(1) (3 pts) Let A be an $m \times n$ matrix, let R be a row-echelon form of A.

(a) If $B \in col(A)$, the linear system $AX = B$ is solvable. True or false?

My answer:__________________

(b) If A is a 4×7 matrix and R has precisely two rows of zeros, then the set of solutions of $AX = B$ depends on how many parameters? (Assume that the system is solvable)

My answer:__________________

(c) If A is a 5×9 matrix and every row of R has a leading one, then $AX = 0$ has infinitely many solutions. True or false?

My answer:__________________

Solution: (a) (True) By definition $col(A) = im(A) = AX : X \in \mathbb{R}^n$. Thus, if $B \in col(A)$ there exists $X \in \mathbb{R}^n$ such that $AX = B$, i.e., the system is solvable.

(b) We have rank(A) = 2. Because the system has 7 variables, we have 7-2=5 free parameters. Thus, if the system is solvable, the set of solution depends on 5 parameters.

(c) (True) R has 5 leading one, thus $Rank(A) = 5$,$\dim(NullA) = 4$ and the system $AX = 0$ depends on 4 parameters then has infinitely many solutions

(2) (3 pts) Let A be an $n \times n$ matrix. State three conditions which are equivalent to the condition that the homogenous linear system $AX = 0$ has infinitely many solutions.

(a) An equivalent condition in terms of the eigenvalues of A:

My answer:__________________

(b) An equivalent condition in terms of linear independence or dependence of the columns of A^T.

My answer:__________________

(c) An equivalent condition in terms of the rank of A:

My answer:__________________

Solution: (a) We have $\{X \in \mathbb{R}^n : Ax = 0\} = null(A) = E_0(A)$. Hence $AX = 0$ is equivalent to $null(A) \neq 0$, i.e., 0 is an eigenvalue.

(b) The given condition means that A is not invertible, hence the rows of A are linearly dependent, i.e., the columns of A^T are dependent.

(c) Since A is not invertible, we have $rank(A) < n$.

(3) (5 pts)
(a) If A and B are matrices for which AB can be formed and equals zero, i.e., $AB = 0$, then $A = 0$ or $B = 0$. True or false?

My answer:________________

(b) Let γ be the last digit of your student number. Then the inverse of the matrix $\begin{bmatrix} 1 & \gamma \\ 3 & \sqrt{3} \end{bmatrix}$ is

My answer:________________

(c) A triangular matrix is always invertible and its inverse is again a triangular matrix. True or false?

My answer:________________

(d) If A and B are square matrices, then $\det(A + B) = \det(A) + \det(B)$. True or false?

My answer:________________

(e) Give the formula which describes the inverse of an invertible matrix A in terms of the adjoint matrix $adj(A)$.

My answer:________________

Solution: (a) This is false, for example, let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $AB = 0$ but $A \neq 0$ and $B \neq 0$.

(b) The inverse is

$$\frac{1}{\sqrt{3} - 3\gamma} \begin{bmatrix} \sqrt{3} & -\gamma \\ -3 & 1 \end{bmatrix}.$$

(c) This is false. A triangular matrix is invertible precisely when all the diagonal entries are nonzero. For example, the following upper triangular matrix is not invertible

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(d) This is false. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$. Then $A+B = 0$, $\det(A+B) = 0$, $\det A = \det B = 1$.

(e)

$$A^{-1} = \frac{1}{\det A} adj(A)$$

(4) (3 pts) If A^T is a 6×11 matrix of rank 4, then the dimensions of the row, column and null space of A are

$\dim row(A) =$ __________, $\dim col(A) =$ __________, $\dim null(A) =$ __________.

Solution: Since A^T is a 6×11 matrix, A is a 11×6 matrix. We know that $rank(A^T) = rank(A) = 4$. Therefore

$$\dim row(A) = rank(A) = 4, \qquad \dim col(A) = rank(A) = 4,$$
$$\dim null(A) = 6 - rank(A) = 6 - 4 = 2$$

(5) (5 pts)

(a) If a vector space V is spanned by 7 vectors, then any other spanning set of V cannot have more than 7 elements. True or false?

My answer:______________

(b) The dimension of the vector space $\mathbb{M}_{46}$ of 4×6 matrices is

My answer:______________

(c) The functions 1, $\sin^2(x)$ and $\cos^2(x)$ are linearly independent in the vector space $\mathbb{F}[\mathbb{R}]$. True or false?

