(1) (6 pts) Let α be the last digit of your student number. Which of the following is (are) subspaces? Support your answer with details.

(A) $U_1 = \{ \begin{bmatrix} x & y & z \end{bmatrix}^T : x \neq 0 \}$,

(B) $U_2 = \{X \in \mathbb{R}^3 : AX = 3X\}$, where A is a 3×3 matrix.

(C) $U_3 = \{ \begin{bmatrix} a & b & c \end{bmatrix}^T : a + 2b + c = 3\alpha \}$

Solution: (A) Since $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ is not in U_1, it is not a subspace.

(B) This is precisely the eigenspace $E_3(A)$, so it is a subspace.

(C) If $\alpha = 0$, then U_3 is a subspace, namely the null space of the matrix $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. This can also be seen directly: Given any two vectors $X, Y \in U_3$ with $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, $Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$, then $x_1 + 2x_2 + x_3 = 0 = y_1 + 2y_2 + y_3$. Similarly, for any scalar $t \in \mathbb{R}$,

$$(x_1 + y_1) + 2(x_2 + y_2) + x_3 + y_3 = 0, tx_1 + 2(tx_2) + tx_3 = 0$$

Hence, $X + Y \in U_3$ and $tX \in U_3$. Since clearly $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \in U_3$, it follows that U_3 is a subspace.

Let now $\alpha \neq 0$. Then repeating the argument above we see that tX does not belong to U_3 when $t \neq 0$. So in this case, U_3 is **not** a subspace. Another reason is that U_3 for $\alpha \neq 0$ does not contain the zero vector.

Marking: 2 points per problem, requires a correct justification. Correct answer only = 1 point

(2) (3 pts) Let β be the **second last** digit of your student number. Determine whether the following set is linearly independent. Support your answer with details.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \beta \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right\}$$

Solution: By the Invertible Matrix Theorem, it suffices to look at the matrix

$$A = \begin{bmatrix} 1 & \beta & 3 \\ 2 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}.$$

This set of vectors is linearly independent if and only if $\det A \neq 0$. But, by developing along the first column, we get

$$\det A = \det \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} - 2 \det \begin{bmatrix} \beta & 3 \\ -1 & -2 \end{bmatrix} = -2 - 2(-2\beta + 3) = 4\beta - 8.$$

So if $\beta = 2$, the set is linearly dependent. Otherwise, it is linearly independent.

One can also check that in any trivial linear combination, all coefficients are zero.

(3) (6 pts) Let α the last digit of your student number, and let

$$\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ \alpha \\ \alpha \end{bmatrix} \quad \vec{z} = \begin{bmatrix} -4\alpha - 1 \\ -\alpha \\ 2\alpha + 1 \end{bmatrix}$$

(a) Is $\vec{w} \in \operatorname{Span}\{\vec{u}, \vec{v}\}$? (b) Is $\vec{z} \in \operatorname{Span}\{\vec{u}, \vec{v}\}$?

Solution: (a) The vector $\vec{w}$ lies in the span of $\vec{u}$ and $\vec{v}$ if and only if there exist $a, b \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} = \vec{w}$. This is equivalent to solvability of the linear system $AX = \vec{w}$ where A is formed out of the vectors $\vec{u}$ and $\vec{v}$ as columns. We row-reduce the corresponding augmented matrix:

$$\begin{bmatrix} -1 & 3 & 2 \\ 2 & 4 & \alpha \\ 3 & 2 & \alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 2 & 4 & \alpha \\ 3 & 2 & \alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & \alpha + 4 \\ 0 & 11 & \alpha + 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & \alpha - 16 \end{bmatrix}$$

(Hint: in the last step we first subtracted row 2 from row 3, then switched row 2 and 3, and then continued with the usual Gaussian algorithm). By looking at the last row, we see that the system is not compatible: Because $0 \le \alpha \le 9$, $\alpha - 16 \neq 0$. Hence $\vec{w}$ is not in $\operatorname{Span}\{\vec{u}, \vec{v}\}$.

