(1) (2 pts) In the matrix below replace β with the **second last digit of your student number** and calculate the determinant:

$$\begin{vmatrix} 1 & 2 & \beta \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

Solution: Expanding along the first column, we get

$$\begin{vmatrix} 1 & 2 & \beta \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & \beta \\ 1 & 2 \end{vmatrix} = -1 + (4 - \beta) = 3 - \beta.$$

My answer:________________

(2) (4 pts) Let A be an $n \times n$ matrix with $\det(A) = 0$. For each of the following statements determine if it is true or false. Answer with T (for true) and F (for false)

(a) A is invertible.

My answer:________________

(b) 0 is an eigenvalue.

My answer:________________

(c) The columns of A span $\mathbb{R}^n$.

My answer:________________

(d) The rows of A are linearly dependent.

My answer:________________

Solution: (a) F (b) T (c) F (d) T .
Reference: §5.2 Th.3.

(3) (2 pts) Write the complex number below in the form $a + bi$ for $a, b, \in \mathbb{R}$:

$$\frac{11 - 2i}{1 - 2i}.$$

Solution: We multiply both numerator and denominator by the complex conjugate of the denominator:

$$\frac{11 - 2i}{1 - 2i} = \frac{(11 - 2i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{15 + 20i}{5} = 3 + 4i.$$

Reference: Exercise App.A, 2d. DGD (March 8): App.A, 2c

My answer:________________

(4) (4 pts) Determine if the statements below are true or false. Answer with T (for true) or F (for false). No justification required.

(a) $U = \{\begin{bmatrix} 2 & 3s & 4t \end{bmatrix}^T : s, t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

My answer:_________________

(b) $U = \{\begin{bmatrix} s+3t & 4r+5s & r-2s+4t \end{bmatrix}^T : r, s, t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

My answer:_________________

(c) If U is a subspace of $\mathbb{R}^n$ and $rX \in U$ for all $r \in \mathbb{R}$, then $X \in U$.

My answer:_________________

(d) If $U = \text{Span}\{X, Y\}$ and $Z \in U$, then also $U = \text{Span}\{X, Y, Z\}$.

My answer:_________________

Solution: (a) False: $0 \notin U$. (b) True. For example, $U = \text{Span}\{X_1, X_2, X_3\}$ where

$$X_1 = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T, \quad X_2 = \begin{bmatrix} 1 & 5 & -2 \end{bmatrix}^T, \quad X_3 = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}^T.$$

(c) True: Take $r = 1$. (d) True: $\text{Span}\{X, Y, Z\}$ is the smallest subspace containing X, Y and Z. Since $Z \in \text{Span}\{X, Y\}$ we get $\text{Span}\{X, Y, Z\} \subseteq \text{Span}\{X, Y\}$. But the other inclusion is obvious.

Reference: §5.1 : (a) #1a DGD (March 22), (b) #1c DGD (March 22), (c) #16b (suggested exercise), (d) #16d (suggested exercise).

(5) (3 pts) Find the area of the triangle OAB where $O(0,0,0)$, $A(-4,1,)$ and $B(-4,2,3)$.

Solution: The cross product of the position vectors $\overrightarrow{OA} \times \overrightarrow{OB}$ is

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} = \begin{vmatrix} \vec{i} & -4 & -4 \\ \vec{j} & 1 & 2 \\ \vec{k} & 1 & 3 \end{vmatrix}$$

$$= \left[\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} -4 & -4 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} -4 & -4 \\ 1 & 2 \end{vmatrix} \right] = [1, 8, -4]^T$$

The area of the triangle is therefore $\|\overrightarrow{OA} \times \overrightarrow{OB}\|/2 = \sqrt{1 + 64 + 16} = \sqrt{81}/2 = 9/2$.

Reference: Homework 4, problem 5. Exercise §4.3 #4b. DGD (March 15) §4.3 #4a

Marking: 2 point for $\overrightarrow{OA} \times \overrightarrow{OB}$, 1 point for area.

My answer:_________________

(6) (4 pts) Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and corresponding eigenvectors $X_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and $X_2 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$.
(a) What is the general solution of the linear system of differential equations $f' = Af$?
(b) Find the solution of $f' = Af$ satisfying the boundary conditions $f_1(0) = 3$, $f_2(0) = 1$.

Solution: (a) (2 pts) The general solution is

$$f(x) = c_1 X_1 e^{\lambda_1 x} + c_2 X_2 e^{\lambda_2 x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\lambda_1 x} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{\lambda_2 x},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary scalars.

(b) (2 pts) For such a f the boundary conditions give the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 3c_2 & = & 3 \\ c_1 & + & 2c_2 & = & -1 \end{array}$$

which we solve by reducing the augmented matrix

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Hence $c_1 = 3$ and $c_2 = -1$ are the solutions of the linear system. The function satisfying the given boundary conditions is therefore

$$f(x) = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\lambda_1 x} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{\lambda_2 x}.$$

Reference: Exercise §3.5 #1b. DGD (March 8) #1c.

