(1) (6 pts) If the **last entry** of your student number is even take A_{even}, otherwise take A_{odd}.

$$A_{even} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ -1 & -3 & 2 \end{bmatrix} \qquad A_{odd} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Find the inverse of your matrix and check your answer by verifying the equation $AA^{-1} = I_3$.

Solution: For A_{even} we get

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{bmatrix} \overset{\text{row3+row1}}{\Rightarrow} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{bmatrix} \overset{\text{row3+row2}}{\Rightarrow}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{bmatrix} \overset{(-1)\cdot\text{row3}}{\Rightarrow} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \underset{\text{row1+row3}}{\overset{\text{row2+2}\cdot\text{row3}}{\Rightarrow}}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \overset{\text{row1}-2\cdot\text{row2}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & 4 & 1 & 3 \\ 0 & 1 & 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$$

So we found

$$A_{even}^{-1} = \begin{bmatrix} 4 & 1 & 3 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$$

We now check

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution for A_{odd} is

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \overset{\text{row3}-2\cdot\text{row1}}{\Rightarrow} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix}$$

$$\underset{\text{row1+row3}}{\overset{\text{row2}-2\cdot\text{row3}}{\Rightarrow}} \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 4 & 1 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \overset{\text{row1}-\cdot\text{row2}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & -5 & -1 & 3 \\ 0 & 1 & 0 & 4 & 1 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix}$$

So we found

$$A_{odd}^{-1} = \begin{bmatrix} -5 & -1 & 3 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

We check:

$$AA^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -1 & 3 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Marking: 5 points for applying the inversion algorithm the process with correct answer, and 1 point for the verification $AA^{-1} = I$.

(2) (4 pts) Compute $\det A$ (replace α by the last digit of your student number) with

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 9 & -2 & 1 \\ -2 & -6 & 3 & 0 \\ 3 & 9 & \alpha & 3 \end{bmatrix}$$

Solution: We add $(-3) \times C_1$ to C_2 where C_i is the ith column. This gives

$$\det A = \det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & -2 & 1 \\ -2 & 0 & 3 & 0 \\ 3 & 0 & \alpha & 3 \end{bmatrix}$$

We then use cofactor expansion along the second column

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42} = a_{22}C_{22} = 3C_{22}$$

$$C_{22} = (-1)^{2+2} \det \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 0 \\ 3 & \alpha & 3 \end{bmatrix}$$

Then we use cofactor expansion along the third column, we get

$$C_{22} = 2 \cdot (-1)^{1+3} \det \begin{bmatrix} -2 & 3 \\ 3 & \alpha \end{bmatrix} + 3 \cdot (-1)^{3+3} \det \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= 2 \cdot (-2 \cdot \alpha - 3 \cdot 3) + 3 \cdot (1 \cdot 3 - (-2) \cdot (-1)) = -4\alpha - 15.$$

So

$$\det A = 3 \cdot (-4\alpha - 15) = -12\alpha - 45.$$

(3) (10 pts) If the **second-last entry** of your student number is even take A_{even}, otherwise take A_{odd}:

$$A_{even} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \qquad A_{odd} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

(a) Find the characteristic polynomial and the eigenvalues of your matrix A. As well, for each eigenvalue find the corresponding eigenspace and basic eigenvectors.

(b) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (Justify your answer)

Solution: For A_{even} (a) The characteristic polynomial is $c_A(x) = \det(xI_3 - A)$:

$$\begin{aligned}
\det(xI_3 - A) &= \begin{vmatrix} x-3 & 1 & 2 \\ -2 & x & 2 \\ -2 & 1 & 1+x \end{vmatrix} = \begin{vmatrix} x-1 & 1-x & 0 \\ -2 & x & 2 \\ -2 & 1 & 1+x \end{vmatrix} \\
&= (x-1)\begin{vmatrix} 1 & -1 & 0 \\ -2 & x & 2 \\ -2 & 1 & 1+x \end{vmatrix} = (x-1)\begin{vmatrix} 1 & 0 & 0 \\ -2 & x-2 & 2 \\ -2 & -1 & x+1 \end{vmatrix} = \\
&= (x-1)\begin{vmatrix} x-2 & 2 \\ -1 & x+1 \end{vmatrix} \\
&= (x-1)\big[(x-2)(1+x)+2\big] = (x-1)(x^2-x) = (x-1)^2x.
\end{aligned}$$

In this calculation we have first subtracted row R_2 from row R_1, then pulled out the factor $x-1$ from the first row, then added the first column to the second column, then used cofactor expansion along the first row, and finally calculated the 2×2 determinant.

