University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Assignment 2: due Feb. 26, 2009, 11:30 in the classroom

Family name (CAPITALS)	______________________
First name (CAPITALS)	______________________
Signature	______________________
Student number	______________________

Please read these instructions carefully:

- The table below is for the TA. Do not write in it.
- The assignment has to be submitted with the two cover pages.
- For privacy reasons, this page of the assignment will be detached, and you will only get back the remaining pages. Therefore, **fill in your name on both pages and your student number on this page only.**

Question	1	2	3	4	5	6	7	Total
Score								
Max. score	2	3	2	2	4	5	4	22

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FIRST NAME (CAPITALS)	______________________

Good luck! Bonne chance !

(1) (2 pts) Calculate the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 8 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 13 & -8 & 2 & 2 & 1 & 7 \\ 5 & 5 & -5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 4 & 3 & 0 & 0 \end{vmatrix}$$

without using any elementary operation by choosing at each step an appropriate column or row. Show your calculations.

Solution: We use cofactor expansion along the 5th row (the only non-zero entry has position 54, hence the sign $(-1)^{5+4} = -1$), second row (the only non-zero entry has position 22, hence the sign is 1), 3rd column (the sign is $(-1)^{2+3} = -1$), third column (sign is -1)

$$\begin{vmatrix} 1 & 1 & 2 & 8 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 13 & -8 & 2 & 2 & 1 & 7 \\ 5 & 5 & -5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 4 & 3 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 13 & -8 & 2 & 1 & 7 \\ 5 & 5 & -5 & 0 & 1 \\ 3 & 1 & 4 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 0 & 0 \\ 13 & 2 & 1 & 7 \\ 5 & -5 & 0 & 1 \\ 3 & 4 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 0 \\ 5 & -5 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 5 & -5 & 1 \\ 3 & 4 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -(4-6) = 2.$$

(2) (3 pts) Calculate the determinant $\begin{vmatrix} 2 & -3 & 1 & 4 \\ 12 & -5 & 11 & 3 \\ 7 & 6 & -9 & 4 \\ 2 & -7 & 6 & 9 \end{vmatrix}$.

Any correct method is allowed. Show all details.

Solution: Perform $R_4 - R_1, R_3 - R_1, R_2 - 6R_1$ on the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 4 \\ 12 & -5 & 11 & 3 \\ 7 & 6 & -9 & 4 \\ 2 & -7 & 6 & 9 \end{bmatrix} \quad \text{and get the matrix} \quad B = \begin{bmatrix} 2 & -3 & 1 & 4 \\ 0 & 13 & 5 & -21 \\ 5 & 9 & -10 & 0 \\ 0 & -4 & 5 & 5 \end{bmatrix}.$$

We note (by a theorem from the course) that $\det(A) = \det(B)$. We expand across the first column of B and get

$$\det(B) = 2(-1)^{1+1} \det \begin{bmatrix} 13 & 5 & -21 \\ 9 & -10 & 0 \\ -4 & 5 & 5 \end{bmatrix} + 5(-1)^{3+1} \det \begin{bmatrix} -3 & 1 & 4 \\ 13 & 5 & -21 \\ -4 & 5 & 5 \end{bmatrix}$$

$$= 2\Big\{(-21)(-1)^{3+1} \det \begin{bmatrix} 9 & -10 \\ -4 & 5 \end{bmatrix} + 5(-1)^{3+3} \det \begin{bmatrix} 13 & 5 \\ 9 & -10 \end{bmatrix}\Big\}$$

$$+ 5\Big\{(-3)(-1)^{1+1} \det \begin{bmatrix} 5 & -21 \\ 5 & 5 \end{bmatrix} + 13(-1)^{2+1} \det \begin{bmatrix} 1 & 4 \\ 5 & 5 \end{bmatrix} + (-4)(-1)^{3+1} \det \begin{bmatrix} 1 & 4 \\ 5 & -21 \end{bmatrix}\Big\}$$

$$= 2\{(-21)(45 - 40) + 5(-130 - 45)\} + 5\{(-3)(25 + 105) - 13(5 - 20) - 4(-21 - 20)\}$$

$$= 2\{-105 - 875\} + 5\{-390 + 195 + 164\} = -2115$$

Second solution: Perform $R_4 - R_1, R_3 - R_1, R_2 - 6R_1$ on the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 4 \\ 12 & -5 & 11 & 3 \\ 7 & 6 & -9 & 4 \\ 2 & -7 & 6 & 9 \end{bmatrix} \quad \text{and get the matrix} \quad B = \begin{bmatrix} 2 & -3 & 1 & 4 \\ 0 & 13 & 5 & -21 \\ 5 & 9 & -10 & 0 \\ 0 & -4 & 5 & 5 \end{bmatrix}.$$

