mathbf{v})) = \pm\Big\{ \dots \Big\{\big\{e_i^\sigma, v_{\tau(m_i+2)}^{-\sigma}, u_{m_i+2}^\sigma\big\} \dots \Big\} v_{\tau(n)}^{-\sigma} u_n^\sigma \Big\}.$$

Repeating this argument shows $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v})) = 0$ unless $\tau = 1$, in which case $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \mathbf{v}) = \pm e_i^\sigma$.

Assume now that $\mathscr{G} = \mathscr{Q}_o$ is an odd quadratic form grid [N2, II.1.1]: $\mathscr{Q}_o = \{g_0\} \cup \mathscr{Q}_e$, where g_0 governs every $g \in \mathscr{Q}_e$ ($g \in V_2(g_0)$ and $g_0 \in V_1(g)$, denoted $g_0 \vdash g$) and $\mathscr{Q}_e = \{g_{\pm i}; 1 \le i \le m\}$, $m = (n-1)/2$, is an even quadratic form grid, i.e., $g_{+i} \perp g_{-i}$ and $g_{\pm i} \top g_{\pm j}$ for $i \ne j$. In this case we can explicitly list a Capelli sequence. Our choice is analogous to the orthocollinear case: we choose $e_1 = g_{+1}$, let $(u_1; v_1) = (g_{+1}^\sigma; g_{+1}^{-\sigma})$, enumerate $\mathscr{G}_1(g_{+1}) = \mathscr{Q}_o \setminus \{g_{\pm 1}\}$, and build in a wrinkle at the end of the sequence. In precise terms, we let

$$\begin{array}{llll} u_1 = g_{+1}^\sigma, \dots, & u_m = g_{+m}^\sigma, & u_{m+1} = g_{-m}^\sigma, & \dots, \\ u_{n-2} = g_{-2}^\sigma, & u_{n-1} = g_{-1}^\sigma = u_n, & & \\ v_1 = g_{+1}^{-\sigma}, \dots, & v_m = g_{+m}^{-\sigma}, & v_{m+1} = g_{-m}^{-\sigma}, & \dots, \\ v_{n-2} = g_{-2}^{-\sigma}, & v_{n-1} = g_0^{-\sigma}, & v_n = g_{-1}^{-\sigma}. & \end{array}$$

In $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v}))$ the product $Q(u_1)v_{\tau(1)}$ is nonzero only if $\tau(1) = 1$. For $\tau(1) = 1$ we have $Q(u_1)v_1 = u_1$, and for $2 \le j \le n-2$ a product $\{u_1 v_{\tau(j)} u_j\}$ is nonzero only if $\tau(j) = j$ since $\{g_{\pm i}^\sigma g_0^{-\sigma} g_{\pm j}^\sigma\} = 0$ for $i \ne j$. Therefore $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v}))$ vanishes unless $\tau(j) = j$ for $1 \le j \le n-2$, and in this case $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v})) = \{\{g_1^\sigma, v_{\tau(n-1)}, g_{-1}^\sigma\} v_{\tau(n)} g_{-1}^\sigma\}$ where $\{v_{\tau(n-1)}, v_{\tau(n)}\} = \{g_{-1}^{-\sigma}, g_0^{-\sigma}\}$ (equality of sets). Since $g_1 \perp g_{-1}$ we must have $v_{\tau(n-1)} = g_0^{-\sigma}$ and $v_{\tau(n)} = g_{-1}^{-\sigma}$ for $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \tau(\mathbf{v}))$ to be nonzero. Thus $\tau = 1$ and $\mathrm{DDQ}_n^\sigma(\mathbf{u}, \mathbf{v}) = \{g_0^\sigma g_{-1}^{-\sigma} g_{-1}^\sigma\} = g_0^\sigma$.

Finally we consider a hermitian grid $\mathscr{H} = \{h_{ij};\ 1 \le i \le j \le r\}$ of rank $r \ge 2$ [N2, II.1.2]. But since, for $r = 2$, $\mathscr{H}$ is an odd quadratic form grid we can assume $r \ge 3$. We recall that the relations and multiplication rules of the idempotents $h_{ij} \in \mathscr{H}$ are an axiomatization of the relations and multiplication rules satisfied by the "hermitian matrix units" $h_{ii} = (E_{ii}, E_{ii})$ and $h_{ij} = (E_{ij} + E_{ji}, E_{ij} + E_{ji})$, $i \ne j$, in the Jordan pair of hermitian matrices.

In particular, if we put $h_{ij} = h_{ji}$, we have the following relations for distinct i, j, k, l:

$$h_{ij} \vdash h_{ii} \perp h_{jj}, \qquad h_{ij} \top h_{ik}, \qquad h_{ij} \perp h_{kl}. \tag{3}$$