My answer:______________

(d) Give an example of an infinite-dimensional vector space:

(e) Suppose $T \colon \mathbb{P}_6 \to \mathbb{M}_{58}$ is a linear transformation with $\dim(kerT) = 3$. What is $\dim(imT)$?

My answer:______________

Solution: (a) This is false. For example, if $V = span\{X_1, ..., X_7\}$, then V has a spanning set

$$\{X_1, ..., X_7, X_1 + X_2\}.$$

(b) The dimension is $4 \cdot 6 = 24$.

(c) This is false. We have

$$\sin^2(x) + \cos^2(x) = 1$$

So

$$1 - \sin^2(x) - \cos^2(x) = 0.$$

So they are linearly dependent.

(d) The vector space $\mathbb{P}$ of polynomials is infinite-dimensional.

(e) By the Dimension Theorem $\dim(imT) = \dim \mathbb{P}_6 - \dim(\ker T) = 7 - 3 = 4$.

(6) (2 pts) Find A if

$$\left(A - 2\begin{bmatrix} 1 & 4 & \alpha \\ \beta & 0 & i \end{bmatrix}\right)^T = \begin{bmatrix} 4 & 0 \\ 9 & 11 \\ -5 & \sqrt{2} \end{bmatrix}$$

where α and β are the third-last and second-last digits of your student number and $i \in \mathbb{C}$.

Solution: The left hand side is

$$A^T - \begin{bmatrix} 2 & 2\beta \\ 8 & 0 \\ 2\alpha & 2i \end{bmatrix}.$$

So $A^T = \begin{bmatrix} 2 & 2\beta \\ 8 & 0 \\ 2\alpha & 2i \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 9 & 11 \\ -5 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 6 & 2\beta \\ 17 & 11 \\ 2\alpha - 5 & 2i + \sqrt{2} \end{bmatrix}$. Hence, $A = \begin{bmatrix} 6 & 17 & 2\alpha - 5 \\ 2\beta & 11 & 2i + \sqrt{2} \end{bmatrix}$.

(7) (2 pts) Find the determinant of the following matrix:

$$\begin{bmatrix} -2 & 2 & 0 & 4 \\ 0 & 2 & 0 & 4 \\ 5 & 0 & -1 & 0 \\ 0 & \beta & 0 & \gamma \end{bmatrix}$$

where β, γ are the second-last and last digit of your student number.

My answer:________________

Solution: We use cofactor expansion along the third column:

$$\det A = (-1)^{3+3}(-1) \cdot \det \begin{bmatrix} -2 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & \beta & \gamma \end{bmatrix} = (-1) \cdot (-2) \cdot \det \begin{bmatrix} 2 & 4 \\ \beta & \gamma \end{bmatrix} = 2 \cdot (2\gamma - 4\beta) = 4(\gamma - 2\beta).$$

(8) (3 pts) Let U be a subset of a vector space V. State the conditions defining that U is a subspace of V.

Solution: U is a subspace of V if and only if it satisfies the following conditions:
(0) The zero vector 0 lies in V (equivalently, $U \neq \emptyset$),
(1) For any two vectors $X, Y \in U$, $X + Y \in U$;
(2) For any vector $X \in U$ and any scalar $c \in \mathbb{R}$, $cX \in U$.

(9) (4 pts) (a) Find the standard matrix of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 - 4x_2 + \beta x_3 \\ -5x_1 + \gamma x_2 + 4x_3 \end{bmatrix}$$

where α, β and γ are as usual (no justification required).

(b) Let $U \in \mathbb{R}^n$ with $\|U\| = 1$. Show that $T\colon \mathbb{R}^n \to \mathbb{R}$, $T(X) = X \cdot U$, is a linear transformation. (Include all details!)

Solution: (a) The column i of the standard matrix $A \in \mathbb{M}_{23}$ is given by applying T to the ith standard basis vector:

$$A = \begin{bmatrix} \alpha & -4 & \beta \\ -5 & \gamma & 4 \end{bmatrix}$$

(b) That T is linear follows from

$$T(X + Y) = (X + Y) \cdot U = (X \cdot U) + (Y \cdot U) = T(X) + T(Y)$$
$$T(sX) = (sX) \cdot U = s(X \cdot U) = sT(X)$$

where $X, Y \in \mathbb{R}^n$ and $s \in \mathbb{R}$.