We note (by a theorem from the course) that $\det(A) = \det(B)$. We expand across the first column of B and get

$$\det(B) = 2(-1)^{1+1} \det \begin{bmatrix} 13 & 5 & -21 \\ 9 & -10 & 0 \\ -4 & 5 & 5 \end{bmatrix} + 5(-1)^{3+1} \det \begin{bmatrix} -3 & 1 & 4 \\ 13 & 5 & -21 \\ -4 & 5 & 5 \end{bmatrix}.$$

Perform on $\begin{bmatrix} 13 & 5 & -21 \\ 9 & -10 & 0 \\ -4 & 5 & 5 \end{bmatrix}$ the following operation: subtract row 1 from R_3. We get the following matrix $\begin{bmatrix} 13 & 5 & -21 \\ 9 & -10 & 0 \\ -17 & 0 & 26 \end{bmatrix}$ with the same determinant: $(-21)(-1)^{1+3} \det \begin{bmatrix} 9 & -10 \\ -17 & 0 \end{bmatrix} + (26)(-1)^{3+3} \det \begin{bmatrix} 13 & 5 \\ 9 & -10 \end{bmatrix} = (-21)(-170) + 26(-130 - 45) = -980$. Next we note that if we subtract from R_3 row 2, the matrix $\begin{bmatrix} -3 & 1 & 4 \\ 13 & 5 & -21 \\ -4 & 5 & 5 \end{bmatrix}$ is transformed into the matrix $\begin{bmatrix} -3 & 1 & 4 \\ 13 & 5 & -21 \\ -17 & 0 & 26 \end{bmatrix}$. If from R_2 we subtract 5 times row 1 we get the matrix $\begin{bmatrix} -3 & 1 & 4 \\ 28 & 0 & -41 \\ -17 & 0 & 26 \end{bmatrix}$ with the same determinant: $1(-1)^{1+2} \det \begin{bmatrix} 28 & -41 \\ -17 & 26 \end{bmatrix} = -(728 - 697) = -31$. Hence the determinant of the original matrix is given by: $2(-980) + 5(-31) = -2115$.

Third solution: Add $-11R_1$ to R_2, $9R_1$ to R_3 and $-6R_1$ to R_4 (these operations do not change the determinant. Then expand along the their column:

$$\begin{vmatrix} 2 & -3 & 1 & 4 \\ 12 & -5 & 11 & 3 \\ 7 & 6 & -9 & 4 \\ 2 & -7 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 & 4 \\ -10 & 28 & 0 & -41 \\ 25 & -21 & 0 & 40 \\ -10 & 11 & 0 & -15 \end{vmatrix} = \begin{vmatrix} -10 & 28 & -41 \\ 25 & -21 & 40 \\ -10 & 11 & -15 \end{vmatrix} = 5 \begin{vmatrix} -2 & 28 & -41 \\ 5 & -21 & 40 \\ -2 & 11 & -15 \end{vmatrix}.$$

Now we add $2R_3$ to R_2, $2R_2$ to R_1 and $2R_2$ to R_3 and expand along the first column:

$$5 \begin{vmatrix} -2 & 28 & -41 \\ 5 & -21 & 40 \\ -2 & 11 & -15 \end{vmatrix} = 5 \begin{vmatrix} -2 & 28 & -41 \\ 1 & 1 & 10 \\ -2 & 11 & -15 \end{vmatrix} = 5 \begin{vmatrix} 0 & 30 & -21 \\ 1 & 1 & 10 \\ 0 & 13 & 5 \end{vmatrix} = 5 \begin{vmatrix} 30 & -21 \\ 13 & 5 \end{vmatrix}.$$

Now we can calculate $\det(A) = -5(30 \cdot 5 + 21 \cdot 13) = -2115$.

(3) (2 pts) If A is a square matrix such that $\det(A) = 5$ and $\det(-2A) = 80$, what is the size of A?

Solution: If A has size $n \times n$, then $80 = \det(-2A) = (-2)^n \det(A) = (-2)^n \cdot 5$. Hence $(-2)^n = 80/5 = 16$, whence $n = 4$.

My answer:______________

(4) (2 pts) A student performs the following row operations on a 4×4 matrix A:

$$A = \begin{pmatrix} R_1 \\ \hline R_2 \\ \hline R_3 \\ \hline R_4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} R_4 \\ \hline R_1 - 3R_2 \\ \hline 2R_3 \\ \hline R_1 \end{pmatrix} = A'$$

If $\det(A') = 5$, what is $\det(A)$? Show all details.

Solution:

$$5 = \det A' = -\det \begin{pmatrix} R_1 \\ \hline R_1 - 3R_2 \\ \hline 2R_3 \\ \hline R_4 \end{pmatrix} = -2\det \begin{pmatrix} R_1 \\ \hline R_1 - 3R_2 \\ \hline R_3 \\ \hline R_4 \end{pmatrix}$$

$$= -2\det \begin{pmatrix} R_1 \\ \hline -3R_2 \\ \hline R_3 \\ \hline R_4 \end{pmatrix} = (-2)(-3)\det \begin{pmatrix} R_1 \\ \hline R_2 \\ \hline R_3 \\ \hline R_4 \end{pmatrix} = 6\det(A).$$

Hence $\det(A) = 5/6$.

