(1) (a) (1 pt) A linear system with 4 variables and 5 equations always has infinitely many solutions. True or false? (No justification required)

Solution: This is **false**, since the system may not be solvable. For example, the first equation could be $x_1 + x_2 + x_3 + x_4 = 0$ and the second equation $x_1 + x_2 + x_3 + x_4 = 1$.
Reference: Homework 1, problem 4

My answer:__________________

(b) (1 pt) If the linear system $AX = B$ has no solution, any row-echelon form of A has a row of zeros. True or false? (No justification required)

Solution: This is **correct**: Since the system does not have a solution, any row-echelon form of the **augmented** matrix has a row of the form $\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$. The row up to the last entry is a row of a row-echelon form of A.
Reference: §1.2, exercise 12c (DGD)

My answer:__________________

(c) (1 pt) If the linear system has 5 equations and 4 variables, the rank of the augmented matrix is at most (no justification required):

Solution: The augmented matrix has size 5×5, so the rank is **at most 5.**
Reference: §1.2, assigned exercise 12h

My answer:__________________

(d) (1 pt) We consider the homogeneous linear system $AX = 0$. If a row-echelon form of A has a row of zeros, the system has a non-trivial solution. True or false? (No justification required)

Solution: This is **false**. For example the linear system

$$\begin{array}{rcrcl} x & + & y & = & 0 \\ x & - & y & = & 0 \\ 2x & - & 2y & = & 0 \end{array} \quad \text{has coefficient matrix } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

but is uniquely solvable.
Reference: §1.3, assigned exercise 1h

My answer:__________________

(2) (a) (1 pt) Let α be the **last digit** of your student number. Find the matrix A if

$$\left(2A^T + \begin{bmatrix}\alpha \\ 8\end{bmatrix}\right)^T = 4\begin{bmatrix}-1 & 9\end{bmatrix}$$

Solution: We use the basic rules of transposition (Th. 2 in §2.1) and obtain:

$$2A + \begin{bmatrix}\alpha & 8\end{bmatrix} = 4\begin{bmatrix}-1 & 9\end{bmatrix},$$

whence

$$2A = \begin{bmatrix}-4 & 36\end{bmatrix} - \begin{bmatrix}\alpha & 8\end{bmatrix} = \begin{bmatrix}-4-\alpha & 28\end{bmatrix}$$

and therefore

$$A = \begin{bmatrix}-2-\alpha/2 & 14\end{bmatrix}$$

Reference: Homework 2, problem 4; section 2.1, exercise 15a(DGD), 15b (assigned)

My answer:__________________

(b) (1 pt) If A is a 3×7 matrix, then A and its transpose A^T have the same main diagonal. True or false?

Solution: This is true.
Reference: §2.1, assigned exercise 19d

My answer:__________________

(c) (1 pt) Let A be a $m \times n$ matrix and let X be a n-column vector. If AX has a zero entry, then A has a row of zeros. True or false?

Solution: This is not true. For example, if $A = \begin{bmatrix}1 & -1\end{bmatrix}$ (so $m = 1$, $n = 2$) and $X^T = \begin{bmatrix}1 & -1\end{bmatrix}$, then $AX = 0$, but A does not have a row of zeros.
Reference: §2.2 assigned exercise 10b

My answer:__________________

(d) (1 pt) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the y-axis. The standard matrix of this linear transformation is:

Solution: The transformation sends a vector $\begin{bmatrix}x & y\end{bmatrix}^T$ to $\begin{bmatrix}-x & y\end{bmatrix}^T$. Hence the standard matrix is

$$A = \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}$$

Reference: §2.2, exercises 11b(assigned) and 11c (DGD)

My answer:__________________

(3) (a) (1 pt) Consider the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

Which one is or which ones are in reduced row-echelon form?

Solution: B and D; A is not in ref because of columns 2; C is in ref but not in rref because of column 3; E is not in ref because of column 1; F is in ref, but not in rref because of column 2.
Marking: 1/2 point for every correct answer

My answer:__________________

(b) (3 pts) Complete the theorem below by stating 3 conditions which are equivalent to, but not the same as the condition in (a) of the theorem.

Theorem. *For a $n \times n$ matrix A the following conditions are equivalent:*

(a) *A is invertible.*

(b)

(c)

(d)

Remark: The theorem stated in class had more equivalent conditions. But you are only asked to list 3 of them.

Solution: Any combination of three of the following is correct: :

- The linear system $AX = B$ has a unique solution for every column B.
- The homogeneous linear system $AX = 0$ has only the trivial solution.
- The reduced row-echelon form of A is the identity matrix I_n.
- A has rank n.
- The linear system $AX = B$ has a solution for every columns B.
- There exists a $n \times n$ matrix C such that $AC = I_n$.
- A^T is invertible.
- $\det(A) \neq 0$.

