(1) (5 pts) In each case give an example of:
(a) (1 pts) An inconsistent linear system of 2 equations in 2 variables.

Solution: There are many possibilities. For example,

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ x_1 & + & x_2 & = & 1 \end{array}$$

(b) (2 pts) A linear system of 3 equations in 2 variables which has infinitely many solutions. Also, give at least two different solutions.

Solution: For example,

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ x_1 & + & x_2 & = & 0 \\ x_1 & + & x_2 & = & 0 \end{array}$$

has the solutions

$$x_1 = -t, \quad x_2 = t$$

for t any real scalar. So, for example $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, -1)$ are two different solutions.

(c) (2 pts) An example of a linear system which is uniquely solvable. Also, give the unique solution.

Solution: For example, the linear system $x = 1$ has the unique solutions $x = 1$.

(2) (3 pts) Complete the theorem below by stating 3 conditions which are equivalent to, but not the same as the condition in (a).

Theorem. *For a $n \times n$ matrix A the following conditions are equivalent:*

(a) *A is invertible.*

(b)

(c)

(d)

Remarque : The theorem stated in class had more equivalent conditions. But you are only asked to list 3 of them.

Solution: Any combination of three of the following is correct: :

- The linear system $AX = B$ has a unique solution for every column B.
- The homogeneous linear system $AX = 0$ has only the trivial solution.
- The reduced row-echelon form of A is the identity matrix I_n.
- A has rank n.
- The linear system $AX = B$ has a solution for every columns B.
- There exists a $n \times n$ matrix C such that $AC = I_n$.
- A^T is invertible.
- $\det(A) \neq 0$.

(3) (2 pts) Consider the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -5 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 2 & -3 & 5 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

Which one is or which ones are in reduced row-echelon form?

My answer:__________________

Solution: B and D; A is not in ref because of columns 2; C is in ref but not in rref because of column 3; E is not in ref because of column 1; F is in ref, but not in rref because of column 2.

(4) (2 pts) Let $A = \begin{bmatrix} 2 & 7 & 1 \\ 5 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ et $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Calculate the matrix $B(AB)^{-1}A^2$.

My answer:__________________

Solution:

$$B(AB)^{-1}A^2 = B(B^{-1}A^{-1})(AA) = (BB^{-1})(A^{-1}A)A = I_2\, I_2\, A = A = \begin{bmatrix} 2 & 7 & 1 \\ 5 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(5) For the system of linear equations

$$\begin{array}{rcrcrcr} x & & & + & z & = & 1 \\ 3x & + & (a-3)y & + & az & = & 3 \\ 9x & & & + & a^2z & = & a+6 \end{array}$$

(a) (6 pts) determine the values of a for which the system has
 (i) no solution,
 (ii) infinitely many solutions,
 (iii) a unique solution.

(b) (2 pts) In case (ii) above describe give all solutions.

Solution: The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & a-3 & a & 3 \\ 9 & 0 & a^2 & a+6 \end{array}\right]$$

We perform the following operations, where R_i is row i: $-3R_1 + R_2 \to R_2$ et $-9R_1 + R_3 \to R_3$, and obtain :

$$M = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & a-3 & a-3 & 0 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right].$$

Since $a^2 - 9 = (a-3)(a+3)$ we get :

- If $a = -3$, then the last row of M is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -6 \end{array}\right]$. Hence the system is inconsistent.
- If $a = 3$ then $M = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Hence the system has infinitely many solutions.
- If $a \notin \{-3, 3\}$, then $M = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{array}\right]$ where the stars "$*$" are non-zero numbers. Hence the system is uniquely solvable, because there does not exist a free variable.

The answer to question (a) is therefore :
(i) The system in inconsistent if $a = -3$.
(ii) The system has infinitely many solutions if $a = 3$.
(iii) The system is uniquely solvable if $a \notin \{3, -3\}$.

To answer (b), let $a = 3$ in the matrix M above. This yields

$$M = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

a matrix in reduced row-echelon form. The only leading variable is x, while y and z are free variables. Putting $y = s$ and $z = t$ $(s, t \in \mathbb{R})$ gives the following general solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

hence the set of solutions is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad (s, t \in \mathbb{R})}$$

(6) (8 pts) In the matrix below **replace α with the second-last digit of your student number** and find its inverse:

$$A = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Check your answer by verifying $AA^{-1} = I_3$.

Solution: We apply the Inversion Algorithm, i.e., we find the reduced row-echelon form of $[A|I_3]$:

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -5\alpha & 2\alpha \\ 0 & 1 & 0 & 0 & 5 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \end{aligned}$$

Hence the inverse of A is the 3×3-matrix next to the identity matrix I_3 above:

$$A^{-1} = \begin{bmatrix} 1 & -5\alpha & 2\alpha \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}.$$

We check

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -5\alpha & 2\alpha \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & -5\alpha+5\alpha & 2\alpha-2\alpha \\ 0 & 0+5-4 & -2+2 \\ 0 & 10-10 & -4+5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(7) (2 bonus points) (a) Give the definition for a $n \times n$ matrix A to be invertible.
(b) Show that the inverse of an invertible matrix is unique.

Solution: (a) A $n \times n$ matrix A is invertible if there exists a $n \times n$ matrix B such that $AB = I_n = BA$, where I_n is the $n \times n$ identity matrix.

(b) Suppose B and B' satisfy $AB = I_n = BA$ and $AB' = I_n = B'A$. Then $B = BI_n = B(AB') = (BA)B' = I_nB' = B'$.