Also in this case we can list Capelli sequences $(\mathbf{u}; \mathbf{v})$ all at once. The pairs (u_j, v_j) are

$$\begin{aligned}
&(h_{11}^{\sigma}, h_{11}^{-\sigma}), (h_{1r}^{\sigma}, h_{1r}^{-\sigma}), (h_{1,r-1}^{\sigma}, h_{1,r-1}^{-\sigma}), \dots, (h_{13}^{\sigma}, h_{13}^{-\sigma}), (h_{22}^{\sigma}, h_{12}^{-\sigma}),\\
&\dots\\
&(h_{ii}^{\sigma}, h_{ii}^{-\sigma}), (h_{ir}^{\sigma}, h_{ir}^{-\sigma}), (h_{i,r-1}^{\sigma}, h_{i,r-1}^{-\sigma}), \dots,\\
&\qquad (h_{i,i+2}^{\sigma}, h_{i,i+2}^{-\sigma}), (h_{i+1,i+1}^{\sigma}, h_{i,i+1}^{-\sigma}),\\
&\dots\\
&(h_{r-2,r-2}^{\sigma}, h_{r-2,r-2}^{-\sigma}), (h_{r-2,r}^{\sigma}, h_{r-2,r}^{-\sigma}), (h_{r-1,r-1}^{\sigma}, h_{r-2,r-1}^{-\sigma}),\\
&(h_{r-1,r-1}^{\sigma}, h_{r-1,r-1}^{-\sigma}), (h_{rr}^{\sigma}, h_{r-1,r}^{-\sigma}),\\
&(h_{rr}^{\sigma}, h_{rr}^{-\sigma}).
\end{aligned}$$

The reader will notice that this sequence is constructed in a way similar to the orthocollinear case. One proceeds in r steps, with auxiliary idempotents $e_i = h_i$ and subgrids $\mathscr{H}^i = \{h_{pq};\ i \le p \le q \le r\}$, $1 \le i \le r$. The ith step for $1 \le i < r$ corresponds to the orthocollinear Case A. After having chosen $l = (i-1)(r - i/2)$ elements u_j and an enumeration of $\mathscr{H} \setminus \mathscr{H}^i$ one puts $(u_{l+1}, v_{l+1}) = (e_i^{\sigma}, e_i^{-\sigma})$, chooses an enumeration $(g_{l+2}, \dots, g_m)$, $m = i(r - (i+1)/2)$ of $\mathscr{H}_1^i(e_i)$, puts $(u_j, v_j) = (g_j^{\sigma}, g_j^{-\sigma})$, $l + 2 \le j \le m - 1$, and builds in a wrinkle at the end by putting (u_m, v_m) $= (e_{i+1}^{\sigma}, g_m^{-\sigma})$, where $(g_m; e_i; e_{i+1})$ is a triangle in the sense of [N2, I.2.1]. One then continues with the $(i+1)$th step for which $\mathscr{H}^{i+1} = \mathscr{H}_0^i(e_i)$. In the final rth step one has $\mathscr{H}^r = \{e_r\}$, which corresponds to the orthocollinear Case B.

It remains to prove that the sequence $(\mathbf{u}; \mathbf{v})$ above is a Capelli sequence with $\mathrm{IC}_n^{\sigma}(\mathbf{u}, \mathbf{v}) = 2^{r-2} h_{1r}^{\sigma}$. By (3) we have $\mathscr{H}_2(h_{11}) = \{h_{11}\}$, hence $Q(h_{11}^{\sigma}) h_{pq}^{-\sigma} = 0$ unless $pq = 11$, in which case we get h_{11}^{σ}. One now has to consider products $\{h_{11}^{\sigma}, h_{pq}^{-\sigma}, h_{1j}^{\sigma}\}$ for $j > 1$. Since $h_{11} \in \mathscr{H}_2(h_{1j})$ such a product vanishes unless also $h_{pq} \in \mathscr{H}_2(h_{1j})$. But, by (3), $\mathscr{H}_2(h_{1j}) = \{h_{11}, h_{1j}, h_{jj}\}$ and $\{h_{11}^{\sigma}, h_{jj}^{-\sigma}, h_{1j}^{\sigma}\} = 0$. Since $pq = 11$ was already chosen we must have $pq = 1j$, in which case $\{h_{11}^{\sigma}, h_{1j}^{-\sigma}, h_{1j}^{\sigma}\} = 2h_{11}^{\sigma}$. At the end of the first row we have

$$\left\{\left\{ \dots \{Q(u_1) v_1, v_2, u_2\} \dots \right\} v_{\tau(r)} u_r\right\} = 2^{r-2}\left\{h_{11}^{\sigma}, h_{pq}^{-\sigma}, h_{22}^{\sigma}\right\},$$

which vanishes unless $h_{pq} \in \mathscr{H}_2(h_{11} + h_{22}) = \{h_{11}, h_{12}, h_{22}\}$. The only possible choice left is therefore $pq = 12$, in which case we obtain

