(1) (2 pts) In the matrix below replace α by the **last digit** of your student number. Find the matrix A satisfying the following equation:

$$\left(A^T + 2\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & 3 & \alpha \\ -1 & 0 & 2 \end{bmatrix}$$

Solution: Since $(A^T)^T = A$, the left hand side of the equation can be simplified as

$$A + 2\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix} = A + \begin{bmatrix} 2 & 0 & 2 \\ 4 & -2 & -4 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 1 & 3 & \alpha \\ -1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 2 \\ 4 & -2 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & \alpha - 2 \\ -5 & 2 & 6 \end{bmatrix}$$

Reference: Section 1.1, exercise #8

My answer: $A =$_________________

(2) (2 pts) Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

Which one is or which ones are in *reduced* row-echelon form?

My answer:__________________

Solution: A and C; B is not in ref because of columns 3; D is in ref but not in rref because of column 3; E is not in ref because of column 1; F is in ref, but not in rref because of column 4.

Marking: 1 point pour every correct answer.

(3) (3 pts) (a) A linear system with 4 variables and 5 equations always has infinity many solutions. True or False? (No justification required).

My answer:__________________

Solution: This is false. For example, one of the equations could be $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$.

(b) If the linear system $AX = B$ has no solution, any row-echelon form of the augmented matrix of the system has a row of zeros. True or False? (No justification required).

My answer:__________________

Solution: This is false. For example the linear system $0x = 1$ has no solution, yet the augmented matrix $[0\,1]$ is in row-echelon form, even in reduced row-echelon form.

(c) If a linear system has 5 equations and 4 variables, the rank of the augmented matrix is at most equal to ... (no justification required):

My answer:__________________

Solution: The augmented matrix A is a 5×5 matrix: There are 5 equations, 4 variables plus the right sides. The rank of A is therefore at most 5.

(4) (2 pts) Let A and B be two 3×4 matrices. Which of the following formulas is correct? The (23)-entry of $A^T B = (c_{ij})$ is

(A) $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{32} + a_{24}b_{42}$
(B) $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{32}$
(C) $c_{23} = a_{21}b_{31} + a_{22}b_{32} + a_{23}b_{32} + a_{24}b_{34}$
(D) $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{32} + a_{24}b_{34}$
(E) $c_{23} = a_{12}b_{13} + a_{22}b_{23} + a_{32}b_{33}$
(F) $c_{23} = a_{12}b_{12} + a_{22}b_{23} + a_{32}b_{32} + a_{42}b_{42}$

Solution: The correct answer is E, since putting $(d_{ij}) = D = A^T = (a_{ji})$ we have $c_{23} = \sum_{i=1}^{3} d_{2i}b_{i3} = \sum_{i=1}^{3} a_{i2}b_{i3} = a_{12}b_{13} + a_{22}b_{23} + a_{32}b_{33}$

My answer:________________

(5) (2 pts) Let A and B be matrices. True or false?

(a) If AB is defined, then also BA is defined.

Solution: False: Say A is 1×2 and B is 2×3. Then AB is defined and is a 1×3 matrix, while BA is not defined since the product has format $(2 \times 3) \cdot (1 \times 2)$.
Reference: §1.4, Ex. 10a, done in DGD

My answer:_________________

(b) If A^2 can be formed, then A must be square.

Solution: True: Assume that A is a $m \times n$ matrix. Since $A^2 = AA$ can be formed and AA is of type $(m \times n) \cdot (m \times n)$ we must have $m = n$, so A is a square matrix.
Reference: §1.4, Ex.10d, suggested exercise

Marking: 1 point for each correct answer

My answer:_________________

(6) (5 pts) Solve the following system and write the solution as linear combination of basic solutions:

$$\begin{array}{rrrrcl} x_1 & +2x_2 & -x_3 & +x_4 & = & 0 \\ -x_1 & -x_2 & & +2x_4 & = & 0 \\ x_1 & +x_2 & & -2x_4 & = & 0 \end{array}$$

Solution: The matrix of coefficients is

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 0 & -2 \end{bmatrix}.$$

We use Gaussian elimination to get the reduced row-echelon form as follows:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 0 & -2 \end{bmatrix} \overset{\text{row3+row2}}{\Rightarrow} \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\text{row2+row1}}{\Rightarrow} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\text{row1}-2\cdot\text{row2}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This leads to the following equivalent homogeneous linear system

$$\begin{array}{rrrrcl} x_1 & & +x_3 & -5x_4 & = & 0 \\ & x_2 & -x_3 & +3x_4 & = & 0 \end{array}$$

So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s+5t \\ s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

whose basic solutions are

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that basic solutions are not unique! In other words, you may have other basic solutions.

Reference: Section 1.3, Exercise #1

Marking: 3 points for row-reduction, 1 point for general solution, 1 point for basic solution

(7) (8 pts) Consider the system of linear equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & -1 \\ -x & - & y & + & pz & = & 4 \\ 3x & + & 4y & - & z & = & q \end{array}$$

where p and q are real numbers.

(a) (6 pts) determine the values of p and q for which the system has
  (i) a unique solution,
  (ii) no solution,
  (iii) infinitely many solutions.

(b) (2 pts) In case (iii) above give all solutions.

Solution: (a) We apply the Gaussian algorithm to the augmented matrix of the linear system:

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ -1 & -1 & p & 4 \\ 3 & 4 & -1 & q \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & p+1 & 3 \\ 0 & -2 & -4 & q+3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & p+1 & 3 \\ 0 & 0 & 2p-2 & q+9 \end{bmatrix}$$

In the first step we replaced row R_2 by $R_2 + R_1$ and row R_3 by $R_3 = 3R_1$; in the second step we replaced row R_3 by $R_3 + 2R_2$. The last equation of the linear system corresponding to the last matrix is

$$(2p-2)z = q + 9$$

from which we can decide the questions in (a):

(i) a unique solution: This happens if and only if $2p - 2 \neq 0$, i.e., $p \neq 1$.

(ii) no solution: This happens precisely when the equation $(2p-2)z = q+9$ has the from $0z = *$ where $* \neq 0$. Thus the condition is $p = 1$, $q \neq -9$.

(iii) infinitely many solutions: This happens in the remaining case, i.e., the last equation becomes $0z = 0$. Therefore $p = 1$, $q = -9$.

(b) We solve the linear system in case (iii), i.e., $p = 1$, $q = -9$. Observe that in this case we can further row-reduce:

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & p+1 & 3 \\ 0 & 0 & 2p-2 & q+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by replacing row R_1 by $R_1 - 2R_2$. The corresponding linear system is

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & -7 \\ & & y & + & 2z & = & 3 \end{array}, \quad \text{hence} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7+3t \\ 3-2t \\ t \end{bmatrix}$$

with t a free parameter is the general solution of the linear system.

Reference: §1.2, #9ab (done in DGD)

Marking: 1 point for the correct augmented matrix; 3.5 points for the row-reduction; 1.5 points for the correct answers in (i)–(iii) based on the row-reduced matrix, i.e., 0.5 points for the correct answer in (i)–(iii).