University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Final Exam (April 2010) **Time: 3 hours**

Family name (CAPITALS)	______________________
First name (CAPITALS)	______________________
Signature	______________________
Student number	______________________

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. Use the back of pages if necessary, but be sure to indicate to the marker that you have done so. There is an extra page at the end of the exam.
- Questions 1–6 are short answer questions, and no part marks will be given. You must show all the details for questions 7–14, and argue logically. Write legibly.
- Where it is possible to check your work, do so! Read each question carefully – you will save yourself time and unnecessary grief later on.
- **This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Do not detach the pages of the exam. The exam has 14 pages.

Good luck! Bonne chance !

Question	1–6	7–9	10–11	12–13	14	Total
Score						
Max. score	20	14	18	14	4 (bonus)	66

(1) (4 pts) For an $m \times n$ matrix A answer the following questions:

(a) If the linear system $AX = 0$ has a nontrivial solution, then there is no trivial solution. True or false?
Solution: This is false. Any homogenous linear system always has a trivial solution.

Reference: §1.3, #1e, DGD problem

My answer:__________________

(b) If $X = 0$ is a solution of $AX = B$ for some column B in $\mathbb{R}^m$, then $B = 0$. True or False?
Solution: This is true. If $A0 = B$ then $B = 0$.

Reference: §1.3, #1c, DGD problem

My answer:__________________

(c) If there is a column $B \in \mathbb{R}^m$ such that the system $AX = B$ has infinitely many solutions, then $rank(A) < n$. True or False?
Solution: This is true. If the system $AX = B$ has infinitely many solutions, then there is at least one parameter. Thus $rank A \leq n - \#$ of parameters $< n - 1 < n$.

My answer:__________________

(d) If the row-echelon form of A has a row of zeros, then for every $B \in \mathbb{R}^m$ the linear system $AX = B$ is inconsistent. True or false?

Solution: This is false. For example, for $B = 0$ there is always a solution.

My answer:__________________

(2) (3 pts) (a) If A is a $m \times n$ matrix, B is a $p \times q$ matrix and AB and BA can both be formed, then the sizes of A and B are:

Solution: Since AB can be formed we must have $n = p$ and since BA can be formed we have $q = m$. Thus the answer is $n = p$ and $m = q$.

Reference: §2.3, assigned exercise #7b

My answer:__________________

(b) If A is a non-zero matrix such that $AB = A$, then $B = I$ is the identity matrix. True or false?

Solution: This is false. For example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{yet} \quad AB = A.$$

Reference: §2.3, assigned exercise 27b

My answer:__________________

(c) If A and B are matrices such that AB has a row of zeros, then A has a row of zeros. True or false?

Solution: This is false. For example, for $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we have $AB = \begin{bmatrix} 0 \end{bmatrix}$.

Reference: §2.3, assigned exercise 27j

My answer:__________________

(3) (5 pts) Let A be an $n \times n$ matrix with $\det(A) = 1 + (-1)^\gamma$, where γ is your student number. Determine if each of the following statements is true or false. Answer with T for true and F for false.

(a) $\operatorname{rank}(A) < n$.

Solution: If γ is even then $\det(A) = 2 \neq 0$, so A is invertible. If γ is odd, then $\det(A) = 1-1 = 0$, so A is not invertible. Answer: γ even: False and γ odd: True

My answer:__________________

(b) The homogeneous system $AX = 0$ has infinitely many solutions.

Solution: True for γ odd; false for γ even.

My answer:__________________

(c) There exists a vector $B \in \mathbb{R}^n$ such that the system $AX = B$ is inconsistent.

Solution: True for γ odd; false for γ even.

My answer:__________________

(d) The columns of A are linearly independent.

Solution: False for γ odd; true for γ even.

My answer:__________________

(e) The rows of A span $\mathbb{R}^n$.

Solution: False for γ odd; true for γ even.

My answer:__________________

(4) (2 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $\pi/4$ from the x-axis to the y-axis. Give the matrix A such that $T = T_A$, i.e., $T(X) = AX$ for all $X \in \mathbb{R}^2$. (If you have forgotten the formula, calculate to which vectors the standard basis vectors $[1\ 0]^T$ and $[0\ 1]^T$ are mapped.)