My answer:______________

(5) (4 pts) Let A and B be two square matrices of the same size such that

$$\det(A^2 B^T) = 250 \quad \text{and} \quad \det(A^T B^2) = -1/2.$$

What are the determinants of A and B? Show your calculations.

Solution: Let $a = \det(A)$ and $b = \det(B)$. Then $\det(A^2 B^T) = \det(A)\det(A)\det(B^T) = \det(A)\det(A)\det(B) = a^2 b$, and similarly $\det(A^T B^2) = ab^2$. Hence

$$a^2 b = 250 \quad \text{and} \quad ab^2 = -1/2.$$

Therefore $250(-1/2) = (a^2 b)(ab^2) = a^3 b^3 = (ab)^3$, hence $(ab)^3 = -125$, and then $ab = -5$. Finally,

$$a = \frac{a^2 b}{ab} = \frac{250}{-5} = -50, \qquad b = \frac{ab^2}{ab} = \frac{(-1/2)}{-5} = \frac{1}{10}.$$

(6) (5 pts) Which of the following are subspaces? Support your answer with details.

(a) $U_1 = \{X \in \mathbb{R}^n : AX = 3X\}$.

Solution: This is a subspace. There are two ways to see this. First, notice that $AX = 3X \iff (A - 3I_n)X = 0$. Hence $U_1 = \{X \in \mathbb{R}^n : (A - 3I_n)X = 0\} = (A - 3I_n)$. As a null space of a matrix, U_1 is a subspace.

Second solution: Check the 3 conditions of the subspace test. (1) The zero vector 0 lies in U_1 since $A0 = 0 = 3X$. (2) If $X_1 \in U_1$ and $X_2 \in U_1$, then also $X_1 + X_2 \in U_1$ since $A(X_1 + X_2) = AX_1 + AX_2 = 3X_1 + 3X_2 = 3(X_1 + X_2)$ using that $AX_1 = 3X_1$ and $AX_2 = 3X_2$. (3) If $X \in U_1$, then also $sX \in U$ for any scalar $s \in \mathbb{R}$: $A(sX) = s(AX) = s(3X) = 3(sX)$.

Marking: 2 points

(b) $U_2 = \{X = \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 : x + 2y + 3z = \alpha\}$, where α is the last digit of your student number.

Solution: If $\alpha = 0$ then U_2 is the null space of the matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, hence a subspace. The second solution is to check the 3 conditions of the subspace test.

On the other side, if $\alpha \neq 0$ then U_2 is not a subspace since the zero vector $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ does not lie in U_2.

Marking: 1 point

(c) We fix a vector $d \in \mathbb{R}^3$, and define $U_3 = \{X \in \mathbb{R}^3 : \text{proj}_d(X) = 0\}$, where $\text{proj}_d(X)$ is the projection of X onto d.

Solution: This is a subspace. This can be seen by checking the 3 conditions of the subspace test. (1) The zero vector lies in U_3 since $\text{proj}_d(0) = (0 \cdot d)/(d \cdot d)d = 0$. (2) If $X_1 \in U_3$ and $X_2 \in U_3$, then also $X_1 + X_2 \in U_1$: Indeed, by assumption $\text{proj}_s(X_1) = 0 = \text{proj}_d(X_2)$. Hence $\text{proj}_d(X_1 + X_2) = ((X_1 + X_2) \cdot d)/(d \cdot d)d = (X_1 \cdot d)/(d \cdot d)d + (X_2 \cdot d)/(d \cdot d)d = \text{proj}_d(X_1) + \text{proj}_d(X_2) = 0 + 0 = 0$. (3) If $X \in U_1$, then also $sX \in U$ for any scalar $s \in \mathbb{R}$: $\text{proj}_d(sX) = ((sX) \cdot d)/(d \cdot d)d = s((X \cdot d)/(d \cdot d)d) = s\,\text{proj}(X) = s0 = 0$.

Marking: 2 points

(7) (4 pts) Determine α and β such that

$$\begin{bmatrix} 2 \\ 1 \\ \beta \end{bmatrix} \quad \text{lies in the span of } X_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 3 \\ \alpha \end{bmatrix}.$$

Solution: The given vector lies in the span if and only if the linear system

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & \alpha z & = & \beta \end{array}$$

is solvable. We write down the augmented matrix of this system and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 4 & 5 & \alpha & \beta \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & \alpha - 4 & \beta - 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & \alpha - 6 & \beta - 7 \end{array}\right]$$

This system is solvable if and only if $\alpha \neq 6$ or $\alpha = 6$ and $\beta = 7$. Thus, the given vector lies in the span exactly under the following two conditions:
(i) $\alpha \neq 6$, or
(ii) $\alpha = 6$, $\beta = 7$.