$\{h_{11}^{\sigma}, h_{12}^{-\sigma}, h_{22}^{\sigma}\} = h_{12}^{\sigma}$ by the multiplication rules for hermitian grids. At the end of the $(i-1)$th row we arrive at a product $\{\{\ldots\{Q(u_1)v_1, v_2, u_2\}\ldots\}v_l u_l\} = 2^{r-2}h_{1i}^{\sigma}$: each term $\mathrm{DDQ}_n^{\sigma}(\mathbf{u}, \tau(\mathbf{v}))$ vanishes unless τ fixes the first $i-1$ rows of hs. The same continues to hold in the ith row: $\{h_{1i}^{\sigma}, h_{pq}^{-\sigma}, h_{ii}^{\sigma}\}$ for $i \le p, q$ is zero unless $pq = ii$, in which case it reproduces h_{1i}^{σ}, and $\{h_{1i}^{\sigma}, h_{pq}^{-\sigma}, h_{ij}^{\sigma}\}$ for $i \le p, q$ and $j \ge i+2$ vanishes by rigid collinearity of h_{1i} and h_{ij} unless $pq = 1i$ or $pq = ij$. Therefore $pq = ij$ and in this case we get $\{h_{1i}^{\sigma}, h_{ij}^{-\sigma}, h_{ij}^{\sigma}\} = h_{1i}^{\sigma}$. At the end of the ith row we have to consider $\{h_{1i}^{\sigma}, h_{pq}^{-\sigma}, h_{i+1,i+1}^{\sigma}\}$, $i \le p, q$. Since $h_{1i} \perp h_{i+1,i+1}$ this product is 0 except for $(pq) = (i, i+1)$, when it produces $h_{1,i+1}^{\sigma}$. Continuing in this way proves that $(\mathbf{u}; \mathbf{v})$ is a Capelli sequence with $\mathrm{IC}_n^{\sigma}(\mathbf{u}; \mathbf{v}) = 2^{r-2}h_{1r}^{\sigma}$.

Remark. The proof above is inspired by the proofs of Propositions 16–19 of [RR] and of Proposition 1 of [RR2], where special cases of the theorem were proven. Indeed, suppose $T = V^+ \oplus V^-$ is the polarized Jordan triple system of a Jordan pair $V = (V^+, V^-)$. Any Jordan triple polynomial f on T has the form $f = f^+ \oplus f^-$ for a Jordan pair polynomial (f^+, f^-) of V. Hence, the theorem also holds for T and $\mathrm{IC}_n = \mathrm{IC}_n^+ \oplus \mathrm{IC}_n^-$. Interpreted in this way, it yields Propositions 16, 17, and 19 of [RR]. The theorem can also be interpreted for the Jordan pair (T, T) associated with a Jordan triple system T. In this way, one obtains Proposition 18 of [RR] and Proposition 1 of [RR2].

3. SPLIT JORDAN PAIRS

For the purpose of this paper, it is appropriate to call a Jordan pair V over some base ring k *split*, or *split of type* $\mathscr{G}$ in case we need to be more precise, if V is freely spanned by a finite connected grid $\mathscr{G}$: $V^{\sigma} = \bigoplus_{g \in \mathscr{G}} k \cdot g^{\sigma}$ for $\sigma = \pm$. In this case, we can assume that $\mathscr{G}$ is a finite connected standard grid [N3, 3.8], and hence V is obtained by base ring extension from the Jordan pair $\langle \mathscr{G} \rangle = \bigoplus_{g \in \mathscr{G}} (\mathbb{Z}g^+, \mathbb{Z}g^-)$ over $\mathbb{Z}$: $V = \langle \mathscr{G} \rangle \otimes_{\mathbb{Z}} k$. All base ring extensions of V are then again split of type $\mathscr{G}$, i.e., for any commutative unital k-algebra K we have $V \otimes_k K \approx \langle \mathscr{G} \rangle \otimes_{\mathbb{Z}} K$.

Any finite-dimensional simple Jordan pair over an algebraically closed field is split. This is obvious from the classification of [L, 17.12] and is proven without classification in [N3, 3.11]. The classification of standard grids [N3] shows that over any given base ring k there are the following six types of split Jordan pairs which we describe using the notation of [L, 17.12]. We also give the dimension and indicate if the Jordan pair has invertible elements:

Split V	Grid $\mathscr{G}$	Dimension	Inv. elem.
$\mathrm{I}_{pq}(1 \le p \le q)$	Rectangular grid $\mathscr{R}(p, q)$	pq	If $p = q$
$\mathrm{II}_n(n \ge 5)$	Symplectic grid $\mathscr{S}(n)$	$\frac{n(n-1)}{2}$	If $n \equiv 0(2)$
$\mathrm{III}_n(n \ge 2)$	Hermitian grid $\mathscr{H}(n)$	$\frac{n(n+1)}{2}$	Yes
$\mathrm{IV}_n(n \ge 5)$	Quadratic form grid $\mathscr{Q}_e(n)$ or $\mathscr{Q}_0(n)$	n	Yes
V	Bi-Cayley grid $\mathscr{B}$	16	No
VI	Albert grid $\mathscr{A}$	27	Yes

The types IV_n can of course be defined for every n, but become isomorphic to other types for small n. In particular, we have

$$\mathrm{III}_2 \approx \mathrm{IV}_3, \qquad \mathrm{I}_{22} \approx \mathrm{IV}_4. \tag{1}$$

An integral Jordan polynomial $f \in \mathrm{FJP}(\mathbf{X})^\sigma$ is a *strict identity* of some Jordan pair V over k if f is an identity for all base ring extensions $V \otimes_k K$ of V. For split Jordan pairs of type $\mathscr{G}$ the following conditions are equivalent:

(2.a) f is a strict identity of the integral Jordan pair $\langle \mathscr{G} \rangle$, i.e., an identity for all split Jordan pairs of type $\mathscr{G}$;

(2.b) f is an identity of the complex Jordan pair $\langle \mathscr{G} \rangle \otimes \mathbb{C}$.