Marking: (a) 2 points, (b) 2 points

(10) (8 pts) Determine the values of a so that the linear system

$$\begin{array}{ccccccc} x & + & 3y & + & z & = & 0 \\ 2x & + & 7y & + & (a+2)z & = & 1 \\ x & + & (3-a)y & + & (a+1)z & = & 1 \end{array}$$

(a) has no solution,
(b) has a unique solution,
(c) has infinitely many solutions.
In case (c) determine all solutions.

Solution: The augmented matrix associated to this system is:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & a+2 & 1 \\ 1 & 3-a & a+1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & -a & a & 1 \end{bmatrix}$$

(a) The system has no solution if and only of $a = 0$.
(b) Assume now that $a \neq 0$. Then we can divided the third row by $-a$ and obtain

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 1 & -1 & \frac{-1}{a} \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & -1-a & \frac{-1-a}{a} \end{bmatrix}$$

Hence if $a \neq 0$ and also $a \neq -1$ we obtain

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & \frac{1}{a} \end{bmatrix}$$

Therefore, if $a \neq 0$ and $a \neq -1$ this system has one unique solution $x = \frac{-1}{a}$, $y = 0$ and $z = \frac{1}{a}$
(c) Now if $a = -1$ then we obtain

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the variable z is free, we put $z = t$ and $y = 1 - t$ and $x = -3 + 2t$ and the general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3+2t \\ 1-t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad (\ t \text{ is a free parameter}),$$

Marking: 2 pts for correct row reduction in the first part, 1 pt for correct row reduction in (c); 3 points for correct identification of the three cases; 1 point for the particular solution in (c), 1 point for the general solution in (c).

(11) (10 pts) Let A be the matrix

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial $c_A(x) = \det(xI_3 - A)$, and conclude that the eigenvalues of A are 0, 3 and -1.

(b) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(You have two pages to complete this problem.)

Solution: To find all eigenvalues we calculate the characteristic polynomial

$$\begin{aligned} c_A(x) &= \det(xI_3 - A) \\ &= \det \begin{bmatrix} x-2 & -2 & -4 \\ -1 & x-1 & -3 \\ 0 & 0 & x+1 \end{bmatrix} \\ &= (x+1)((x-2)(x-1) - (-1)(-2)) = (x+1)x(x-3). \end{aligned}$$

The eigenvalues are the roots of $c_A(x)$. Therefore the eigenvalues of A are 0, 3 and -1.

(b) To find a basis of the eigenspace corresponding to the eigenvalue 0 we need to find a basis of the null space Nul $(0I_3 - A)$, where I_3 is the 3×3-identity matrix. We do this by row reducing $0I_3 - A$:

$$0I_3 - A = \begin{bmatrix} -2 & -2 & -4 \\ -1 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \underset{R2+3R3}{\overset{R1+4R3}{\sim}} \begin{bmatrix} -2 & -2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R1-2R2}{\sim} \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the corresponding homogeneous linear system is $x_1 + x_2 = 0$ and $x_3 = 0$. Its general solution is $x_3 = 0$, $x_2 = t$, $x_1 = -t$, t a free parameter. Hence a basis of the eigenspace $E_0 = \text{Nul}\,(0I_3 - A)$ is $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$.

Similarly, to find a basis of the eigenspace corresponding to the eigenvalue 3 we need to find a basis of the null space Nul $(3I_3 - A)$, which we do by row reducing $3I_3 - A$:

$$3I_3 - A = \begin{bmatrix} 1 & -2 & -4 \\ -1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} \overset{\frac{1}{4}R3}{\sim} \begin{bmatrix} 1 & -2 & -4 \\ -1 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \underset{R2+3R3}{\overset{R1+4R3}{\sim}} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R2+R1}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the corresponding homogeneous linear system is $x_1 - 2x_2 = 0 = x_3$ and its general solution is $x_3 = 0$, $x_2 = s$, $x_1 = 2s$, s a free parameter. Hence a basis of the eigenspace $E_3 = \text{Nul}\,(3I_3 - A)$ is $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$.

To find a basis of the eigenspace corresponding to the eigenvalue -1 we need to find a basis of the null space Nul $(-I_3 - A)$, which we do by row reducing $-I_3 - A$:

$$-I_3 - A = \begin{bmatrix} -3 & -2 & -4 \\ -1 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \overset{R1-3R2}{\sim} \begin{bmatrix} 0 & 4 & 5 \\ -1 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \underset{-R2}{\overset{\frac{1}{4}R1}{\sim}} \begin{bmatrix} 0 & 1 & \frac{5}{4} \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \overset{R1-2R2}{\sim} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the corresponding homogeneous linear system is $x_1 + \frac{x_3}{2} = 0 = x_2 + \frac{5x_3}{4}$, and its general solution is $x_3 = r, x_2 = \frac{-5r}{4}, x_1 = \frac{-r}{2}$, r a free parameter. Hence a basis of the eigenspace $E_{-1} = \text{Nul}\,(-I_3 - A)$ is $\begin{bmatrix} \frac{-1}{2} & \frac{-5}{4} & 1 \end{bmatrix}^T$.

