University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Final Exam (April 2012) Time: 3 hours

Family name (CAPITALS)	______________________
First name (CAPITALS)	______________________
Signature	______________________
Student number	______________________

The last three digits of my student number are

third last digit $\alpha =$
second last digit $\beta =$
last digit $\gamma =$

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. Use the back of pages if necessary, but be sure to indicate to the marker that you have done so.
- Questions 1–5 are true/false or short answer questions. No part marks will be given. You can earn part marks for the other questions. You must show all the details for questions 8–13, and argue logically. Write legibly.
- Where it is possible to check your work, do so! Read each question carefully – you will save yourself time and unnecessary grief later on.
- **This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Do not detach the pages of the exam. The exam has 15 pages.

Good luck! Bonne chance!

Question	1–5	6–9	10–12	13–15	Total
Score					
Max. score	19	11	23	23	78

(1) (3 pts) Let A be an $m \times n$ matrix, let R be a row-echelon form of A.

(a) If $B \in col(A)$, the linear system $AX = B$ is solvable. True or false?

My answer:________________

(b) If A is a 4×7 matrix and R has precisely two rows of zeros, then the set of solutions of $AX = B$ depends on how many parameters? (Assume that the system is solvable)

My answer:________________

(c) If A is a 5×9 matrix and every row of R has a leading one, then $AX = 0$ has infinitely many solutions. True or false?

My answer:________________

(2) (3 pts) Let A be an $n \times n$ matrix. State three conditions which are equivalent to the condition that the homogenous linear system $AX = 0$ has infinitely many solutions.

(a) An equivalent condition in terms of the eigenvalues of A:

My answer:________________

(b) An equivalent condition in terms of linear independence or dependence of the columns of A^T.

My answer:________________

(c) An equivalent condition in terms of the rank of A:

My answer:________________

(3) (5 pts)

(a) If A and B are matrices for which AB can be formed and equals zero, i.e., $AB = 0$, then $A = 0$ or $B = 0$. True or false?

My answer:__________________

(b) Let γ be the last digit of your student number. Then the inverse of the matrix $\begin{bmatrix} 1 & \gamma \\ 3 & \sqrt{3} \end{bmatrix}$ is

My answer:__________________

(c) A triangular matrix is always invertible and its inverse is again a triangular matrix. True or false?

My answer:__________________

(d) If A and B are square matrices, then $\det(A+B) = \det(A) + \det(B)$. True or false?

My answer:__________________

(e) Give the formula which describes the inverse of an invertible matrix A in terms of the adjoint matrix $adj(A)$.

My answer:__________________

(4) (3 pts) If A^T is a 6×11 matrix of rank 4, then the dimensions of the row, column and null space of A are

$\dim row(A) =$ ____________, $\dim col(A) =$ ____________, $\dim null(A) =$ ____________.

(5) (5 pts)

(a) If a vector space V is spanned by 7 vectors, then any other spanning set of V cannot have more than 7 elements. True or false?

My answer:__________________

(b) The dimension of the vector space $\mathbb{M}_{46}$ of 4×6 matrices is

My answer:__________________

(c) The functions 1, $\sin^2(x)$ and $\cos^2(x)$ are linearly independent in the vector space $\mathbb{F}[\mathbb{R}]$. True or false?

My answer:__________________

(d) Give an example of an infinite-dimensional vector space:

(e) Suppose $T\colon \mathbb{P}_6 \to \mathbb{M}_{58}$ is a linear transformation with $\dim(kerT) = 3$. What is $\dim(imT)$?

My answer:__________________

(6) (2 pts) Find A if

$$\left(A - 2\begin{bmatrix} 1 & 4 & \alpha \\ \beta & 0 & i \end{bmatrix}\right)^T = \begin{bmatrix} 4 & 0 \\ 9 & 11 \\ -5 & \sqrt{2} \end{bmatrix}$$

where α and β are the third-last and second-last digits of your student number and $i \in \mathbb{C}$.

(7) (2 pts) Find the determinant of the following matrix:

$$\begin{bmatrix} -2 & 2 & 0 & 4 \\ 0 & 2 & 0 & 4 \\ 5 & 0 & -1 & 0 \\ 0 & \beta & 0 & \gamma \end{bmatrix}$$

where β, γ are the second-last and last digit of your student number.

My answer:__________________

(8) (3 pts) Let U be a subset of a vector space V. State the conditions defining that U is a subspace of V.

(9) (4 pts) (a) Find the standard matrix of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 - 4x_2 + \beta x_3 \\ -5x_1 + \gamma x_2 + 4x_3 \end{bmatrix}$$

where α, β and γ are as usual (no justification required).

(b) Let $U \in \mathbb{R}^n$ with $\|U\| = 1$. Show that $T\colon \mathbb{R}^n \to \mathbb{R}$, $T(X) = X \cdot U$, is a linear transformation. (Include all details!)

(10) (8 pts) Determine the values of a so that the linear system

$$\begin{array}{rcccccl} x & + & 3y & + & z & = & 0 \\ 2x & + & 7y & + & (a+2)z & = & 1 \\ x & + & (3-a)y & + & (a+1)z & = & 1 \end{array}$$

(a) has no solution,
(b) has a unique solution,
(c) has infinitely many solutions.

In case (c) determine all solutions.

(11) (10 pts) Let A be the matrix

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial $c_A(x) = \det(xI_3 - A)$, and conclude that the eigenvalues of A are 0, 3 and -1.

(b) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(You have two pages to complete this problem.)

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(12) (5 pts) Show that the functions x, $\sin x$ and $\cos x$ are linearly independent in $\mathbb{F}[\mathbb{R}]$.

(13) (10 pts) (a) Show that the vectors

$$X_1 = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 & 5 & 5 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -2 & -3 & 0 & 3 \end{bmatrix}$$

in $\mathbb{R}^4$ are linearly independent.

(b) Find a vector X_4 which does not lie in the span of X_1, X_2, X_3.

(c) Enlarge X_1, X_2, X_3 to a basis of $\mathbb{R}^4$. (You must give concrete vectors with numbers.)

(You have two pages to complete this problem.)

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(14) (8 pts) The vectors

$$X_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T, \quad X_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$$

are a basis of a subspace U of $\mathbb{R}^4$.

(a) Find an orthogonal basis of U.

(b) Write $X = \begin{bmatrix} \alpha & 0 & 4 & 4 \end{bmatrix}^T$ as a sum of a vector in U and a vector in $U^\perp$, where α is the third-last digit of your student number.

(c) Find the dimension of $U^\perp$, the orthogonal complement of U.

(You have two pages to complete this problem.)

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(15) (5 pts) Find a basis of the vector space $\{p \in \mathbb{P}_2 : p(-x) = -p(x)\}$