Indeed, there exist polynomials f_g over $\mathbb{Z}$ in a finite number of variables (depending on f and $\mathscr{G}$) such that for every k and every evaluation of f on $V = \langle \mathscr{G} \rangle \otimes k$ we have $f(\mathbf{u}, \mathbf{v}) = \sum_{g \in \mathscr{G}} g^\sigma \otimes f_g(\mathbf{u}, \mathbf{v})$. If f vanishes identically on $\langle \mathscr{G} \rangle \otimes \mathbb{C}$, then because $\mathbb{C}$ is infinite the polynomials f_g are the zero polynomials, hence $f = 0$ on V.

4. THE POLYNOMIAL ICQ_l

PROPOSITION. *Let* $\mathrm{ICQ}_l(x_1, \dots, x_l, z_1, \dots, z_l; y) \in \mathrm{FJP}(\{x_1, \dots, x_l, z_1, \dots, z_l\}, \{y\})^+$ *be defined by*

$$\mathrm{ICQ}_l(x_1, \dots, x_l, z_1, \dots, z_l; y) = \mathrm{IC}_l^+\left(x_1, \dots, x_l, Q_y z_1, \dots, Q_y z_l\right).$$

Then ICQ_l *is an identity for all split Jordan pairs of type* $\mathscr{G}$ *in the following cases*:

(a) $\mathscr{G} = \mathscr{R}(p, q)$, $p^2 < l$;

(b) $\mathscr{G} = \mathscr{S}(n)$, $n \equiv 1(2)$, $(n-1)(n-2) < 2l$.

However, ICQ_l *is not an identity of a split Jordan pair of dimension* l *containing invertible elements. In particular, if* $T(V)$ *denotes the T-ideal of identities of a Jordan pair* V *we have*

$$p \le p' \quad \textit{and} \quad q \le q' \quad \Leftrightarrow \quad I_{pq} \subset I_{p'q'} \quad \Leftrightarrow \quad T\big(I_{pq}\big) \supset T\big(I_{p'q'}\big). \tag{1}$$

Proof. ((a) and (b)). By (3.2) it suffices to show that ICQ_l is an identity for the Jordan pair $V = \langle \mathscr{G} \rangle \otimes \mathbb{C}$. Since $\mathrm{IC}_l^+(x_1, \dots, x_l, Q_y z_1, \dots, Q_y z_l)$ is alternating multilinear in $Q_y z_1, \dots, Q_y z_l$, it is enough to prove that every inner ideal $Q_v V^+$, $v \in V^-$, has dimension $< l$. But V is simple nondegenerate and hence regular, v is part of an idempotent $c = (c_+, v)$ of V. Therefore $Q_v V^+ = V_2^-(c) \subset V_2^-(e)$ for a maximal idempotent e of V. By the conjugacy theorem [L, 17.1], $V_2(e)$ has dimension p^2 in case (a) and dimension $\frac{1}{2}(n-1)(n-2)$ in case (b).

If V contains invertible elements there exists $v \in V^-$ such that $Q_v V^+ = V^-$ and hence by Section 2 a substitution for which ICQ_l does not vanish. Finally, with respect to (1), it is clear that $p \le p'$ and $q \le q'$ implies $\mathrm{I}_{pq} \subset \mathrm{I}_{p'q'}$, which in turn implies $T(\mathrm{I}_{pq}) \supset T(\mathrm{I}_{p'q'})$. Assuming $T(\mathrm{I}_{pq}) \supset T(\mathrm{I}_{p'q'})$ we will show $p \le p'$ and $q \le q'$: if $p > p'$ then $\mathrm{ICQ}_{p^2} \in T(\mathrm{I}_{p'q'})$ but $\mathrm{ICQ}_{p^2} \notin T(\mathrm{I}_{pq})$ since it is not an identity of $\mathrm{I}_{pp} \subset \mathrm{I}_{pq}$. Therefore $p \le q$, and because $\mathrm{I}_{pq} \approx \mathrm{I}_{qp}$ and $\mathrm{I}_{p'q'} \approx \mathrm{I}_{q'p'}$ we then also have $q \le q'$.

Remark. A different proof for the equivalences (1) is given in [I, Theorem 1].

5. JORDAN PAIR POLYNOMIALS OBTAINED FROM JORDAN ALGEBRA POLYNOMIALS

Let X be a set and let $\mathrm{FJA}(X)$ be the free nonunital Jordan algebra on X over $\mathbb{Z}$. We put $\mathbf{X} = (X, \{y\})$ for some $y \notin X$ and denote by $\mathrm{FJP}(\mathbf{X})_y^+$ the y-homotope of $\mathrm{FJP}(\mathbf{X})$ [L, 1.9]. By the universal property of $\mathrm{FJA}(X)$ there exists a unique (nonunital) Jordan algebra homomorphism ψ: $\mathrm{FJA}(X) \to \mathrm{FJP}(\mathbf{X})_y^+$ mapping every $x \in X \subset \mathrm{FJA}(X)$ onto $x \in X \subset \mathrm{FJP}(\mathbf{X})^+$. (It is easily seen that ψ is an isomorphism but we do not need this.) We define $g^{\mathrm{JP}} := \psi(g)$ for $g \in \mathrm{FJA}(X)$ and call it the *Jordan homotope polynomial associated with* g.

