

- (1) (6 pts) Are the following statements true or false? Answer with T for “true” and F for “false”.
- (a) One calls vectors X_1, X_2, \dots, X_k linearly independent if the following condition holds: Whenever all scalars s_1, \dots, s_k are zero then $s_1X_1 + s_2X_2 + \dots + s_kX_k = 0$.

My answer: _____

- (b) Any linearly independent subset of \mathbb{R}^n is orthogonal.

My answer: _____

- (c) Every set of 3 vectors in \mathbb{R}^n is a spanning set of a three-dimensional subspace of \mathbb{R}^n .

My answer: _____

- (d) If A is an $n \times n$ matrix whose columns span \mathbb{R}^n , then A^T is invertible.

My answer: _____

- (e) Any 3 vectors in \mathbb{R}^2 will be linearly dependent.

My answer: _____

- (f) If X_1, X_2 are linearly independent and Y is not a linear combination of X_1, X_2 , then $\{X_1, X_2, Y\}$ can be linearly dependent.

My answer: _____

Solution: (a) This is false. The definition of linear independence is the opposite: Whenever $s_1X_1 + s_2X_2 + \dots + s_kX_k = 0$, then all scalars s_1, \dots, s_k are zero.

(b) This is false. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent but not orthogonal.

(c) This is false. If the three vectors are linearly dependent, then they will not span a three-dimensional space.

(d) This is true. The condition implies that $\text{rank}A = n$, so A is invertible, hence A^T is invertible.

(e) This is true by the Fundamental Theorem.

(f) This is false by the Independent Lemma.

Marking: 1 point for each correct answer.

- (2) (7 pts) (a) (1 pt) 1 Assume that A is a 4×5 matrix for which the associated homogeneous linear system $AX = 0$ has 3 basic solutions. Then the dimension of the row space of A equals ... (complete the sentence).

My answer: _____

- (b) (2 pts) Let $U \subset \mathbb{R}^6$ be a 4-dimensional space. Then ... (give the dimensions)

$$\dim U^\perp = \underline{\hspace{2cm}} \quad \text{and} \quad \dim U^{\perp\perp} = \underline{\hspace{2cm}}$$

- (c) (1 pt) 1 Suppose X_1, X_2, X_3 is a basis of a subspace U in \mathbb{R}^5 . Then the orthogonal basis F_1, F_2, F_3 of U obtained from X_1, X_2, X_3 by the Gram-Schmidt algorithm is ... (give the formula)

- (d) (3 pts) Let α, β, γ be the third last, second last and last digit of your student number. Which one(s) of the following subsets of \mathbb{R}^3 is (are) a basis of \mathbb{R}^3 ?

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \\ 1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \gamma \end{bmatrix} \right\}$$

No justification required.

My answer: _____

Solution: (a) Since the homogeneous linear system $AX = 0$ has 5 variables and 3 basic solutions, this means that the rank of the matrix A is 2. But the rank of A equals the dimension of the row space of A . Thus: $\dim \text{row}(A) = 5 - 3 = 2$.

(b) By Theorem 3 in §4.6,

$$\dim U^\perp = n - \dim U = 6 - 4 = 2 \quad \text{and} \quad \dim U^{\perp\perp} = \dim U = 4$$

since $U^{\perp\perp} = U$.

Marking: 1 point for each correct answer

(c) By Theorem 8 in §4.5 we have

$$\begin{aligned} F_1 &= X_1 \\ F_2 &= X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 \\ F_3 &= X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 \end{aligned}$$

(d) B_1 is not a basis: The second vector is a scalar multiple of the first, whence the set is not linearly independent. Alternative solution: If we put the vectors as columns of a matrix, we get $\det(B_1) = 0$, so B_1 is not a basis of \mathbb{R}^3 by the Invertible Matrix Theorem.

B_2 is a basis, since for example the matrix formed out of the vectors in the set B_2 is invertible. For example, its determinant is not zero:

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & \beta \\ 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2.$$

Finally, B_3 contains more than 3 vectors and is therefore not a basis of \mathbb{R}^3 .

Marking: 1 point for each correct answer.

(3) (6 pts) Let $U = \text{Span}\{X_1, X_2, X_3, X_4\}$ where the vectors $X_1, \dots, X_4 \in \mathbb{R}^4$ are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

Find a basis of U and $\dim U$.

Solution: We write the vectors X_1, X_2, X_3, X_4 into a matrix A with X_i the i -th row,

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 2 & 0 & 0 \\ 2 & 1 & -1 & 2 \\ 3 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{R2+R1 \\ R3-2R1 \\ R4-3R1}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 4 & 1 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -5 & -2 & -1 \end{bmatrix} \xrightarrow{\frac{R3}{-3}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 4 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -5 & -2 & -1 \end{bmatrix} \\
 &\xrightarrow{\substack{R1-2R3 \\ R2-4R3 \\ R4+5R3}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix} \xrightarrow{\substack{\text{switch } R2 \text{ and } R4}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \\
 &\xrightarrow{\substack{\text{add } R3 \text{ to } R4 \\ \text{multiply row } R3 \text{ by } -1}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R
 \end{aligned}$$

Since $U = \text{row}A$, so a basis of U are the rows of R with leading 1's, that is, the following is a basis:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \end{bmatrix}.$$

So $\dim U = 3$. Note the basis is not unique. Alternative solution: Interpret U as the column space of a matrix, and apply the Rank Theorem.

Marking: 2 points for correct method, 2 points for correct row reduction, 1 point for correct basis, 1 point for correct dimension.

