University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Test 2

FAMILY NAME (CAPITALS)	
FIRST NAME (CAPITALS)	
Signature	
Student number	

Please read these instructions carefully:

- The table below is for the TA. Do not write in it.
- For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages. Therefore, **fill in your name on both pages and your student number on this page only.**

Question	1	2	3	4	5	6	7	8	Total
Score									
Maximal score	3	3	3	3	5	5	5	2 bonus	27

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Test 2

Family name (CAPITALS) ______________________________

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Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. For questions 5 – 7 you may use the back of pages if necessary, but be sure to indicate to the marker that you have done so.
- No part marks will be given for questions 1 – 4. However, you must show some work to obtain the point. Simply writing the correct answer will earn you 0.
- Question 8 is a bonus proof question. You can get 2 extra points.
- No books or notes are allowed. **Calculators are not permitted.**

Good luck! Bonne chance !

(1) (3 pts) Let U be a subspace of $\mathbb{R}^7$ which is spanned by 6 vectors. Are the following claims true or false? Answer with T for "true" or F for "false".

(a) $\dim U \leq 6$.

My answer:__________________

(b) Any set of $\dim U$ vectors in U is linearly independent.

My answer:__________________

(c) Every spanning set of U has at most 6 vectors.

My answer:__________________

Solution: (a) = T: Theorem 3(3) in §4.3. (b) = F since there are no specifications about the set; for example, if $\dim U = 3$ and $0 \neq X \in U$ then $\{X, 2X, 3X\}$ is a set of 3 vectors in U which is not linearly independent. (c)= F: For example, if U is spanned by $\{X_1, X_2, \ldots, X_6\}$ then U is also spanned by the set $\{X_1, X_2, \ldots, X_6, Y\}$ of 7 vectors where Y is any vector in U which is not contained in $\{X_1, \ldots, X_6\}$.

(2) (3 pts) Let A be a 4×9 matrix. Are the following claims true or false? Answer with T for "true" or F for "false".

(a) If A has rank 4, then the columns of A^T are linearly independent.

My answer:__________________

(b) If A has rank 3, the null space of A is a subspace of $\mathbb{R}^6$.

My answer:__________________

(c) If A has rank 4, then every set of 4 columns of A is linearly independent.

My answer:__________________

Solution: (a)=T since the columns of A^T are the rows of A and the 4 rows of A are linearly independent because of rank $(A) = 4$.

(b)= F: The null space is a subspace of $\mathbb{R}^9$ (recall that the corresponding linear system $AX = 0$ has 9 variables); the null space has dimension $9 - 3 = 6$ but this does not mean that the null space is a subspace of $\mathbb{R}^6$.

(c)= F: For instance $A = \begin{pmatrix} 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$ is of rank 4 but the set consisting of its first four columns is linearly dependant.

(3) (3 pts) (a) The dimension of the vector space $\mathbb{M}_{34}$ of 3×4-matrices is

My answer:_________________

(b) The dimension of the vector space $\mathbb{P}_5$ of polynomials of degree ≤ 5 is

My answer:_________________

(c) Give a basis of $\mathbb{M}_{22}$ consisting of non-invertible matrices:

Solution: $\dim \mathbb{M}_{34} = 3 \cdot 4 = 12$, $\dim \mathbb{P}_5 = 6$. (c) There are many possible answer. For example, the standard basis of $\mathbb{M}_{22}$ is such a basis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4) (3 pts) Are the following subsets U subspaces of the indicated vector spaces V? Answer with Y for "yes" or N for "no".

(a) $U = \{f \in V : f(0) = 0 = f(1)\}$ in $V = \mathbb{F}[0,3]$.

My answer:_________________

(b) $U = \{A \in V : A^T = A\}$ in $V = \mathbb{M}_{2,2}$.

My answer:_________________

(c) $U = \{p \in V : p(0) = 0, p(1) = 1\}$ in $V = \mathbb{P}_3$.

My answer:_________________

Solution: (a)=Y: One verifies the conditions of the subspace test.