Intuitively, g^{JP} is obtained as follows: write g as a sum of monomials where each monomial is a composition of maps U_x, squaring operators $x \mapsto x^2$, and left multiplications $V_{x,z}$ defined by $V_{x,z}u = \{xzu\} = Q_{x+u}z - Q_x z - Q_u z$ for $x, z, u \in \mathrm{FJA}(X)$; then replace each factor U_x by $U_x^{(y)} = Q_x Q_y$, squaring operators x^2 by $x^{(2,y)} = Q(x)y$, and the left multiplications $V_{x,z}$ by $V_{x,z}^{(y)} = D(x, Q_y z)$. For example, the polynomial $\mathrm{ICQ}_l(\mathbf{x}, \mathbf{z}; y)$ of Proposition 4 is a Jordan homotope polynomial since the term $\mathrm{DDQ}^+(\mathbf{x}, \mathbf{z}; y) = D(x_n, Q_y z_n) \cdots D(x_2, Q_y z_2) Q_{x_1} Q_y z_1$ is the image under ψ of the Jordan algebra polynomial $\mathrm{VVU}(\mathbf{x}, \mathbf{z}) = V_{x_n, z_n} \cdots V_{x_2, z_2} U_{x_1} z_1$. We will later use the following examples of homotope polynomials:

(a) (Racine's central polynomials) Let $n \geq 3$. By [R, Theorem 3] there exist homogeneous integral polynomials $\mathrm{R}_n(x_1, x_2) \in \mathrm{FJA}(\{x_1, x_2\})$ which are central polynomials of the Jordan algebra $J = \mathrm{H}_n(\mathscr{C})$ of hermitian matrices over an associative composition algebra $\mathscr{C}$ over a field k, i.e., $\mathrm{R}_n(J, J) \subset k \cdot 1$, where 1 is the identity element of J. Since the map $V_{x_3, x_4} - V_{x_4, x_3}$ is a derivation, the derived version of $\mathrm{R}_n(x_1, x_2)$,

$$\mathrm{DR}_n(x_1, x_2, x_3, x_4) = \{x_3, x_4, \mathrm{R}_n(x_1, x_2)\} - \{x_4, x_3, \mathrm{R}_n(x_1, x_2)\},$$

is then a homogeneous integral polynomial identity of J. The associated integral homotope polynomials will be denoted $\mathrm{R}_n(x_1, x_2; y) := \mathrm{R}_n(x_1, x_2)^{\mathrm{JP}}$ and $\mathrm{DR}_n(x_1, x_2, x_3, x_4; y) := \mathrm{DR}(x_1, x_2, x_3, x_4; y)^{\mathrm{JP}}$ and called, respectively, the *Racine homotope polynomial* and the *derived Racine homotope polynomial.*

(b) We recall from [McZ, (0.25) and (7.6)] that in a Jordan algebra the *commutator square* is defined as $\mathrm{C}(x_1, x_2) = x_1 \circ U_{x_2} x_1 - U_{x_1} x_2^2 - U_{x_2} x_1^2$, and the *standard Clifford polynomial* is

$$\mathrm{SC}(x_1, x_2, x_3, x_4) = \{\mathrm{C}(x_1, x_2), x_3, x_4\} - \{x_3, \mathrm{C}(x_1, x_2), x_4\}.$$

In the setting of Jordan triple systems the associated homotope polynomials were introduced in [RR]. We denote them by

$$\begin{aligned} \mathrm{C}(x_1, x_2; y) &:= \mathrm{C}(x_1, x_2)^{\mathrm{JP}} \\ &= \{x_1, y, Q_{x_2} Q_y x_1\} - Q_{x_1} Q_y Q_{x_2} y - Q_{x_2} Q_y Q_{x_1} y, \\ \mathrm{SC}(x_1, \ldots, x_4; y) &:= \mathrm{SC}(x_1, x_2, x_3, x_4)^{\mathrm{JP}} \\ &= \{\mathrm{C}(x_1, x_2; y), Q_y x_3, x_4\} - \{x_3, Q_y \mathrm{C}(x_1, x_2; y), x_4\}, \end{aligned}$$

and call $\mathrm{SC}(x_1, x_2, x_3, x_4; y)$ the *standard Clifford homotope polynomial.* As we will show in the following lemma, SC is in fact a Clifford homotope polynomial in the spirit of [DMc2]: it vanishes on quadratic form pairs but not on the Jordan pair of hermitian matrices of rank ≥ 3.

6. STANDARD CLIFFORD HOMOTOPE POLYNOMIALS

LEMMA [RR]. *The standard Clifford homotope polynomial SC does not vanish on any Jordan pair containing a rectangular grid $\mathscr{R}(2,3)$ or a hermitian grid $\mathscr{H}(3)$. On the other hand, SC is an identity of rectangular matrix pairs of size $1 \times q$ and of quadratic form pairs.*

In particular, SC *divides the split Jordan pairs into two groups: it does not vanish on* $\mathrm{I}_{pq}(2 \le p \le q,\ 3 \le q)$, $\mathrm{II}_n(5 \le n)$, $\mathrm{III}_n(3 \le n)$, V, *and* VI, *but it vanishes on* $\mathrm{I}_{1q}(1 \le q)$ *and* $\mathrm{IV}_n(3 \le n)$.

