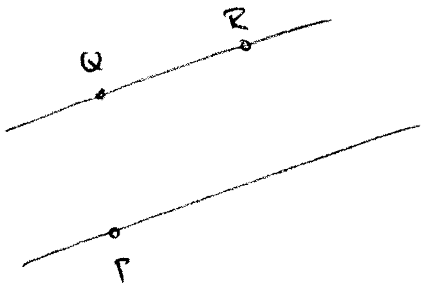
1. Find the scalar equations of the line passing through $P(2,-5,7)$ and parallel to the line through $Q(-1,1,-1)$ and $R(1,-1,1)$.

A. $x = 9+2t$, $y = -7-2t$, $z = 4+2t$, t scalar.
B. $x = -2+2t$, $y = -5-2t$, $z = 5+2t$, t scalar.
C. $x = 2+3t$, $y = -3-2t$, $z = 7+2t$, t scalar.
D. $x = 2+2t$, $y = -5-2t$, $z = 7+2t$, t scalar.
E. $x = 2+2t$, $y = -5-2t$, $z = 7+2t$, t scalar.
F. $x = 2+t$, $y = -5+9t$, $z = 1+t$, t scalar.

My answer: D



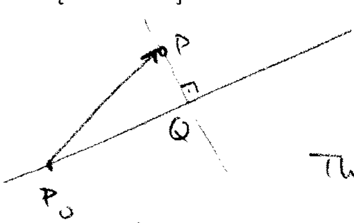
Parallel lines have the same direction vector. The direction vector of the line through Q and R is $\vec{QR} = [1,-1,1]^T - [-1,1,-1]^T = [2,-2,2]^T$. Hence scalar equations of the line through $P(2,-5,7)$ with direction vector $[2,-2,2]^T$ are

$x = 2+2t$, $y = -5-2t$, $z = 7+2t$, t scalar

2. Find the direction vector of the line passing through $P(3,-3,3)$, intersecting the line $[x \ \ y \ \ z]^T = [9 \ \ -5 \ \ 7]^T + t[-1 \ \ 2 \ \ -1]^T$ and perpendicular to it.

A. $[-1 \ \ 2 \ \ -1]^T$
B. $[9 \ \ -5 \ \ 7]^T$
C. Such a line does not exist.
D. $[0 \ \ -4 \ \ -9]^T$
E. $[-11 \ \ -8 \ \ -5]^T$
F. $[1 \ \ 4 \ \ 7]^T$.

My answer: E



The point $P_0(9,-5,7)$ lies on the given line.

The vector $\vec{P_0P}$ is $[3,-3,3]^T - [9,-5,7]^T = [-6,2,-4]^T$

We calculate $\mathrm{proj}_{\vec{d}}(\vec{P_0P})$ for $d = [-1,2,-1]^T$ = direction vector of the given line: $\mathrm{proj}_{\vec{d}}(\vec{P_0P}) = \frac{1}{6}[-6,2,-4]^T \cdot [-1,2,-1]^T (d) =$

$= \frac{1}{6}(6+4+4)\, d = \frac{14}{6}[-1,2,-1]^T = \frac{7}{3}[-1,2,-1]^T$.

$\vec{P_0P} - \mathrm{proj}_{\vec{d}}(\vec{P_0P}) = [-6,2,-4]^T - \frac{7}{3}[-1,2,-1]^T = \frac{1}{3}[-18+7, 6-14, -12+7]$

$= \frac{1}{3}[-11,-8,-5]$ is orthogonal to $\mathrm{proj}_{\vec{d}}(\vec{P_0P})$, and is a direction vector of the line we are looking for. So is any non-zero scalar multiple.

3. Find the scalar equation of the line passing through $P(3,-9,0)$ and $Q(-1,2,7)$.

A. $x = 3 + 4t$, $y = -9 + 11t$, $z = 7t$, t scalar.

B. $x = 3 - 4t$, $y = -9 + 11t$, $z = 7t$, t scalar.

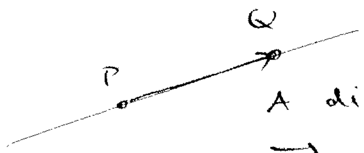
C. $x = 1 - 3t$, $y = 4 + 11t$, $z = 1 + 2t$, t scalar.

D. $x = 1 + 2t$, $y = 1 + 4t$, $z = 2t$, t scalar.

E. $x = 3 + 4t$, $y = -4 + t$, $z = 1 + 7t$, t scalar.

F. $x = -3 + 4t$, $y = 9 + 11t$, $z = 7t$, t scalar.

My answer: B



A direction vector of the line passing through P and Q is $\overrightarrow{PQ} = [-1,2,7]^T - [3,-9,0]^T = [-4,11,7]^T$. Scalar equations are therefore as follows

$x = 3 - 4t$, $y = -9 + 11t$, $z = 7t$

4. Find the equation of the plane passing through $A(1,0,-7)$ and perpendicular to the line $[x \quad y \quad z]^T = [5 \quad 0 \quad 5]^T + t[2 \quad -1 \quad 7]^T$.

