University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Final Exam (April 2010)   Time: 3 hours

FAMILY NAME (CAPITALS)	______________________
FIRST NAME (CAPITALS)	______________________
Signature	______________________
Student number	______________________

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. Use the back of pages if necessary, but be sure to indicate to the marker that you have done so. There is an extra page at the end of the exam.
- Questions 1–6 are short answer questions, and no part marks will be given. You must show all the details for questions 7–14, and argue logically. Write legibly.
- Where it is possible to check your work, do so! Read each question carefully – you will save yourself time and unnecessary grief later on.
- **This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Do not detach the pages of the exam. The exam has 14 pages.

Good luck! Bonne chance !

Question	1–6	7–9	10–11	12–13	14	Total
Score						
Max. score	20	14	18	14	4 (bonus)	66

(1) (4 pts) For an $m \times n$ matrix A answer the following questions:

(a) If the linear system $AX = 0$ has a nontrivial solution, then there is no trivial solution. True or false?

My answer:__________________

(b) If $X = 0$ is a solution of $AX = B$ for some column B in $\mathbb{R}^m$, then $B = 0$. True or False?

My answer:__________________

(c) If there is a column $B \in \mathbb{R}^m$ such that the system $AX = B$ has infinitely many solutions, then $rank(A) < n$. True or False?

My answer:__________________

(d) If the row-echelon form of A has a row of zeros, then for every $B \in \mathbb{R}^m$ the linear system $AX = B$ is inconsistent.

My answer:__________________

(2) (3 pts) (a) If A is a $m \times n$ matrix, B is a $p \times q$ matrix and AB and BA can both be formed, then the sizes of A and B are:

My answer:__________________

(b) If A is a non-zero matrix such that $AB = A$, then $B = I$ is the identity matrix.

My answer:__________________

(c) If A and B are matrices such that AB has a row of zeros, then A has a row of zeros.

My answer:__________________

(3) (5 pts) Let A be an $n \times n$ matrix with $\det(A) = 1 + (-1)^\gamma$, where γ is your student number. Determine if each of the following statements is true or false. Answer with T for true and F for false.

(a) $\operatorname{rank}(A) < n$.

My answer:__________________

(b) The homogeneous system $AX = 0$ has infinitely many solutions.

My answer:__________________

(c) There exists a vector $B \in \mathbb{R}^n$ such that the system $AX = B$ is inconsistent.

My answer:__________________

(d) The columns of A are linearly independent.

My answer:__________________

(e) The rows of A span $\mathbb{R}^n$.

My answer:__________________

(4) (2 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $\pi/4$ from the x-axis to the y-axis. Give the matrix A such that $T = T_A$, i.e., $T(X) = AX$ for all $X \in \mathbb{R}^2$. (If you have forgotten the formula, calculate to which vectors the standard basis vectors $[1\ 0]^T$ and $[0\ 1]^T$ are mapped.)

My answer:__________________

(5) (3 pts) Let $T : \mathbb{R}^2 \to R^3$ be a linear transformation such that $T\left([4,3]^T\right) = [2,1,1]^T$ and $T\left([1,1]^T\right) = [-1,\beta,1]^T$ where β is the **second last digit of your student number.** Find

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

My answer:__________________

(6) (3 pts) Let U be a subspace of $\mathbb{R}^n$. Answer the following questions with T for "true" and F for "false".

(a) Every spanning set of U can be extended to a basis of $\mathbb{R}^n$.

My answer:__________________

(b) Every set of n vectors in U is linearly independent.

My answer:__________________

(c) Every set of n linearly independent vectors in U is a basis of U.

My answer:__________________

(7) (3 pts) Determine if the matrix A below is invertible or not. If the matrix is invertible, give its inverse. If it is not invertible, say why.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$

(8) (5 pts) Find a basis and calculate the dimension of the subspace $U = \{ \begin{bmatrix} a & a-b & a+b & b \end{bmatrix} : a, b \in \mathbb{R}\}$. Justify all your claims!

(9) (6 pts) Let U be the subspace $U = \text{Span}\{\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} -2 & 1 & -1 & 0 \end{bmatrix}^T\}$.

(a) Find an orthogonal basis of U.

(b) Let $X = \begin{bmatrix} -1 & 2 & 0 & 1 \end{bmatrix}^T$. Find X_1 and X_2 such that $X = X_1 + X_2$ and $X_1 \in U$, $X_2 \in U^\perp$.

(10) (8 pts) In the matrix A below replace α by the **last digit of your student number**.

$$A = \begin{bmatrix} 1 & -1 & 2 & 5 & 1 \\ 3 & 1 & 4 & 2 & 7 \\ 1 & 1 & 0 & 0 & \alpha \\ 5 & 1 & 6 & 7 & 8 \end{bmatrix}.$$

(a) Find the rank of A.
(b) Find a basis for $row(A)$ and the dimension of $row(A)$.
(c) Find a basis for $col(A)$ and the dimension of $col(A)$.
(d) What is the dimension of $null(A)$? Support your answer.

(11) (10 pts) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. (The problem has two pages)

(a) Find the characteristic polynomial of A.
(b) Find all eigenvalues of A.
(c) For each eigenvalue determine the corresponding eigenspace.

(continued on the next page)

(d) Diagonalize the matrix A, i.e., find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Check your answer.

(e) Does there exists a basis of $\mathbb{R}^2$ consisting of eigenvectors of A? If yes, give such a basis. If no, justify why not.

(f) Calculate A^4 using (d).

(12) (8 pts) For the system of linear equations

$$\begin{array}{rcrcrcr} -x & - & 2y & + & 3z & = & -4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

(a) (6 pts) determine the values of a for which the system has

(i) no solution,

(ii) infinitely many solutions,

(iii) a unique solution.

(b) (2 pts) In case (ii) above describe give all solutions.

(13) (6 pts) Let $B \in \mathbb{M}_{33}$ be a fixed matrix. Show that $U = \{A \in \mathbb{M}_{33} : AB = BA\}$ is a subspace of $\mathbb{M}_{33}$.

(14) (4 bonus points) Show that $U = \{p \in \mathbb{P}_2 : p(2) = 0\}$ is a subspace of $\mathbb{P}_2$ and find a basis of U and its dimension.

(extra page)