(b) We proceed in the same way as in (a):

$$\begin{bmatrix} -1 & 3 & -4\alpha - 1 \\ 2 & 4 & -\alpha \\ 3 & 2 & 2\alpha + 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4\alpha + 1 \\ 2 & 4 & -\alpha \\ 3 & 2 & 2\alpha + 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \alpha + 1 \\ 0 & 1 & -\alpha \\ 0 & 0 & \alpha - 2 \end{bmatrix}$$

Then if $\alpha = 2$ the system is solvable, in fact $\vec{z} = 3\vec{u} - 2\vec{v}$ and so $\vec{z}$ lies in the span of $\vec{u}$ and $\vec{v}$. However, if $\alpha \neq 2$, the system is not solvable and thus $\vec{z}$ is not in $\operatorname{Span}(\vec{u}, \vec{v})$

Marking: 3 points per problem

(4) (6 pts) Let α the last digit of your student number and let β be the second last digit of your student number. We consider

$$E = \left\{ \begin{bmatrix} \alpha s & \alpha s - \beta t & \alpha s + \beta t & -\beta t \end{bmatrix}^T \in \mathbb{R}^4 : s \text{ and } t \in \mathbb{R} \right\}$$

(a) Show that E is a subspace of $\mathbb{R}^4$.
(b) Find a basis of E.
(c) Calculate the dimension of E.

Solution: (a) The first method of solving (a) is to check that E satisfies the 3 conditions defining a subspace: E contains $\vec{0} = (0, 0, 0, 0)$, E is closed under addition of vectors, and E is closed under scalar multiplication. The second method of solving (a) is to write E as a span and then use the fact that any span is a subspace. In fact, it is immediate that E is spanned by

$$\vec{u} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ -\beta \\ \beta \\ -\beta \end{bmatrix}$$

(b) If $\alpha = 0$ and $\beta = 0$, then $E = \{0\}$, and so E does not have a basis.
If $\alpha = 0$ and $\beta \neq 0$ then a basis of E is

$$\vec{v} = \begin{pmatrix} 0 \\ -\beta \\ \beta \\ -\beta \end{pmatrix};$$

If $\alpha \neq 0$ and $\beta = 0$ then a basis of E is

$$\vec{u} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \end{pmatrix};$$

If $\alpha \neq 0$ and $\beta \neq 0$ then a basis of E is

$$\vec{u} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \end{pmatrix}; \vec{v} = \begin{pmatrix} 0 \\ -\beta \\ \beta \\ -\beta \end{pmatrix};$$

(c) The dimension of E is 2 if $\alpha \neq 0$ and $\beta \neq 0$

The dimension of E is 1 if $\alpha \neq 0$ and $\beta = 0$ or if $\alpha = 0$ and $\beta \neq 0$.

The dimension of E is 0 if $\alpha = 0$ and $\beta = 0$.

Marking: (a) 2 points, (b) 3 points, (c) 1 point.

(5) (9 pts) In the matrix A below replace α by the **last digit** and β by the **second last** digit of your student number.

$$A = \begin{bmatrix} 1 & 1 & 0 & \beta \\ 1 & 1-\alpha & -1 & \beta-1 \\ 2 & 2-2\alpha & -2 & 2\beta-2 \\ 3 & 3+2\alpha & 2 & 3\beta+2 \end{bmatrix}.$$

(a) Find the rank of A.
(b) Find a basis for $row(A)$ and the dimension of $row(A)$.
(c) Find a basis for $col(A)$ and the dimension of $col(A)$.
(d) Find a basis of the space of solutions of the homogeneous linear system $AX = 0$.

Solution: (a) We row-reduce the matrix A:

$$\begin{bmatrix} 1 & 1 & 0 & \beta \\ 1 & 1-\alpha & -1 & \beta-1 \\ 2 & 2-2\alpha & -2 & 2\beta-2 \\ 3 & 3+2\alpha & 2 & 3\beta+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & \beta \\ 0 & -\alpha & -1 & -1 \\ 0 & -2\alpha & -2 & -2 \\ 0 & 2\alpha & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & \beta \\ 0 & \alpha & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Regardless of $\alpha = 0$ or $\alpha \neq 0$, the rank of A is 2.

(b) A basis of the row space if given by the non-zero rows of R, that is

$$\begin{bmatrix} 1 & 1 & 0 & \beta \end{bmatrix}, \quad \begin{bmatrix} 0 & \alpha & 1 & 1 \end{bmatrix}$$

The dimension of the row space is 2.