A. $2x + 3y + 6z = -1$

B. $x - 2y + 4z = 2$

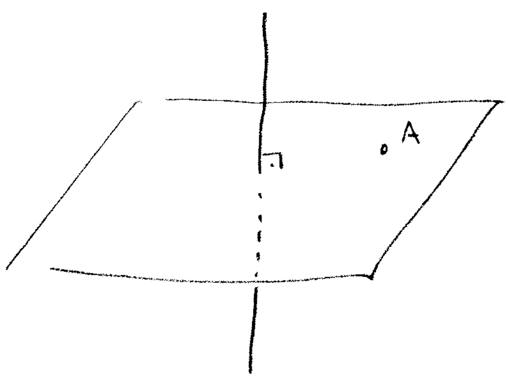
C. $2x - y + 7z = -47$

D. $4x - y + 2z = -4$

E. $2x + y + z = -41$

F. $6x - y + 2z = -40$

My answer: C



Any direction vector of the line is a normal to the plane. Hence $[2,-1,7]^T$ is a normal, and thus an equation of the plane is

$2x - y + 7z = [2,-1,7]^T \cdot [1,0,-7]^T =$

$= 2 + 0 - 49 = -47$

5. Find the equation of the plane containing the point $P(1,2,-3)$ and also containing the line through $Q(2,0,1)$ with direction vector $[1 \quad 1 \quad 1]^T$.

A. $2x - y - z = 3$

B. $2x - y - z = 0$

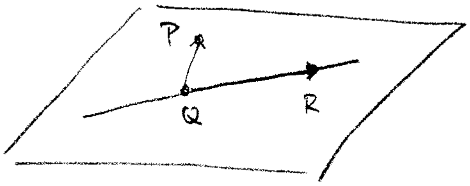
C. $x + y + z = 3$

D. $x + y + z = 0$

E. $6x + 3y + z = 9$

F. $6x + 3y + z = 0$

My answer: A



The vector $\overrightarrow{QP} = [1,2,-3]^T - [2,0,1]^T = [-1,2,-4]^T$ lies in the plane, as does the direction vector $[1\ 1\ 1]^T$ of the line. A normal of the plane is therefore $[-1,2,-4]^T \times [1,1,1]^T = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \left(\begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & -4 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} \right)$

$= [6, 1-4, -1-2]^T = [6,-3,-3]^T = 3\,[2,-1,-1]^T$. Hence the plane has the equation $2x - y - z = c$ where the constant c is determined by the condition that $P(1,2,-3)$ lies on the plane: $c = 2\cdot 1 - 2 - (-3) = 3$. Thus $2x - y - z = 3$ is an equation of the plane.

6. What is the distance between the point $P(1,1,1)$ and the plan $2x - 2y + z = 0$?

A. $\frac{1}{3}$

B. $\frac{1}{9}$

C. $\frac{1}{\sqrt{3}}$

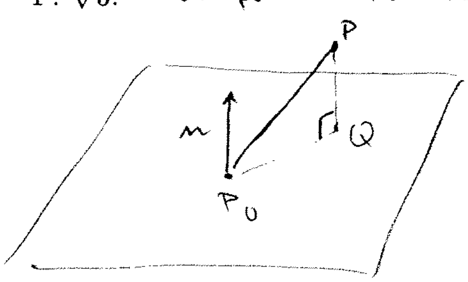
D. 9

E. 3

F. $\sqrt{3}$.

My answer: A

We follow the 1st approach of Example 12 in §3.3, and choose a point in the plane: $P_0(0,0,0)$. The projection of $\overrightarrow{P_0P} = \overrightarrow{OP} = [1,1,1]$ onto a normal $\vec{n}$ of the plane is, taking $\vec{n} = [2,-2,1]$,



$$\frac{\overrightarrow{OP} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{2-2+1}{4+4+1} \vec{n} = \frac{1}{9} \vec{n}.$$

The distance is $\| \mathrm{proj}_{\vec{n}} (\overrightarrow{P_0P}) \| = \| \frac{1}{9} \vec{n} \| = \frac{1}{9} \sqrt{4+4+1} = \frac{\sqrt{9}}{9} = \frac{3}{9} = \frac{1}{3}$

7. Let $u = [1 \quad 1 \quad 1]^T$ and $v = [2 \quad -2 \quad 1]^T$. Suppose w and x are vectors satisfying the following three conditions:

- $u = w + x$,
- w is a scalar multiple of v, i.e., $w = sv$ for some scalar $s \in \mathbb{R}$,
- x is orthogonal to v.