Proof. The first part follows essentially from [RR, Proposition 2] and the remark on pages 976–978 of [RR]. Indeed, for any triangle $(g; e_1, e_2)$ of idempotents in a Jordan pair V one easily calculates $\mathrm{C}(e_1^\sigma, e_2^\sigma; g^{-\sigma}) = g^\sigma$. If V contains a rectangular grid $\mathscr{R}(2,3) = (e_{ij};\ 1 \le i \le 2,\ 1 \le j \le 3)$ then $(e_{12} + e_{21}; e_{11}, e_{22})$ is a triangle, and since $Q(e_{12}^{-\sigma} + e_{21}^{-\sigma})e_{13}^\sigma = 0$ and $e_{13} \perp e_{21}$ we obtain

$$\mathrm{SC}(e_{11}^\sigma, e_{22}^\sigma, e_{13}^\sigma, e_{22}^\sigma; e_{12}^{-\sigma} + e_{21}^{-\sigma}) = 0 - \{e_{13}^\sigma, e_{12}^{-\sigma}, e_{22}^\sigma\} = -e_{23}^\sigma.$$

Similarly, if $\mathscr{H}(3) = (h_{ij},\ 1 \le i \le j \le 3) \subset V$ is a hermitian grid then $(h_{12}; h_{11}, h_{22})$ is a triangle and one verifies, using $Q(h_{12}^{-\sigma})h_{13}^\sigma$ by (2.3),

$$\mathrm{SC}(h_{11}^\sigma, h_{22}^\sigma, h_{13}^\sigma, h_{23}^\sigma; h_{12}^{-\sigma}) = 0 - \{h_{13}^\sigma, h_{12}^{-\sigma}, h_{22}^\sigma\} = -h_{23}^\sigma,$$

That SC vanishes on rectangular matrix pairs of size $1 \times q$ and on quadratic form pairs is shown in [RR, Propositions 1 and 2].

Concerning split Jordan pairs, one only has to observe that the first group contains $\mathscr{R}(2,3)$ or $\mathscr{H}(3)$ while in view of the isomorphisms (3.1) the second group is made up of special types of rectangular matrix pairs and quadratic form pairs.

7. IDENTITIES OF J AND OF (J, J)

LEMMA. *Let k be an infinite field of characteristic $\neq 2$, and let J be a finite-dimensional unital Jordan algebra over k. Then a homogeneous integral polynomial $g \in \mathrm{FJA}(X)$ is a polynomial identity of J if and only if the integral homotope polynomial g^{JP} is an identity of the Jordan pair $V = (J, J)$.*

Proof. Under a specialization $X^+ = X \to V^+ = J$ and $y \mapsto v \in V^- = J$, the polynomial g^{JP} becomes the polynomial g evaluated on the Jordan algebra V_v^+, which is nothing else but the v-homotope of J. Since $J = V_1^+$ for the unit element 1 of $J = V^-$, it is clear that g is an identity of J if g^{JP} is an identity of V. To prove the converse we may after a base field extension assume that k is algebraically closed. It suffices to prove that g^{JP} vanishes under any specialization of type $y \mapsto v$, where v belongs to the Zariski-dense subset of invertible elements of J. But for such a v the Jordan algebra V_v^+ is isomorphic to J since v is a square in J [J, p. 60] and [J, VI.7 Lemma, p. 242]. Therefore g vanishes on V_v^+.

8. THE DERIVED RACINE HOMOTOPE POLYNOMIAL

PROPOSITION. *Let $n \geq 3$ and let k be a ring without 2-torsion. The derived Racine homotope polynomial* $\mathrm{DR}_n(x_1, x_2, x_3, x_4; y)$ (*see* (a), *Section* 5) *is an identity of the Jordan pairs* I_{nn}, II_{2n}, *and* III_n *over k, but not of* I_{mm}, II_{2m}, *and* III_m *for any $m > n$.*

Proof. Let $\mathscr{C} = k$, $k \oplus k$, and $\mathrm{Mat}_2(k)$. The canonical involution of $\mathscr{C}$ considered as a composition algebra, i.e., Id_k for $\mathscr{C} = k$, the exchange involution for $\mathscr{C} = k \oplus k$, and the symplectic involution for $\mathscr{C} = \mathrm{Mat}_2(k)$, extends naturally to an involution on the associative algebra of $m \times m$ matrices over $\mathscr{C}$. Let $\mathrm{H}_m(\mathscr{C})$ be the Jordan algebra of hermitian matrices with respect to this involution. The Jordan pairs III_m, I_{mm}, and II_{2m} are the Jordan pairs of the Jordan algebras $\mathrm{H}_m(\mathscr{C})$. Since $\mathrm{DR}_n(x_1, \ldots, x_4) \in$ FJA vanishes on the Jordan algebra $\mathrm{H}_n(\mathscr{C})$ over fields, the corresponding homotope polynomial $\mathrm{DR}_n(x_1, \ldots, x_4; y)$ vanishes on the Jordan pair $(\mathrm{H}_n(\mathscr{C}), \mathrm{H}_n(\mathscr{C}))$ for $k = \mathbb{C}$ by the Lemma, Section 7, and then for arbitrary k by (3.2).