- (4) (6 pts) Let $X_1 = [1 \ 1 \ 1]^T$ and $X_2 = [1 \ -1 \ 0]^T$.
 (a) (2 pts) Show that $\{X_1, X_2\}$ is an orthogonal set. Is it also orthonormal?
 (b) (4 pts) Let $U = \text{Span}\{X_1, X_2\}$. Find the point in U closest to $X = [1 \ 2 \ \alpha]^T$ where α is the **third last** digit of your student number.

Solution: (a) The dot product $X_1 \cdot X_2 = 0$, whence $\{X_1, X_2\}$ is an orthogonal set. But it is not an orthonormal set because $\|X_1\| \neq 1$.

(b) The point in U closest to $X = [1 \ 2 \ \alpha]^T$ is given by

$$Z = \text{proj}_U(X) = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2$$

But $X \cdot X_1 = 3 + \alpha$; $X \cdot X_2 = -1$; $\|X_1\|^2 = 3$, and $\|X_2\|^2 = 2$. Consequently

$$Z = \frac{3 + \alpha}{3} X_1 - \frac{1}{2} X_2 = \left[\frac{\alpha}{3} + \frac{1}{2} \quad \frac{\alpha}{3} + \frac{3}{2} \quad \frac{\alpha}{3} + 1 \right]^T$$

Marking: (b) 2 points for realizing that $Z = \text{proj}_U(X)$ and writing down the correct formula for $\text{proj}_U(Z)$. 2 points for calculating Z .

- (5) (a) (3 pts) Let $\mathbb{F}[0, 9]$ be the vector space of functions $f: [0, 9] \rightarrow \mathbb{R}$. In the following definition replace β with the **second last** digit of your student number. Show that

$$U = \{f \in \mathbb{F}[0, 9] : f(\beta) = 0\}$$

is a subspace of $\mathbb{F}[0, 9]$.

(b) (3 pts) Let s, c, h be the functions in $\mathbb{F}[\mathbb{R}]$ defined by

$$s(x) = \sin^2(x), \quad c(x) = \cos^2(x), \quad h(x) = x.$$

Is $h \in \text{Span}\{s, c\}$?

In both parts you must include all details to get full marks.

Solution: (a) We check the 3 conditions of the subspace test:

(1) $0_{\mathbb{F}[0,9]} \in U$: Indeed, the zero “vector” of $F[0, 9]$ is the zero function which is zero on ALL points in $[0, 9]$, in particular it vanishes at the point $\beta \in [0, 9]$.

(2) U is closed under addition: Given any two elements f, g in U , then

$$f(\beta) = 0, g(\beta) = 0.$$

Now, using the definition of addition in $\mathbb{F}[0, 9]$, we get

$$(f + g)(\beta) = f(\beta) + g(\beta) = 0 + 0 = 0.$$

Hence, $f + g \in U$.

(3) U is closed under scalar multiplication: Again, let $f \in U$, so $f(\beta) = 0$ by definition of U . But then

$$(cf)(\beta) = cf(\beta) = c \cdot 0 = 0 \text{ for any } c$$

so $cf \in U$.

Now (1)–(3) together show that U is a subspace.

(b) Suppose $h \in \text{Span}\{c, s\}$. Then there exist scalars a, b such that $h = ac + bs$. This means that

$$x = h(x) = a \cos^2(x) + b \sin^2(x) \quad *$$

holds for all $x \in \mathbb{R}$. We evaluate (*) for $x = 0$ and get $0 = h(0) = a$ since $\cos^2(0) = 1$ and $\sin^2(0) = 0$. Next, we evaluate (*) for $x = \pi/2$ and get $\pi/2 = h(\pi/2) = b$ since $\cos^2(\pi/2) = 0$ and $\sin^2(\pi/2) = 1$. Thus $x = \frac{\pi}{2} \sin^2(x)$ holds for all $x \in \mathbb{R}$, which is not true: For example, for $x = \pi$ we get $\pi = h(\pi) = \frac{\pi}{2} \sin^2(\pi) = 0$. Therefore $h \notin \text{Span}\{c, s\}$.

Another argument: c and s are bounded functions, hence so is every linear combination. However, h is unbounded.

- (6) (2 bonus points) Let v_1 and v_2 be elements of a vector space V . Show that $\text{Span}\{v_1, v_2\}$ is a subspace of V . Recall that $\text{Span}\{v_1, v_2\}$ is the set of all linear combinations of v_1, v_2 . In other words, $\text{Span}\{v_1, v_2\} = \{s_1v_1 + s_2v_2 : s_1, s_2 \in \mathbb{R}\}$.

Solution: This has been proven in class twice.

Let $K = \text{Span}\{v_1, v_2\}$. We have just to prove the three conditions defining a subspace.

(1) $0 \in K$ because $0 = 0v_1 + 0v_2$.

(2) Let u_1 and u_2 be elements of K , then $u_1 = s_1v_1 + s_2v_2$ for suitable scalars $s_1, s_2 \in \mathbb{R}$ and $u_2 = t_1v_1 + t_2v_2$ for scalars $t_1, t_2 \in \mathbb{R}$. Consequently $u_1 + u_2 = (s_1 + t_1)v_1 + (s_2 + t_2)v_2 = av_1 + bv_2$, where $a = s_1 + t_1 \in \mathbb{R}$ and $b = s_2 + t_2 \in \mathbb{R}$. Thus $u_1 + u_2$ also is in K .

(3) Let $u \in K$ and $a \in \mathbb{R}$. Then $au = (as_1)v_1 + (as_2)v_2$ is in K .

Altogether, this proves that K is a subspace of V .