(b) = Y: A 2×2-matrix A lies in U if and only if A has the form

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where there are no conditions on a, b and d. Hence U is the span of the three indicated matrices and is therefore a subspace of V.

(c)= N since the zero vector of $\mathbb{P}_3$, which is the zero polynomial = zero function, does not lie in U: It does not satisfy $p(1) = 1$.

(5) (5 pts) Find all $x \in \mathbb{R}$ such that $\begin{bmatrix} x+7 & 2 & -3 \\ -4 & x+1 & 3 \\ -3 & 3 & x+1 \end{bmatrix}$ is NOT invertible.

Solution: Let A be the matrix in the problem. If we add row 1 row 2 we get a matrix, say B, with the same determinant. Hence

$$\det A = \det B = \det \begin{bmatrix} x+7 & 2 & -3 \\ x+3 & x+3 & 0 \\ -3 & 3 & x+1 \end{bmatrix} = (x+3)\det \begin{bmatrix} x+7 & 2 & -3 \\ 1 & 1 & 0 \\ -3 & 3 & x+1 \end{bmatrix}$$

where in the last step we "pulled out" a factor $(x+3$ from the second row. We now subtract column 2 from column 1 (observe that this does not change the determinant) and then expand along the second row:

$$\begin{aligned}(x+3)\det \begin{bmatrix} x+7 & 2 & -3 \\ 1 & 1 & 0 \\ -3 & 3 & x+1 \end{bmatrix} &= (x+3)\det \begin{bmatrix} x+5 & 2 & -3 \\ 0 & 1 & 0 \\ -6 & 3 & x+1 \end{bmatrix} \\ &= (x+3)\det \begin{bmatrix} x+5 & -3 \\ -6 & x+1 \end{bmatrix} = (x+3)\big((x+5)(x+1)-(-6)(-3)\big) \\ &= (x+3)(x^2+6x-13)\end{aligned}$$

Instead of the column operation above, we can also calculate the determinant by "brute force": Expanding across the second row (and using properties of determinants) we get:

$$\begin{aligned}\det A &= (x+3)\{1(-1)^{2+1}\det \begin{bmatrix} 2 & -3 \\ 3 & x+1 \end{bmatrix} + 1(-1)^{2+2}\det \begin{bmatrix} x+7 & -3 \\ -3 & x+1 \end{bmatrix}\} \\ &= (x+3)\{-(2x+2+9)+(x+7)(x+1)-9\} \\ &= (x+3)\{-2x-2-9+x^2+8x+7-9\} \\ &= (x+3)(x^2+6x-13).\end{aligned}$$

In any case, the result is (of course) the same. We now use the result that the matrix A is not invertible if and only if $\det(A) = 0$. Since the roots of $x^2+6x-13$ are $-3 \pm \sqrt{22}$, the matrix is not invertible when either $x=-3$, $x=-3-\sqrt{22}$, or $x=-3+\sqrt{22}$.

(6) (5 pts) Consider the subspace U of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3, v_4, v_5\}$ where:

$$v_1 = \begin{bmatrix} 1 \\ -4 \\ 7 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ -4 \\ 5 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 5 \\ 4 \\ -3 \\ -6 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 3 \\ -2 \\ -7 \\ 6 \end{bmatrix}.$$

Find a basis of U and give the dimension of U.

Solution: We notice that $U = \operatorname{Col} A$ where A is the matrix whose columns are the given 5 vectors:

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ -4 & -4 & 0 & 4 & -2 \\ 7 & 5 & 2 & -3 & -7 \\ -4 & 1 & -5 & -6 & 6 \end{bmatrix}.$$