Marking: 1 point for each answer. The answers must be mathematically correct. For example, the answer "$AC = I_n$" is not correct (the correct answer is "there exists a ... …").

(4) For the system of linear equations

$$\begin{array}{ccccccc} x & + & 2y & + & 4z & = & 2 \\ 2x & + & 7y & + & 8z & = & -2 \\ x & + & ay & + & a^2 z & = & a \end{array}$$

(a) (6 pts) determine the values of a for which the system has
  (i) no solution,
  (ii) infinitely many solutions,
  (iii) a unique solution.

(b) (2 pts) In case (ii) above describe give all solutions.

Solution: The augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 2 & 7 & 8 & -2 \\ 1 & a & a^2 & a \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - R_1$, $R_2 \rightsquigarrow (1/3)R_2$, $R_3 \rightsquigarrow R_3 - (a-2)R_2$, and obtain

$$A \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 3 & 0 & -6 \\ 0 & a-2 & a^2-4 & a-2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & a-2 & a^2-4 & a-2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & a^2-4 & 3(a-2) \end{array}\right] = M \qquad (*).$$

Since $a^2 - 4 = (a-2)(a+2)$ we get :

- If $a = -2$, then the last row of M is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -12 \end{array}\right]$. Hence the system is inconsistent.
- If $a = 2$ then

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Hence the system has infinitely many solutions.
- If $a \notin \{-2, 2\}$, then $a^2 - 4 \neq 0$ and so

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & * \end{array}\right]$$

  for $* = 3(a-2)/(a^2-4)$. Hence the system is uniquely solvable.

The answer to question (a) is therefore :

(i) The system in inconsistent if $a = -2$.
(ii) The system has infinitely many solutions if $a = 2$.
(iii) The system is uniquely solvable if $a \notin \{2, -2\}$.

(b) We have seen in (a) above that in this case

$$A \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is a matrix in reduced row-echelon form. The corresponding linear system is

$$\begin{array}{rcrcl} x & & + \; 4z & = & 6 \\ & y & & = & -2 \end{array}$$

Thus z is a free variable. Putting $z = t$, the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6-4t \\ -2 \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \quad (t \text{ a free parameter})$$

Reference: Homework 1, problem 5, §2.1; also exercise 9c (DGD) and 9f (assigned)

Marking: : (a) 3 pts for the row-reduction until (*), 1 pt for (i), (ii) and (iii).

(5) (a) (6 pts) In the matrix below **replace** β **with the second-last digit of your student number** and find its inverse:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & \beta \\ 2 & 4 & -1 \end{bmatrix}$$

(b) (2 pts) Check your answer by verifying $AA^{-1} = I_3$.

Solution: We apply the Inversion Algorithm, i.e., we find the reduced row-echelon form of $[A|I_3]$:

$$\begin{aligned}
[A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & \beta & 0 & 1 & 0 \\ 2 & 4 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & \beta+3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -(\beta+3) & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2\beta-3 & -1 & \beta+3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4\beta+5 & 2 & -2\beta-5 \\ 0 & 1 & 0 & -2\beta-3 & -1 & \beta+3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]
\end{aligned}$$

Hence the inverse of A is the 3×3-matrix next to the identity matrix I_3 above:

$$A^{-1} = \begin{bmatrix} 4\beta+5 & 2 & -2\beta-5 \\ -2\beta-3 & -1 & \beta+3 \\ -2 & 0 & 1 \end{bmatrix}.$$

We check

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & \beta \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 4\beta+5 & 2 & -2\beta-5 \\ -2\beta-3 & -1 & \beta+3 \\ -2 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4\beta+5-4\beta-6+2 & -2+2+0 & -2\beta-5+2\beta+6-1 \\ 12\beta+15-10\beta-15-2\beta & 6-5 & -6\beta-15+5\beta+15+\beta \\ 8\beta+10-8\beta-12+2 & 4-4 & -4\beta-10+4\beta+12-1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Marking: finding the correct rref: 5 pts; identifying A^{-1} : 1 pt; verifying $AA^{-1} = I_3$: 2 pts

(6) (2 bonus points) (a) Give the definition for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.
(b) Let A be an $n \times m$ matrix. Show that the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, given by $T(X) = AX$ is linear.

Solution: (a) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if it satisfies the two following rules for $X, Y \in \mathbb{R}^n$ and $s \in \mathbb{R}$:
(T1) $T(X + Y) = T(X) + T(Y)$,
(T2) $T(sX) = sT(X)$ for $s \in \mathbb{R}$ and $X \in \mathbb{R}^n$.

(b) We use the rules of matrix multiplication to verify (T1) and (T2):
(T1) $T(X + Y) = A(X + Y) = AX + AY = T(X) + T(Y)$, and
(T2) $T(sX) = A(sX) = s(AX) = sT(X)$.

Marking: (a) and (b) each 1 point, no partial marks. Only mathematically correct solutions are accepted.