(1) Replace β be the **second last** digit of your student number. Find the scalar equation of the line parallel to $\begin{bmatrix}1 & 0 & \beta\end{bmatrix}^T$ and passing through $P(3, -2, 5)$.

My answer:____________________________

Solution: Since the line is parallel to the vector $\overrightarrow{d} = \begin{bmatrix}1 & 0 & \beta\end{bmatrix}^T$, so $\overrightarrow{d}$ is a direction vector for the line. The vector $\overrightarrow{p_0} = \begin{bmatrix}3 & -2 & 5\end{bmatrix}^T$ is a position vector, so the vector equation is

$$\overrightarrow{p} = \overrightarrow{p}_0 + t\overrightarrow{d}.$$

The scalar equation is

$$\begin{cases} x = 3 + t \\ y = -2 \\ z = 5 + \beta t \end{cases}$$

Reference: This is the same as the suggested exercise 2b in §3.3; parts of exercise 2 were done in the DGD on February 27.

(2) Let γ be the **last digit** of your student number, find the equation of the plane passing through the points $P(1, 0, -1)$ and perpendicular to the line $\begin{bmatrix}x & y & z\end{bmatrix}^T = \begin{bmatrix}11 & 3 & -1\end{bmatrix}^T + t\begin{bmatrix}1 & \gamma & -3\end{bmatrix}^T$.

My answer:____________________________

Solution: Since the plane is perpendicular to the line, so the direction vector of the line $\overrightarrow{n} = \begin{bmatrix}1 & \gamma & -3\end{bmatrix}^T$ is a normal vector of the plane.

The vector $\overrightarrow{p_0} = \overrightarrow{OP} = \begin{bmatrix}1 & 0 & -1\end{bmatrix}^T$ is a position vector of the plane. So the vector equation of the plane is

$$\overrightarrow{n} \cdot (\overrightarrow{p} - \overrightarrow{p_0}) = 0.$$

The scalar equation is:

$$1(x - 1) + \gamma(y) + (-3)(z + 1) = 0.$$

or

$$x + \gamma y - 3z - 4 = 0.$$

Reference: This is the same as the suggested exercise 13d in §3.3; also done in the DGD on February 27.

(3) Let α be the **third last** digit of your student number. Given two vectors $\overrightarrow{v} = \begin{bmatrix} 2 & \alpha & 5 \end{bmatrix}^T$ and $\overrightarrow{w} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, write $\overrightarrow{v} = \overrightarrow{v_1} + \overrightarrow{v_2}$ such that $\overrightarrow{v}_1$ is parallel to $\overrightarrow{w}$ and $\overrightarrow{v_2}$ is perpendicular to $\overrightarrow{w}$.

My answer: $\overrightarrow{v_1} =$__________ **My answer:** $\overrightarrow{v_2} =$__________

Solution: From the Projection Theorem,

$$\overrightarrow{v_1} = proj_{\overrightarrow{w}}(\overrightarrow{v}) = (\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{\|\overrightarrow{w}\|^2})\overrightarrow{w}.$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = 2 \cdot 1 + \alpha \cdot 0 + 5 \cdot (-1) = -3.$$

$$\|\overrightarrow{w}\|^2 = 1^2 + 0^2 + (-1)^2 = 2.$$

So $\overrightarrow{v}_1 = (\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{\|\overrightarrow{w}\|^2})\overrightarrow{w} = (\frac{-3}{2})\overrightarrow{w} = \begin{bmatrix} \frac{-3}{2} & 0 & \frac{3}{2} \end{bmatrix}^T$.

$$\overrightarrow{v}_2 = \overrightarrow{v} - \overrightarrow{v}_1 = \begin{bmatrix} \frac{7}{2} & \alpha & \frac{7}{2} \end{bmatrix}^T.$$

Reference: This is the same as the suggested exercise 10b in §3.2

(4) Let β the **second last digit** of your student number. If β is odd, do question (a), otherwise do question (b):

(a) Let

$$\vec{u} = \begin{bmatrix} 1 \\ \beta \\ 0 \end{bmatrix}; \quad \vec{v} = \begin{bmatrix} 2\beta \\ 4 \\ -2 \end{bmatrix}; \quad \vec{w} = \begin{bmatrix} -2 \\ 1 \\ -\beta \end{bmatrix}$$

Find the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

(b) Find the area of the triangle with vertices:

$$A = \begin{bmatrix} -1 \\ \beta \\ 3 \end{bmatrix}; \quad B = \begin{bmatrix} -\beta \\ 1 \\ 3 \end{bmatrix}; \quad C = \begin{bmatrix} -\beta \\ \beta \\ -5 \end{bmatrix}$$

My answer:____________________________

Solution: (a)

$$V = |\det(\vec{u}, \vec{v}, \vec{w})| = \begin{vmatrix} 1 & 2\beta & -2 \\ \beta & 4 & 1 \\ 0 & -2 & -\beta \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 \\ -2 & -\beta \end{vmatrix} - \beta \begin{vmatrix} 2\beta & -2 \\ -2 & -\beta \end{vmatrix} = -4\beta + 2 - \beta(-2\beta^2 - 4) = 2(1 + \beta^3)$$

In this calculation we have just used cofactor expansion along the first column.