(c) The matrix is diagonalizable since it has three distinct eigenvalues. Let

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, P = \begin{bmatrix} -1 & 2 & \frac{-1}{2} \\ 1 & 1 & \frac{-5}{4} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $P^{-1}AP = D$.

Marking: 4 points for (a): 1 point for knowing that the eigenvalues of A are the roots of the characteristic polynomial, 2 points for calculating the $c_A(x)$ correctly, 1 point for finding the roots. (b) 2 points for each of the two eigenspaces. (c) 2 points, justification required.

(12) (5 pts) Show that the functions x, $\sin x$ and $\cos x$ are linearly independent in $\mathbb{F}[\mathbb{R}]$.

Solution: Assume there exist $a, b, c \in \mathbb{R}$ such that $ax + b\sin x + c\cos x = 0$ for any $x \in \mathbb{R}$. Then for $x = 0$ we get $c = 0$ so that now $ac + b\sin x = 0$ for all $x \in \mathbb{R}$. Then for $x = \pi$ we obtain $a = 0$ from $0 = a\pi + b0$. Thus $b\sin x = 0$ for any $x \in \mathbb{R}$. But then $b = 0$, for example by evaluating this for $x = \pi/2$. Consequently we have $a = b = c = 0$, which proves that the functions x, $\sin x$ and $\cos x$ are linearly independent in $\mathbb{F}[\mathbb{R}]$.

Marking: 1 point for writing down the definition of linear independence, 2 to 3 points for showing some knowledge of the function $\sin, \cos,$.

(13) (10 pts) (a) Show that the vectors

$$X_1 = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 & 5 & 5 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -2 & -3 & 0 & 3 \end{bmatrix}$$

in $\mathbb{R}^4$ are linearly independent.

(b) Find a vector X_4 which does not lie in the span of X_1, X_2, X_3.

(c) Enlarge X_1, X_2, X_3 to a basis of $\mathbb{R}^4$. (You must give concrete vectors with numbers.)

(You have two pages to complete this problem.)

Solution: (a) We put these vectors as columns of a matrix A and find a row-echelon form of it:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -3 \\ 1 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $rank(A) = 3$, i.e., the columns of A are independent.

(b) A vector $X = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ lies in the space of the 3 vectors if and only if there are scalars s_1, s_2, s_3 such that $s_1X_1 + s_2X_2 + s_3X_3 = X$, which is equivalent to the fact that the linear system with coefficient matrix A and constant side $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$ is solvable. By negation, a vector X does not lie in the span if and only if the linear system is not solvable. Hence, we row-reduce the matrix $[A|X]$:

$$\begin{bmatrix} 1 & 2 & -2 & a \\ 2 & 5 & -3 & b \\ 1 & 5 & 0 & c \\ 0 & 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 3 & 2 & c-b \\ 0 & 1 & 3 & d \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & -1 & c-4b+6a \\ 0 & 0 & 2 & b-2a-d \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 1 & 4b-6a-c \\ 0 & 0 & 0 & 10a-7b+2c-d \end{bmatrix}$$

Hence $X = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ does not lie in the span exactly when $10a - 7b + 2c - d \neq 0$. For example, for

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

we get $10a - 7b + 2c - d = 1$ so that $X_4 \notin \text{Span}\,\{X_1, X_2, X_3\}$.

(c) Since $\dim \mathbb{R}^4 = 4$, any basis of $\mathbb{R}^4$ will have 4 vectors. We know that any $X_4 \in \mathbb{R}^4$ with $X_4 \notin \text{Span}\,\{X_1, X_2, X_3\}$ together with $\{X_1, X_2, X_3\}$ is linearly independent, thus providing a basis of $\mathbb{R}^4$. For example, enlarging X_1, X_2, X_3 by the vector X_4 found in (b) will be a basis of $\mathbb{R}^4$.