The problem is therefore reduced to finding a basis for $\operatorname{Col} A$. To do so, we row-reduce:

$$\begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ -4 & -4 & 0 & 4 & -2 \\ 7 & 5 & 2 & -3 & -7 \\ -4 & 1 & -5 & -6 & 6 \end{bmatrix} \xrightarrow{R_2+4R_1, R_3-7R_1, R_4+4R_1} \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ 0 & -12 & 12 & 24 & 10 \\ 0 & 19 & -19 & -38 & -28 \\ 0 & -7 & 7 & 14 & 18 \end{bmatrix}$$

$$\xrightarrow{\frac{-1}{12}R_2, \frac{1}{19}R_3, \frac{-1}{7}R_4} \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ 0 & 1 & -1 & -2 & -5/6 \\ 0 & 1 & -1 & -2 & -28/19 \\ 0 & 1 & -1 & -2 & -18/7 \end{bmatrix} \xrightarrow{R_4-R_2, R_3-R_2} \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ 0 & 1 & -1 & -2 & -5/6 \\ 0 & 0 & 0 & 0 & -53/114 \\ 0 & 0 & 0 & 0 & -73/42 \end{bmatrix}$$

$$\xrightarrow{\frac{-42}{73}R_4, \frac{-114}{53}R_3} \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ 0 & 1 & -1 & -2 & -5/6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 1 & -2 & 3 & 5 & 3 \\ 0 & 1 & -1 & -2 & -5/6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence Column 1, Column 2 and Column 5 of A form a basis, and so $\{v_1, v_2, v_5\}$ is a basis of the subspace spanned by the given 5 vectors.

(7) (5 pts) Show that

$$U = \{p \in \mathbb{P}_2 : p(2) = 0\}$$

is a subspace of $\mathbb{P}_2$, find a basis of U and determine its dimension.

Solution: The polynomials p in $\mathbb{P}_2$ has the form $p(x) = a_0 + a_1x + a_2x^2$ for unique $a_i \in \mathbb{R}$. Since $p(2) = a_0 + 2a_1 + 4a_2$ a polynomial $p \in \mathbb{P}_2$ lies in U if and only if p can be written in the form $p(x) = a_0 + a_1x + a_2x^2$ with $a_0 = -2a_1 - 4a_2$, i.e.,

$$p(x) = (-2a_1 - 4a_2) + a_1x + a_2x^2 = a_1(x-1) + a_2(x^2 - 4)$$

for arbitrary $a_1, a_2 \in \mathbb{R}$. This equation says that $U = \text{Span}\,\{p_1, p_2\}$ where

$$p_1(x) = x - 1 \quad \text{and} \quad p_2(x) = x^2 - 4.$$

In particular, U is a subspace since it can be written as a span.

Since the polynomials p_1 and p_2 have different degrees, they are linearly independent (seen in class). Therefore $\{p_1, p_2\}$ is a basis of U and $\dim U = 2$.

My answer for basis:________________________________

My answer for the dimension:________________________________

(8) (2 bonus points) (a) Give the definition of a basis of an arbitrary vector space V.

(b) Prove that the vector space $\mathbb{P}$ of all polynomials does not have a basis.

Solution: (a) A basis of a vector space V is a finite subset $\{v_1, \ldots, v_n\}$ of V which has the following two properties:

(i) $\{v_1, \ldots, v_n\}$ is linearly independent, and

(ii) $\operatorname{Span}\{v_1, \ldots, v_n\} = V$.

(b) Proof by contradiction: Suppose $\{p_1, \ldots, p_n\}$ is a basis of $\mathbb{P}$. Since there are only finitely many polynomials, their degrees are bounded by above by some natural number N, i.e. $\deg(p_i) \leq N$ for all $1 \leq i \leq n$. This means that all $p_i \in \mathbb{P}_N$. But $\mathbb{P}_N$ is a subspace of $\mathbb{P}$. Therefore $\operatorname{Span}\{p_1, \ldots, p_n\} \subset \mathbb{P}_N$. Hence $\mathbb{P} = \operatorname{Span}\{p_1, \ldots, p_n\} \subset \mathbb{P}_N$. Any polynomial whose degree is bigger than N, for example $q = x^{N+1}+1$, does not lie in $\mathbb{P}_N$, hence not in the span of $\{p_1, \ldots, p_n\}$, contradicting $q \in \mathbb{P}$.