Solution: In general, rotation by an angle θ is given by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Since $\cos(\pi/4) = \frac{1}{2}\sqrt{2} = \sin(\pi/4)$ we get

$$A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Reference: suggested exercise 2.6 , 10 b

My answer:__________________

(5) (3 pts) Let $T : \mathbb{R}^2 \to R^3$ be a linear transformation such that $T\left([4,3]^T\right) = [2,1,1]^T$ and $T\left([1,1]^T\right) = [-1,\beta,1]^T$ where β is the **second last digit of your student number.** Find

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Solution: We have to express $[1,-1]^T$ as $a[4,3]^T + b[1,1]^T$, i.e., solve the linear system

$$\begin{aligned} 4a + b &= 1 \\ 3a + b &= -1 \end{aligned}$$

The unique solution is $a = 2, b = -7$. Hence

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) + (-7)T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 7\begin{bmatrix} -1 \\ \beta \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2-7\beta \\ -5 \end{bmatrix}.$$

Reference: Assigned exercises §2.6, #1b

My answer:__________________

(6) (3 pts) Let U be a subspace of $\mathbb{R}^n$. Answer the following questions with T for "true" and F for "false".

(a) Every spanning set of U can be extended to a basis of $\mathbb{R}^n$.

My answer:__________________

(b) Every set of n vectors in U is linearly independent.

My answer:__________________

(c) Every set of n linearly independent vectors in U is a basis of U.

My answer:________________

Solution: (a) False, since a spanning set need not be linearly independent. (b) False: An arbitrary subset of n vectors need not be linearly independent. (c) True: If U contains n linearly independent vectors, its dimension is $\geq n$. But as a subspace of $\mathbb{R}^n$, its dimension is also len. Hence $\dim U = n$ and any set of n linearly independent vectors of U is a basis of U.

(7) (3 pts) Determine if the matrix A below is invertible or not. If the matrix is invertible, give its inverse. If it is not invertible, say why.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$

Solution: Let I_3 be the 3×3 identity matrix. We row-reduce the 3×6-matrix $[A|I_3]$:

$$[A|I_3] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + 2R_1}}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$

$$\overset{R_2 \to R_3 - 2R_2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}.$$

Since the matrix has rank 2, the matrix A is not invertible.

Second solutions: Calculate the determinant of A, using elementary row operations:

$$\begin{vmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

So A is not invertible.

Marking: Correct method, but small mistakes so that the answer becomes A is invertible: (2 pts) Calculating the determinant with mistakes (2 pts) ; calculating the determinant with mistakes so that $\det(A) \neq 0$ and then concluding that the matrix is not invertible: (1 pt)

(8) (5 pts) Find a basis and calculate the dimension of the subspace $U = \{\begin{bmatrix} a & a-b & a+b & b \end{bmatrix} : a, b \in \mathbb{R}\}$. Justify all your claims!

Solution: The equation

$$\begin{bmatrix} a & a-b & a+b & b \end{bmatrix} = a \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix}$$

says that the two vectors

$$v_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix}$$

are a spanning set of U. We verify that they are also linearly independent: Suppose $s_1v_1 + s_2v_2 = 0$, i.e.,

$$s_1 \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Calculating the left hand side of this equation we get

$$\begin{bmatrix} s_1 & s_1 - s_2 & s_1 + s_2 & s_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

We now see that $s_1 = 0$ from the first coefficient and that $s_2 = 0$ from the last coefficient. Hence v_1, v_2 are linearly independent. Since they are also a spanning set, $\{v_1, v_2\}$ is a basis of U. Its dimension is therefore 2.

Marking: (2 pts) for spanning set, (2 pts) for linearly independent, basis (0 pts), dimension (1 pt) Correct basis but no justification (3 pts)

(9) (6 pts) Let U be the subspace $U = \text{Span}\{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} -2 & 1 & -1 & 0 \end{bmatrix}^T\}$.
(a) Find an orthogonal basis of U.
(b) Let $X = \begin{bmatrix} -1 & 2 & 0 & 1 \end{bmatrix}^T$. Find X_1 and X_2 such that $X = X_1 + X_2$ and $X_1 \in U$, $X_2 \in U^{\perp}$.