The eigenvalues of A are the zeros of $c_A(x)$, hence $\lambda_1 = 0$ and $\lambda_2 = 1$ (multiplicity 2). To calculate the corresponding eigenvectors we solve the homogeneous linear system $\lambda I_3 - A$, of course by applying the Gaussian algorithm. For $\lambda = 0$ we obtain

$$0I_3 - A = -A = \begin{bmatrix} -3 & 1 & 2 \\ -2 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution and the eigenspace $E_0(A) = \{X \in \mathbb{R}^3 : AX = 0\}$ are therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (t \text{ a free parameter}), \quad E_0(A) = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = y = z\}$$

and a basic eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$ we get

$$I_3 - A = \begin{bmatrix} -2 & 1 & 2 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & \end{bmatrix}$$

The general solution and the eigenspace $E_0(A) = \{X \in \mathbb{R}^3 : AX = 0\}$ are therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2(s+2t) \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (s \text{ and } t \text{ are free parameters}),$$

$$E_1(A) = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 1/2s + t, \text{ for } s, t \in \mathbb{R}\}$$

and basic eigenvectors are

$$\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Note that instead of the first vector we can also take $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$.

(b) A is diagonalizable because $\lambda_2 = 1$ (double) has two basic eigenvectors. The matrix P of basic eigenvectors is

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that P is automatically invertible.

Solution for A_{odd}: (a) The characteristic polynomial is $c_A(x) = \det(xI_3 - A)$:

$$\det(xI_3 - A) = \begin{vmatrix} x-1 & -2 & 0 \\ -2 & x-1 & 0 \\ -2 & 1 & 1+x \end{vmatrix} =$$

$$= (x+1) \begin{vmatrix} x-1 & -2 \\ -2 & x-1 \end{vmatrix} = (x+1)\big[(x-1)^2 - 4\big]$$

$$= (x+1)[x^2 - 2x - 3] = (x+1)(x+1)(x-3).$$

In this calculation we have just used cofactor expansion along the third column, and finally calculated the 2×2 determinant.

The eigenvalues of A are the zeros of $c_A(x)$, hence $\lambda_1 = 3$ and $\lambda_2 = -1$ (multiplicity 2). To calculate the corresponding eigenvectors we solve the homogeneous linear system $\lambda I_3 - A$, of course by applying the Gaussian algorithm. For $\lambda = 3$ we obtain

$$3I_3 - A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ -2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -4 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution and the eigenspace are therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4t \\ 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \quad (t \text{ a free parameter}), \qquad E_3(A) = \{ \begin{bmatrix} 4t \\ 4t \\ t \end{bmatrix} : t \in \mathbb{R}\}$$

and a basic eigenvector is

$$\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

For $\lambda = -1$ we get

$$-I_3 - A = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution and the eigenspace is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \quad (t \text{ a free parameter}), \qquad E_{-1}(A) = \{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z = t \text{ for } t \in \mathbb{R}\}$$

and basic eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) A is not diagonalizable because $\lambda_2 = -1$ (double) has only one basic eigenvector.

Marking: 3 points for $c_A(x)$, 1 point for its solution, 4 points for eigenspaces and basic eigenvectors; (b) 1 point for P and 1 point for D in case A_{even} and 2 points for correct answer for A_{odd}.

(4) (2 bonus points) If A, B are idempotent matrices, i.e., $A^2 = A$ and $B^2 = B$, and $AB = BA = 0$, show that $I - A$ and $A + B$ are also idempotent.

Solution:

$$(I - A)^2 = (I - A)(I - A) = I^2 - IA - AI + AA = I - 2A + A = I - A,$$

$$(A + B)^2 = (A + B)(A + B) = AA + AB + BA + BB = AA + BB = A + B.$$

So both $I - A$ and $A + B$ are idempotents.

Marking: 1 point for each correct equation