(c) A basis of the column space is given by the columns of A corresponding to the columns of R with leading 1's. These are the columns 1 and 2 of A if $\alpha \neq 0$. Hence a basis of the column space of A is

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1-\alpha \\ 2-2\alpha \\ 3+2\alpha \end{bmatrix} \qquad (\alpha \neq 0)$$

while for $\alpha = 0$ these are the columns 1 and 3 of A:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ -2 \\ 3 \end{bmatrix} \qquad (\alpha = 0).$$

The dimension of the column space is 2 in both cases (and equals the dimension of the row space and the rank).

(d) The space of solutions of the homogeneous linear system is the same as the null space of A. It is therefore also the same as the null space of R.

Case $\alpha = 0$: Then

$$R = \begin{bmatrix} 1 & 1 & 0 & \beta \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the corresponding linear system is $x_1 = -x_2 - \beta x_4$, $x_3 = -x_4$. The general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-\beta t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\beta \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{basis:} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\beta \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

A basis of null(A) is given by the basic solutions.

Case $\alpha \neq 0$: We proceed as in case $\alpha = 0$.

$$R = \begin{bmatrix} 1 & 1 & 0 & \beta \\ 0 & \alpha & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the corresponding linear system is $x_1 = -x_2 - \beta x_4 = \frac{1}{\alpha}x_3 - (\frac{1}{\alpha} + \beta)x_4$, $x_2 = -\frac{1}{\alpha}(x_3 + x_4)$. The general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}s - (\frac{1}{\alpha} + \beta)t \\ -\frac{1}{\alpha}(s+t) \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{\alpha} \\ -\frac{1}{\alpha} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -(\frac{1}{\alpha} + \beta) \\ -\frac{1}{\alpha} \\ 0 \\ 1 \end{bmatrix} \qquad \text{basis: } \begin{bmatrix} \frac{1}{\alpha} \\ -\frac{1}{\alpha} \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -(\frac{1}{\alpha} + \beta) \\ -\frac{1}{\alpha} \\ 0 \\ 1 \end{bmatrix}$$

Marking: (a) 2 points for the correct row-reduction, 1 point for rank; (b) 1 point for basis and 1 point for dimension; (c) 1 point for basis and 1 point for dimension; (d) 2 points

(6) (2 bonus points) Use the theorems in sections 4.1 – 4.3 to prove the following: Let $U \subset \mathbb{R}^n$ be a non-zero subspace, spanned by some subset $S \subset U$ and let $L \subset S$ be a linearly independent set. Then there exists a basis B of U satisfying $L \subset B \subset S$.

Solution: If $\text{Span}\,(L) = U$, we are done: L is a basis. Otherwise, $\text{Span}\,(S)$ is a proper subspace of U, i.e., $\text{Span}\,(S) \subsetneq U$. In this case, we claim there exists $X \in S$ which does not lie in the span $\text{Span}\,(L)$. Indeed, otherwise $S \subset \text{Span}\,(L)$ and therefore also $\text{Span}\,(S) \subset \text{Span}\,(L)$ by Theorem 1(2) in section §4.1. But this leads to the contradiction $U = \text{Span}\,(S) \subset \text{Span}\,(L) \subsetneq U = \text{Span}\,(S)$. Hence, there exists $X \in S$ with $S \notin \text{Span}\,(L)$. We then know from the Independent Lemma in §4.3.2 that $L_1 = L \cup \{X\} \subset S$ is a linearly independent subset. Thus, we have increased the size of L and now have the situation $L_1 \subset S$.

Continuing in this way we construct an increasing chain of subsets $L \subset L_1 \subset \cdots \subset L_m \subset S$ such that L_m is linealy independent. This process cannot go on indefinitely because of the Fundamental Theorem or its Corollary: $\mathbb{R}^n$ does not contain more than n linearly independent vectors. Hence, for some L_m we get $\text{Span}\,(L_m) = U$ and thus L_m is a basis.

WARNING: The claim does not follow by quoting Theorem 3(2). That result simply says that L can be enlarged to a basis of U, but does not imply that the basis lies in S. Similarly, the result does not follow from Theorem 3(3): That result says that S contains a basis of U but it does not say that the basis will contain L.