Solution: (a) We apply the Gram-Schmidt algorithm to the basis Y_1, Y_2 of U where $Y_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ and $Y_2 = \begin{bmatrix} -2 & 1 & -1 & 0 \end{bmatrix}^T$:

$$\begin{aligned} F_1 &= Y_1 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T, \\ F_2 &= Y_2 - \frac{Y_2 \cdot F_1}{F_1 \cdot F_1} F_1 = \begin{bmatrix} -2 & 1 & -1 & 0 \end{bmatrix}^T - \frac{-4}{4} \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} -2 & 1 & -1 & 0 \end{bmatrix}^T + \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}^T. \end{aligned}$$

Hence

$$\{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}^T\}$$

is an orthogonal basis of U.

(b) The projection of X onto U is

$$\begin{aligned} X_1 = \text{proj}_U(X) &= \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 \\ &= \frac{-4}{4} F_1 + \frac{0}{2} F_2 = -F_1 = \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^T \in U. \end{aligned}$$

The vector X_2 is given by

$$X_2 = X - X_1 = \begin{bmatrix} -1 & 2 & 0 & 1 \end{bmatrix}^T - \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T.$$

Marking: Each part (3 pts)

Reference: §8.1 (a) Assigned exercises #1bd, (b) #2bd

(10) (8 pts) In the matrix A below replace α by the **last digit of your student number**.

$$A = \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 3 & 1 & 4 & 2 & 7 \\ 1 & 1 & 0 & 0 & \alpha \\ 5 & 1 & 6 & 7 & 8 \end{bmatrix}.$$

(a) Find the rank of A.
(b) Find a basis for $row(A)$ and the dimension of $row(A)$.
(c) Find a basis for $col(A)$ and the dimension of $col(A)$.
(d) What is the dimension of $null(A)$? Support your answer.

Solution: We row-reduce the matrix A:

$$\begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 3 & 1 & 4 & 2 & 7 \\ 1 & 1 & 0 & 0 & \alpha \\ 5 & 1 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 4 & -2 & -13 & 4 \\ 0 & 2 & -2 & -5 & \alpha-1 \\ 0 & 6 & -4 & -18 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 2 & -2 & -5 & \alpha-1 \\ 0 & 4 & -2 & -13 & 4 \\ 0 & 6 & -4 & -18 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 2 & -2 & -5 & \alpha-1 \\ 0 & 0 & 2 & -3 & 6-2\alpha \\ 0 & 0 & 2 & -3 & 6-3\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 2 & -2 & -5 & \alpha-1 \\ 0 & 0 & 2 & -3 & 6-2\alpha \\ 0 & 0 & 0 & 0 & -\alpha \end{bmatrix}$$

Hence there are two cases, $\alpha = 0$ and $\alpha \neq 0$.

Case $\alpha = 0$: (a) The matrix A is row-equivalent to

$$\begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 2 & -2 & -5 & -1 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 1 & -1 & -5/2 & -1/2 \\ 0 & 0 & 1 & -3/2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

and therefore has rank 3.

(b) A basis of the row space if given by the non-zero rows of R, that is

$$\begin{bmatrix} 1 & -1 & 2 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & -5/2 & -1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -3/2 & 3 \end{bmatrix}$$

The dimension of the row space is 3.

(c) A basis of the column space is given by the columns of A corresponding to the columns of R with leading 1's. These are the columns $1, 2$ and 3. Hence a basis of the column space of A is

$$\begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix}$$

The dimension of the column space is 3 (and equals the dimension of the row space and the rank).

(d) By the rank theorem $\dim(A) = \text{number of variables} - \text{rank}(A) = 5 - 3 = 2$.