(7) (8 pts) (a) Find all eigenvalues of the matrix $\begin{bmatrix} 4 & 8 \\ 3 & 2 \end{bmatrix}$.

(b) The eigenvalues of the matrix $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = -1$ (you do not have to show this!). For each eigenvalue of B find all eigenvectors.
(c) Decide if B is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer, both for yes and no!

Solution: (a) (2 pts) The characteristic equation is

$$\begin{aligned} 0 = c_A(x) = \det(xI_2 - A) &= \begin{vmatrix} x-4 & -8 \\ -3 & x-2 \end{vmatrix} = (x-4)(x-2) - 24 \\ &= x^2 - 6x - 16 = (x-8)(x+2). \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = -2$.

(b) (4 pts) The eigenspace $E_5(B)$ is null$(5I_2 - B)$, where

$$5I_2 - B = \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence we have to solve $2x - y = 0$. The general solution of this system is therefore

$$\begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (t \in \mathbb{R})$$

The eigenvectors for $\lambda = 5$ are therefore the solutions with $t \neq 0$.

We proceed in the same way with $\lambda = -1$:

$$-I_2 - B = \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence we have to solve $x + y = 0$. The general solution of this system is therefore

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad (t \in \mathbb{R})$$

The eigenvectors for $\lambda = -1$ are therefore the solutions with $t \neq 0$.

(c) (2 pts) Since B has distinct eigenvalues, it is diagonalizable. The matrices P and D are not unique, but one choice is

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

One takes P as the matrix, whose columns are basic eigenvectors and then takes D as the diagonal matrix whose diagonal entries are the eigenvalues corresponding to the eigenvectors in P.

(8) (6 pts) Are the following sets of vectors linearly independent in $\mathbb{R}^3$?
(a) $\{ \begin{bmatrix} 1 & -4 & 7 \end{bmatrix}^T, \begin{bmatrix} -3 & 5 & 6 \end{bmatrix}^T, \begin{bmatrix} 9 & 12 & 4 \end{bmatrix}^T, \begin{bmatrix} 0 & 3 & -2 \end{bmatrix}^T \}$
(b) $\{ \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} -1 & 3 & -5 \end{bmatrix}^T \}$

Solution: (a) (2 pts) No, since 4 vectors in $\mathbb{R}^3$ are always linearly dependent (Fundamental Theorem of Linear Algebra).

(b) (4 pts) To verify that the given three vectors X_1, X_2, X_3 are linearly independent, we assume that we have scalars s_1, s_2, s_3 such that $s_1X_1 + s_2X_2 + s_3X_3 = 0$. This yields an homogeneous linear system whose coefficient matrix is

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ -4 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 6 \\ 0 & 3 & -9 \end{bmatrix}$$

Thus, the coefficient matrix has rank 2. Since there are 3 variables, the general solution has $3 - 2 = 1$ free parameter. This means that there is a non-zero solution. The given vectors are therefore not linearly independent.

Another method: Show that the determinant of the matrix, whose columns are the three given vectors, is zero.

(Or, one can simply "see" that $2\begin{bmatrix} 1 & 3 & -4 \end{bmatrix}^T - 3\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 3 & -5 \end{bmatrix}^T$.)

(9) (4 bonus points) (a) Let $X_1, \ldots, X_k$ be vectors in $\mathbb{R}^n$. Define Span $\{X_1, X_2, \ldots, X_k\}$.
(b) Show that Span $\{X_1, X_2\}$ is a subspace of $\mathbb{R}^n$.

Solution: (a) (1 pt) The span is the subset of all linear combinations of $X_1, X_2, \ldots, X_n$. Equivalently

$$\text{Span}\{X_1, \ldots, X_k\} = \{s_1X_1 + s_2X_2 + \cdots s_kX_k : s_1, \ldots, s_k \in \mathbb{R}\}$$

(b) (3 pts) We abbreviate $U = \text{Span}\{X_1, \ldots, X_k\}$, and check the three axioms (S1)-(S3) for a subspace.

(S1) The zero vector $0 \in U$, since it is obtained by putting all scalars $s_i = 0$: $0 = 0X_1 + 0X_2 + \cdots + 0X_k$.

(S2) Suppose $X, Y \in U$. Thus $X = s_1X_1 + \cdots + s_kX_k$ and $Y = t_1X_1 + \cdots + t_kX_k$ for certain $s_i, t_j \in \mathbb{R}$. But then

$$X + Y = (s_1 + t_1)X_1 + \cdots + (s_k + t_k)X_k$$

shows that $X + Y$ is also a linear combination of $X_1, \ldots, X_k$, i.e., $X + Y \in U$.

(S3) Suppose $X \in U$ and let $s \in \mathbb{R}$ be scalar. Thus $X = s_1X_1 + \cdots + s_kX_k$ for certain $s_i, t_j \in \mathbb{R}$. Then

$$sX = (ss_1)X_1 + \cdots + (ss_k)X_k$$

shows that sX is also a linear combination of $X_1, \ldots, X_k$, i.e., $X + Y \in U$.