Marking: (a) 5 points [justification required; only row reduction 4/5; it is not enough to show that X_3 is not a linear combination of X_1 and X_2 because this does not prove that X_1 and X_2 are linearly independent];

(b) 4 points [justification 3 points, actual vector 1 point] (c) 1 point

(continue on next page)

(14) (8 pts) The vectors

$$X_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T, \quad X_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$$

are a basis of a subspace U of $\mathbb{R}^4$.

(a) Find an orthogonal basis of U.

(b) Write $X = \begin{bmatrix} \alpha & 0 & 4 & 4 \end{bmatrix}^T$ as a sum of a vector in U and a vector in $U^\perp$, where α is the third-last digit of your student number.

(c) Find the dimension of $U^\perp$, the orthogonal complement of U.

(You have two pages to complete this problem.)

Solution: (a) We find an orthogonal basis for U using Gram-Schmidt:

$$\begin{aligned} F_1 &= X_1 \\ F_2 &= X_2 - \tfrac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 \\ F_3 &= X_3 - \tfrac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \tfrac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

(b) We calculate the projection of X onto U:

$$\begin{aligned} \operatorname{proj}_U(X) &= \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 + \frac{X \cdot F_3}{F_3 \cdot F_3} F_3 \qquad (*) \\ &= \frac{\alpha}{2} F_1 + \frac{8}{2} F_2 + \frac{-\alpha/2}{1} F_3 = \frac{\alpha}{2}(F_1 - F_3) + 4F_2 \\ &= \frac{\alpha}{2} \begin{bmatrix} 3/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\alpha/4 \\ \alpha/4 \\ 4 + \alpha/4 \\ 4 - \alpha/4 \end{bmatrix}. \end{aligned}$$

Now we decompose $X = \operatorname{proj}_U(X) + (X - \operatorname{proj}_U(X))$. We know from Th. 2 in §4.6 that $(X - \operatorname{proj}_U(X)) \in U^\perp$, where

$$X - \operatorname{proj}_U(X) = \begin{bmatrix} \alpha \\ 0 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3\alpha/4 \\ \alpha/4 \\ 4 + \alpha/4 \\ 4 - \alpha/4 \end{bmatrix} = \begin{bmatrix} \alpha/4 \\ -\alpha/4 \\ -\alpha/4 \\ \alpha/4 \end{bmatrix} = \alpha \begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{bmatrix}.$$

Thus, the decomposition of X as a sum of a vector in U and a vector in $U^\perp$ is

$$\begin{bmatrix} \alpha \\ 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\alpha/4 \\ \alpha/4 \\ 4 + \alpha/4 \\ 4 - \alpha/4 \end{bmatrix} + \begin{bmatrix} \alpha/4 \\ -\alpha//4 \\ -\alpha/4 \\ \alpha/4 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} 3\alpha/4 \\ \alpha/4 \\ 4 + \alpha/4 \\ 4 - \alpha/4 \end{bmatrix} \in U \text{ and } \begin{bmatrix} \alpha/4 \\ -\alpha//4 \\ -\alpha/4 \\ \alpha/4 \end{bmatrix} \in U^\perp.$$

(c) By Th. 3 in §4.6, $\dim U^\perp = \dim \mathbb{R}^4 - \dim U = 4 - 3 = 1$.

Marking: (a) applying Gram-Schmidt 4 points [2 points for correct formula; 2 points for calculation], (b) finding $\operatorname{proj}_U(X)$ 2 point, decomposing 1 points, [without justification 2 points], (c) 1 point.

(continue on next page)

(15) (5 pts) Find a basis of the vector space $\{p \in \mathbb{P}_2 : p(-x) = -p(x)\}$

Solution: Denote this vector space by U. For any $p(x) \in \mathbb{P}_2$, $p(x) = a + bx + cx^2$. We have $p(-x) = -p(x)$ if and only if $a + bx + cx^2 = -(a - bx + cx^2)$, in other words

$$a + bx + cx^2 = -a + bx - cx^2, \quad \text{so} \quad a = -a, c = -c, b = b.$$

Hence, $a = c = 0$, while b is arbitrary. So U is the subspace of $\mathbb{P}_2$ of elements of the form

$$\{p(x) \in \mathbb{P}_2 : p(x) = bx \text{ for some } b \in \mathbb{R}\}.$$

So $\{x\}$ is a spanning set of U, and $\dim U \leq 1$. Since $U \neq \{0\}$, so $\dim U \geq 1$. Hence, $\dim U = 1$, so $\{x\}$ is a basis of U.