Then x is

A. $\frac{1}{9}(7, 11, 8)$

B. $-\frac{1}{3}(7, 11, 8)$

C. $\frac{1}{9}(1, -1, -4)$

D. $-\frac{1}{3}(1, -1, -4)$

E. $\frac{1}{9}(8, 10, 4)$

F. $-\frac{1}{3}(8, 10, 4)$.

My answer: A

We have $x = u - w = u - sv$ and

$0 = x \cdot v = (u - sv) \cdot v = u \cdot v - s\, v \cdot v$, hence $s = \frac{u \cdot v}{v \cdot v} =$

$= \frac{[1\ 1\ 1] \cdot [2, -2, 1]}{4 + 4 + 1} = \frac{1}{9}(2 - 2 + 1) = \frac{1}{9}$. It then follows that

$x = u - \frac{1}{9} v = [1, 1, 1]^T - \frac{1}{9}[2, -2, 1]^T = [\frac{7}{9}, \frac{11}{9}, \frac{8}{9}]^T = \frac{1}{9}[7, 11, 8]$

8. Find the area of the triangle with vertices $A(2, 0, 1)$, $B(0, 1, 2)$ and $C(1, 1, -1)$.

A. $\frac{1}{2}\sqrt{35}$

B. $\sqrt{35}$

C. 25

D. 25/2

E. 5/2

F. $\frac{1}{2}\sqrt{5}$

My answer: A

We have $\vec{AB} = [0, 1, 2]^T - [2, 0, 1]^T = [-2, 1, 1]^T$,

$\vec{AC} = [1, 1, -1]^T - [2, 0, 1]^T = [-1, 1, -2]^T$. We calculate

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = \left(\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} \right) = [-3, -5, -1]^T$$

The area of the triangle is $\frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \frac{1}{2}\sqrt{9 + 25 + 1} = \frac{1}{2}\sqrt{35}$

9. Find the distance from the point $P(1,0,1)$ to the line through $P_0(0,0,0)$ with direction vector $d = [1 \quad 1 \quad 1]^T$.

A. $\frac{2}{3}$

B. $2\sqrt{5}$

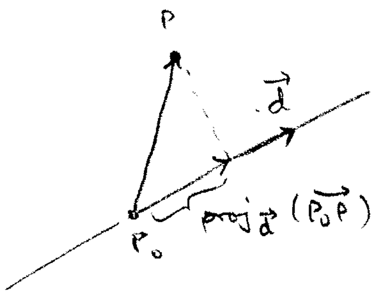
C. 0

D. $\frac{1}{4}$

E. $\frac{1}{7}\sqrt{5}$

F. $\frac{1}{3}\sqrt{6}$

My answer: F



$\overrightarrow{P_0P} = \overrightarrow{OP} = [1,0,1]^T$

$proj_{\vec{d}}(\overrightarrow{P_0P}) = \frac{[1,0,1]^T \cdot [1,1,1]^T}{1+1+1}\vec{d} = \frac{2}{3}[1,1,1]^T$

distance $= \| \overrightarrow{P_0P} - proj_{\vec{d}}(\overrightarrow{P_0P}) \| =$

$= \| [1,0,1]^T - \frac{2}{3}[1,1,1]^T \| = \| [\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}]^T \|$

$= \frac{1}{3} \| [1,-2,1]^T \| = \frac{1}{3}\sqrt{1+4+1} = \frac{1}{3}\sqrt{6}$

10. Write

$$\frac{1+i}{2-i} - \frac{3-4i}{1+2i}$$

in the form $a+ib$ with $a, b, \in \mathbb{R}$.

A. $\frac{1}{4}(7-8i)$

B. $-\frac{1}{2}(9-5i)$

C. $\frac{3}{7}(2-8i)$

D. $\frac{4}{5}(6+i)$

E. $\frac{1}{5}(6+13i)$

F. $-\frac{2}{3}(5-11i)$.

My answer: E

$\frac{1+i}{2-i} - \frac{3-4i}{1+2i} = \frac{(1+i)(2+i)}{4+1} - \frac{(3-4i)(1-2i)}{1+4} = \frac{1}{5}[2+i^2+2i+i-(3+8i^2-4i-6i)]$

$= \frac{1}{5}(1+3i-(-5-10i)) = \frac{1}{5}(6+13i)$

11. The roots of $3x^2 + 2x + 1$ are

A. $\frac{3}{4}(1 \pm 2i)$
B. $\frac{1}{3}(-1 \pm \sqrt{2}i)$
C. This polynomial does not have roots
D. $\frac{1}{6}(-4 \pm \sqrt{11}i)$
E. $\frac{2}{3}(1 \pm \sqrt{5})$
F. $\frac{1}{7}(2 \pm \sqrt{5}i)$

My answer: B

For a quadratic polynomial ax^2+bx+c the roots are given by $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$. We have $a=3$, $b=2$, $c=1$. Hence the roots are

$$\frac{1}{6}\left(-2 \pm \sqrt{4-4\cdot 3}\right) = \frac{1}{6}\left(-2 \pm \sqrt{-8}\right) = \frac{1}{6}\left(-2 \pm 2\sqrt{2}\,i\right) = \frac{1}{3}\left(-1 \pm \sqrt{2}\,i\right)$$

12. Find the correct combination of true/false for the following three statements.

- For a complex number z and its complex conjugate $\bar{z}$ we always have $z\bar{z} \geq 0$.
- The equation $x^3 - 7x^2 + 43x - 13 = 0$ has a solution in the complex numbers.
- For a complex number z and a real number r we always have $|rz| = |r||z|$.

A. true/true/true
B. false/true/true
C. true/false/true
D. true/true/false
D. false/false/true
E. false/true/false
F. false/false /false

all statements are correct.

My answer: A