To prove that the homotope polynomial $\mathrm{DR}_n(x_1, x_2, x_3, x_4; y) = \mathrm{DR}_n^{\mathrm{JP}}$ does not vanish on the Jordan pairs $(\mathrm{H}_m(\mathscr{C}), \mathrm{H}_{\mathrm{m}}(\mathscr{C}))$ for $m > n$, it suffices to establish nonvanishing of the Jordan algebra polynomial DR_n on $\mathrm{H}_m(\mathscr{C})$ since $\mathrm{DR}_n^{\mathrm{JP}}$ evaluated for $y \mapsto 1$ yields DR_n. This in turn will follow from

$$\mathrm{DR}_n \text{ does not vanish on } \mathrm{H}_m(\mathscr{C}),\ m > n, \text{ for fields of characteristic } \neq 2. \tag{1}$$

Indeed, if (1) holds we can proceed as in [RR2, p. 2691]: since k has no 2-torsion, there exists a prime ideal $\wp$ of k not containing 2. The quotient field F of $R = k/\wp$ has characteristic $\neq 2$. By (1), DR_n is not an identity of $\mathrm{H}_m(\mathscr{C})$ over F; clearing denominators by homogeneity then shows that DR_n is not an identity of $\mathrm{H}_m(\mathscr{C})$ over R and hence also not over k.

It remains to prove (1). We use an argument from [DR, p. 312]. By [R, Theorem 2] one knows $\mathrm{R}_n(\mathrm{H}_n(\mathscr{C}), \mathrm{H}_n(\mathscr{C})) = k \cdot 1$ if k is a field of characteristic $\neq 2$. Hence $\mathrm{R}_n(u_1, u_2) = 1$ for some $u_1, u_2 \in \mathrm{H}_n(\mathscr{C})$. Then viewing $u_1, u_2 \in \mathrm{H}_m(\mathscr{C})$ gives $\mathrm{R}_n(u_1, u_2) = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) =:$

$c \in \mathrm{H}_m(\mathscr{C})$, and taking $u_3 = h_{n+1,n+1} = E_{n+1,n+1}$ and $u_4 = h_{n,n+1} = E_{n,n+1} + E_{n+1,n}$ gives $\{u_3 u_4 c\} - \{u_4 u_3 c\} = \frac{1}{4} u_4$. This finishes the proof of the proposition.

9. DISTINGUISHING SPLIT JORDAN PAIRS

THEOREM. *Let k be a ring without* 2*-torsion. Then the split Jordan pairs over k can be distinguished by the following integral polynomial identities*:

(1) *inner Capelli polynomials* IC;

(2) *inner Capelli homotope polynomials* ICQ;

(3) *derived Racine homotope polynomials* DR;

(4) *the standard Clifford homotope polynomial* SC;

(5) *the Jordan pair analogue of the Glennie polynomial.*

In particular, this is so for simple finite-dimensional Jordan pairs over algebraically closed fields of characteristic $\neq 2$.

To be distinguishable by integral polynomial identities means that if V and W are split Jordan pairs of types $\mathscr{G}$ and $\mathscr{G}'$, respectively, with $\mathscr{G} \neq \mathscr{G}'$ then one of the five Jordan polynomials listed above is an identity of one of them but not of the other. We will denote this by $V \leftrightarrow W$. That split Jordan pairs of type I can be distinguished by polynomial identities is also proven in [I].

Proof. We will distinguish between split Jordan pairs of different dimensions by an appropriate inner Capelli polynomial. In particular, V $\leftrightarrow$ VI. [LMc, Theorem 3.10] the Jordan pair version of the Glennie identity does not vanish on the exceptional Jordan pair V, hence neither on VI. On the other hand, it vanishes on all a-special Jordan pairs, in particular on the first four types I−IV. Hence we can distinguish between V, VI, and the a-special types so that in the following it is sufficient to consider only the types I−IV.

Within the classes II, III, and IV we can distinguish by dimensions via IC_ls. If $pq = p'q'$ with $p \neq p'$, say $p < p'$, we have $\mathrm{I}_{pq} \leftrightarrow \mathrm{I}_{p'q'}$ by evaluating ICQ_l for $l = p'^2$: by the Proposition, Section 4, it vanishes on I_{pq} but does not vanish on $\mathrm{I}_{p'p'} \subset \mathrm{I}_{p'q'}$. Thus, in the following we only need to distinguish between Jordan pairs belonging to different classes I, II, III, or IV.

By the Lemma, Section 6, nonvanishing (respectively, vanishing) of the standard Clifford homotope polynomial SC will divide the special split

Jordan pairs into the following two disjoint sets:

$$\left\{\mathrm{I}_{pq}(2 \le p \le q,\, 3 \le q),\, \mathrm{II}_n(5 \le n),\, \mathrm{III}_n(3 \le n)\right\}$$
$$\leftrightarrow \left\{\mathrm{I}_{1q}(1 \le q),\, \mathrm{IV}_n(3 \le n)\right\}.$$

Within the second set we can distinguish $\mathrm{I}_{1n} \leftrightarrow \mathrm{IV}_n$ by ICQ_n (see the Proposition, Section 4). We are therefore left with distinguishing Jordan pairs in different classes $\mathrm{I}_{pq}(2 \le p \le q, 3 \le q)$, $\mathrm{II}_n(5 \le n)$, and $\mathrm{III}_n(3 \le n)$ which have the same dimension.

$\mathrm{I}_{pq} \leftrightarrow \mathrm{III}_n$. We have $pq = n(n+1)/2$. If $p < q$ the polynomial ICQ_{pq} will distinguish between I_{pq} and III_n since $l = pq > p^2$. In case $p = q$ we have $p < n$ and hence $\mathrm{I}_{pp} \leftrightarrow \mathrm{III}_n$ by the Proposition, Section 8.