Case $\alpha \neq 0$: (a) The matrix A is row-equivalent to

$$\begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 2 & -2 & -5 & \alpha-1 \\ 0 & 0 & 2 & -3 & 6-2\alpha \\ 0 & 0 & 0 & 0 & -\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 0 & 1 & -1 & -5/2 & 1/2(\alpha-1) \\ 0 & 0 & 1 & -3/2 & 3-\alpha \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R$$

and therefore has rank 4.

(b) A basis of the row space if given by the non-zero rows of R, that is

$$\begin{bmatrix} 1 & -1 & 2 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & -5/2 & 1/2(\alpha-1) \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 1 & -3/2 & 3-\alpha \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The dimension of the row space is 4.

(c) A basis of the column space is given by the columns of A corresponding to the columns of R with leading 1's. These are the columns $1, 2, 3$ and 5. Hence a basis of the column space of A is

$$\begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ \alpha \\ 8 \end{bmatrix}$$

The dimension of the column space is 4 (and equals the dimension of the row space and the rank).

(d) By the rank theorem $\dim(A) = \text{number of variables} - \text{rank}(A) = 5 - 4 = 1$.

Marking: (a) 1 point for the correct row-reduction, 1 point for rank; (b) 1 point for basis and 1 point for dimension; (c) 1 point for basis and 1 point for dimension; (d) 2 points: 1 point for the correct answer, 1 point for justification.

(11) (10 pts) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. (The problem has two pages)

(a) Find the characteristic polynomial of A.

Solution:

$$\begin{aligned} c_A(\lambda) &= \det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 2)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

(b) Find all eigenvalues of A.

Solution: The eigenvalues are the roots of the characteristic polynomial, hence the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

Marking: (2 pts)

(c) For each eigenvalue determine the corresponding eigenspace.

Solution: The eigenspaces are the null spaces of $\lambda I_2 - A$. For $\lambda = 1$ we get $I_2 - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of the corresponding homogeneous linear system $x_1 - x_2 = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}).$$

Hence the λ_1-eigenspace is

$$E_1(A) = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \mathbb{R} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We proceed in the same way for $\lambda_2 = 3$: $3I_3 - A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of the corresponding homogeneous linear system $x_1 + x_2 = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}).$$

Hence the λ_2-eigenspace is

$$E_3(A) = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \mathbb{R} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Hence, we have 2 eigenspaces, each of dimension 1.

Marking: -1 if the eigenspaces are not correctly written down, e.g. only one eigenvector instead of the full eigenspace.

(continued on the next page)

(d) Diagonalize the matrix A, i.e., find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Check your answer.

Solution: Based on part (c) we get

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

To verify that the solution is correct, we first find the inverse matrix of P which in any case we will need for part (e).

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

It now easy to calculate

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\ &= A. \end{aligned}$$

(e) Does there exists a basis of $\mathbb{R}^2$ consisting of eigenvectors of A? If yes, give such a basis. If no, justify why not.

Solution: Such a basis exists. The matrix P is invertible. Hence its two columns, which are eigenvectors of A, form a basis of $\mathbb{R}^2$. Hence an example for such a basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

(f) Calculate A^4 using (d).

Solution:

$$\begin{aligned} A^4 &= \left(PDP^{-1}\right)^4 \\ &= PD^4P^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -81 \\ 1 & 81 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 41 & -40 \\ -40 & 41 \end{bmatrix}. \end{aligned}$$

Marking: (a) (2 pts) (b) (1 pt) (c) (2 pts) (d) (2 pts) (e) (1 pt) (f) (2 pts)

(12) (8 pts) For the system of linear equations

$$\begin{array}{rcrcrcr} -x & - & 2y & + & 3z & = & -4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

(a) (6 pts) determine the values of a for which the system has
   (i) no solution,
   (ii) infinitely many solutions,
   (iii) a unique solution.

(b) (2 pts) In case (ii) above give all solutions.