(b)

$$\vec{AB} = \begin{bmatrix} 1-\beta \\ 1-\beta \\ 0 \end{bmatrix}; \quad \vec{AC} = \begin{bmatrix} 1-\beta \\ 0 \\ -8 \end{bmatrix};$$

Then area of this triangle is one-half of the norm of the cross product of these two vectors. But

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & 1-\beta & 1-\beta \\ \vec{j} & 1-\beta & 0 \\ \vec{k} & 0 & -8 \end{vmatrix} = \left[\begin{vmatrix} 1-\beta & 0 \\ 0 & -8 \end{vmatrix}, - \begin{vmatrix} 1-\beta & 1-\beta \\ 0 & -8 \end{vmatrix}, \begin{vmatrix} 1-\beta & 1-\beta \\ 1-\beta & 0 \end{vmatrix} \right]^T$$

$$= \begin{bmatrix} -8(1-\beta) \\ 8(1-\beta) \\ -(1-\beta)^2 \end{bmatrix};$$

consequently the area is:

$$Area = \frac{1}{2}\sqrt{64(1-\beta)^2 + 64(1-\beta)^2 + (1-\beta)^4} = \frac{|1-\beta|}{2}\sqrt{128 + (1-\beta)^2}$$

Reference: This is similar to the suggested exercise 3b in §3.5; exercise 3a was done in the DGD on February 27.

(5) Let γ be the **last digit** of your student number. Write $\frac{\gamma+2i}{2+i}$ in the form $a+bi$ with a, b real numbers.

My answer:______________________________

Solution: $\frac{\gamma+2i}{2+i} = \frac{(\gamma+2i)(2-i)}{(2+i)(2-i)} = \frac{2\gamma-2i^2+4i-\gamma i}{4+1} = \frac{1}{5}\big((2\gamma+2)+(4-\gamma)i\big)$.
Reference: Suggested exercise 1h in §2.5; a similar problem was done in the DGD of February 13.

(6) Let α be the **third last** digit of your student number. Find the complex eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 \\ -(\alpha+1) & -1 \end{bmatrix}.$$

My answer:______________________________

Solution: The eigenvalues are the zeros of the characteristic polynomial $c_A(x) = \det(xI_2 - A)$. Hence

$$c_A(x) = \begin{vmatrix} x-1 & -2 \\ \alpha+1 & x+1 \end{vmatrix} = (x-1)(x+1) + 2\alpha + 2 = x^2 + 2\alpha + 1.$$

Since $2\alpha + 1 > 0$, the eigenvalues are therefore $x = \pm i\sqrt{1+2\alpha}$.
Reference: Suggested exercise 12b in §2.5; a similar problem was done in the DGD of February 13.

(7) The characteristic polynomial of the matrix A below is $(x-1)^2(x+2)$.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$$

(a) (1 pt) Find all eigenvalues of A.
(b) (4 pts) For each eigenvalue find the corresponding eigenspace and basic eigenvectors.
(c) (2 pts) Decide if your matrix A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (Justify your answer!)

Solution: (a) The eigenvalues of A are the zeros of the characteristic polynomial, which are $\lambda_1 = -2$ and $\lambda_2 = 1$ (multiplicity 2).

(b) To calculate the corresponding eigenspaces we solve the homogeneous linear system $\lambda I_3 - A$, of course by applying the Gaussian algorithm. For $\lambda = 1$ we obtain

$$I_3 - A = \begin{bmatrix} 0 & 3 & -3 \\ 0 & 6 & -6 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution and the eigenspace $E_1(A) = \{X \in \mathbb{R}^3 : AX = 0\}$ are therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (s \text{ and } t \text{ are free parameters}),$$

$$E_1(A) = \left\{ \begin{bmatrix} s & t & t \end{bmatrix}^T : s, t \in \mathbb{R} \right\}$$

and basic eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -2$ we obtain

$$-2I_3 - A = \begin{bmatrix} -3 & 3 & -3 \\ 0 & 3 & -6 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution and the eigenspace $E_{-2}(A) = \{X \in \mathbb{R}^3 : AX = 0\}$ are therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad (t \text{ a free parameter}), \quad E_{-2}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = y = z \right\}$$

and a basic eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

A is diagonalizable because $\lambda_2 = 1$ (double) has two basic eigenvectors. The matrix P of basic eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Marking: (b) eigenspaces 1 point each; basic eigenvectors 1 point each; (c) correct answer on diagonalizability 1 point; matrix D and P each 1/2 point.
Reference: Suggested exercise 2bdf in §2.3; same as problem 3 of assignment 1; see also the DGD of February 6.

(8) (6 pts) If the last digit of your student number is odd, work on A_{odd}. Otherwise, choose A_{even}. Find the inverse of the following matrix.

$$A_{odd} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}, \quad A_{even} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Show all details of your work.

Solution: For A_{odd}

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

So

$$A_{odd}^{-1} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

For A_{even},

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

So

$$A_{even}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Reference: Suggested exercise 2bdf in §1.5; same as problem 1 of assignment 1; a similar problem was done in the DGD on January 30.

(9) (2 bonus points) Let

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix};$$

Use the general formula for $\vec{u} \times \vec{v}$ to show that the cross product $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$.

Solution:

$$\vec{w} = \vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ z_1 x_2 - z_2 x_1 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Then the dot product $\vec{w}$ and $\vec{u}$ is nil. That proves that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$.