$\mathrm{II}_m \leftrightarrow \mathrm{III}_n$. We have $m(m-1)/2 = n(n+1)/2 =: l$, i.e., $m = n+1$. If $m \equiv 1(2)$ we can use ICQ_l to distinguish between II_m and III_n since $(m-1)(m-2)/2 < l$. If $m \equiv 0(2)$ we have $m/2 < n$ and hence $\mathrm{II}_m \to \mathrm{III}_n$ by the Proposition, Section 8.

$\mathrm{I}_{pq} \leftrightarrow \mathrm{II}_n$. Equality of dimensions leads to

$$2pq = n(n-1) \tag{1}$$

For $p = q$ we will distinguish between I_{pp} and II_n by applying the Proposition, Section 8. Namely, DR_{p-1} does not vanish on I_{pp} while it will vanish on any II_n which is imbeddable in a $\mathrm{II}_{2(p-1)}$, i.e., if $n \le 2p-2$. These two inequalities are fulfilled if $(*)$ $n + 2 \le 2p$. But since $2p^2 = n(n+1)$ the inequality $(*)$ is true for any $n \ge 5$.

We can now assume $p < q$. In this case the polynomial ICQ_l vanishes on I_{pq} for any $l > p^2$ but does not vanish on II_{2m} if $l = \dim \mathrm{II}_{2m} = m(2m-1)$. Hence it does not vanish on II_n for which $\mathrm{II}_{2m} \subset \mathrm{II}_n$, i.e., for $2m \le n$. This yields the inequality

$$p^2 < m(2m-1). \tag{2}$$

For even n we can choose $n = 2m$ and then (2) follows from (1) and $p < q$. Hence in the following we can assume that n is odd. We take $2m = n-1$, and then (2) is equivalent to $2p^2 < (n-1)(n-2)$. Observe that we can distinguish between I_{pq} and II_n using the polynomial ICQ_l in another way: it does not vanish on $\mathrm{I}_{pp} \subset \mathrm{I}_{pq}$ for $l = p^2$ but it vanishes on II_n whenever $(n-1)(n-2) < 2l$ using (b) of the Proposition, Section 4, for n odd, i.e., whenever $2p^2 > (n-1)(n-2)$. Therefore $\mathrm{I}_{pq} \leftrightarrow \mathrm{II}_n$ if we show that the following system does not have an integral solution:

$$2p^2 = (n-1)(n-2), \qquad 2pq = n(n-1), \qquad 2 \le p < q,\ n \equiv 1(2).$$

Since $n-1$ and $n-2$ have different prime divisors, the first equation implies that there exist relatively prime $a, b \in \mathbb{N}$ such that $n-1 = 2a^2$, $n-2 = b^2$, $p = ab$. Substituting this into the second equation yields $2abq = (b^2+2)2a^2$, thus $bq = a(b^2+2)$. Since b is odd it is relatively prime to b^2+2, hence b divides a. On the other hand, $n-1 = 2a^2 = b^2+1$ implies that $a < b$ or that $a = b = 1$, which contradicts that b divides a or that $p \geq 2$.

10. DISTINGUISHING JORDAN ALGEBRAS AND JORDAN TRIPLE SYSTEMS

COROLLARY. *Over algebraically closed fields of characteristic* $\neq 2$, *integral polynomial identities distinguish the isomorphism classes of the following simple finite-dimensional Jordan structures*:

(a) *Jordan algebras*;

(b) *polarized Jordan triple systems*.

For fields of characteristic 0, (a) is proven in [DR, Theorem 1].

Proof. (a) A Jordan pair polynomial $f \in \mathrm{FJP}(\mathbf{X})^\sigma$, $\mathbf{X} = (X^+, X^-)$, is also a Jordan algebra polynomial in the free Jordan algebra over $\mathbb{Z}$ on the generating set $X = X^+ \cup X^-$. If J is a Jordan algebra, f vanishes on the Jordan pair (J, J) if and only if f vanishes on J. Over the algebraic closure, isotopy of Jordan algebras is the same as isomorphism. Hence two simple Jordan algebras are isomorphic if and only if the corresponding Jordan pairs are isomorphic.

(b) Let $S = S^+ \oplus S^-$ and $T^+ \oplus T^-$ be two simple polarized Jordan triple systems. By [N1, Lemma A.1] and [N1, Theorem A.3], the Jordan pairs $\mathscr{S} = (S^+, S^-)$ and $\mathscr{T} = (T^+, T^-)$ are simple, and S and T are isomorphic as Jordan triple systems if and only if $\mathscr{S}$ is isomorphic to $\mathscr{T} = (T^+, T^-)$ or to $\mathscr{T}^{op} = (T^-, T^+)$. Since simple finite-dimensional Jordan pairs always have an involution, we have $S \approx T \Leftrightarrow \mathscr{S} \approx \mathscr{T}$. As in (a), a Jordan pair polynomial $f \in \mathrm{FJP}(\mathbf{X})^\sigma$ can be interpreted as a Jordan triple polynomial in the free Jordan triple system over $\mathbb{Z}$ on the generating set $X = X^+ \cup X^-$. Moreover, if f vanishes on $\mathscr{S}$ but not on $\mathscr{T}$, then f vanishes on S and not on T.

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