Solution: The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & -4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow 3R_1 + R_2$, $R_3 \rightsquigarrow 4R_1 + R_3 \to R_3$; $R_2 \rightsquigarrow R_3 - R_2$; $R_1 \rightsquigarrow -R_1$, $R_2 \rightsquigarrow \frac{-1}{7}R_2$ and obtain:

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right].$$

Since $a^2 - 16 = (a-4)(a+4)$ we get:

- If $a = -4$, then the last row of M is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -8 \end{array}\right]$. Hence the system is inconsistent.
- If $a = 4$ alors $M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Hence the system has infinitely many solutions.
- If $a \notin \{-4, 4\}$, then $M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & * & * \end{array}\right]$ where the stars "$*$" are non-zero numbers. Hence the system is uniquely solvable, because there does not exist a free variable.

The answer to question (a) is therefore:
(i) The system in inconsistent if $a = -4$.
(ii) The system has infinitely many solutions if $a = 4$.
(iii) The system is uniquely solvable if $a \notin \{4, -4\}$.

To answer (b), let $a = 4$ in the matrix M above. This yields

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

a matrix in row-echelon form. The leading variables are x, y while z is a free variable. Putting $z = t$ ($t \in Reals$) gives the following general solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/7 - t \\ 2t + 10/7 \\ t \end{bmatrix} = \begin{bmatrix} 8/7 \\ 10/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

hence the set of solutions is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/7 \\ 10/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R})}$$

Time: 20 minutes

Reference: Test 1

(13) (6 pts) Let $B \in \mathbb{M}_{33}$ be a fixed matrix. Show that $U = \{A \in \mathbb{M}_{33} : AB = BA\}$ is a subspace of $\mathbb{M}_{33}$.

Solution: We check the 3 conditions of the subspace test:

- $0 \in U$ where 0 is the 3×3 zero matrix. Indeed, any product where one factor is the zero matrix is zero, so $0B = 0 = B0$, proving $0 \in U$.
- Suppose $A_1 \in U$ and also $A_2 \in U$. This manes that $A_1B = BA_1$ and also $A_2B = BA_2$. We need to show that $A_1 + A_2 \in U$. This follows from $(A_1+A_2)B = A_1B+A_2B = BA_1+BA_2 = B(A_1 + A_2)$.
- Suppose $A \in U$ and let $s \in RR$ be a scalar. We need to show that $sA \in U$. By assumption on A we have $AB = BA$. Then $(sA)B = s(AB) = s(BA) = B(sA)$, whence $sA \in U$.

We have verified the 3 conditions of the subapace test. Therefore U is a subspace of $\mathbb{M}_{33}$.

Reference: §6.1, assigned exercise #2d

Time: 15 minutes

(14) (4 bonus points) Show that $U = \{p \in \mathbb{P}_2 : p(2) = 0\}$ is a subspace of $\mathbb{P}_2$ and find a basis of U and its dimension.

Solution: An arbitrary polynomial can be written in the form $p = a_2x^2 + a_1x + a_0$. Such a polynomial lies in U if and only if

$$p(2) = 0 \iff 4a_2 + 2a_1 + a_0 = 0 \iff a_0 = -4a_2 - 2a_1.$$

Hence the polynomials in U are exactly those of the form

$$p = a_2a^2 + a_1x - 4a_1 - 2a_1 = a_2(x^2 - 4) + a_1(x - 2)$$

with arbitrary $a_2, a_1 \in \mathbb{R}$. This shows that

$$U = \text{Span}\,\{x^2 - 4, x - 2\}.$$

In particular, it follows that U is a subspace since U is the span of two polynomials. We claim that $\{x^2 - 4, x - 2\}$ is a linearly independent set: Suppose that $s_2(x^2 - 4) + s_1(x - 2) = 0$, the zero polynomial, where $s_1, s_2 \in \mathbb{R}$ are some real scalars. Then $s_2x^2 + s_1x + (-4s_2 - 2s_1) = 0$. Therefore all coefficients $s_2 = 0 = s_1$. This proves that $\{x^2 - 4, x - 2\}$ is a linearly independent set. Since it also is a spanning set, it is a basis of U. Finally, the dimension of U is the number of elements in a basis of U, which is 2: $\dim U = 1$.

Reference: #6.2, Exercise 6c, DGD

Time: 20 minutes

Marking: subspace and spanning 2 points, linear independence 1 point (students can quote the result instead that a set of polynomials with distinct degrees is linearly independent, dimension 1 point