

University of Ottawa
Department of Mathematics and Statistics

MAT 1341C: Introduction to Linear Algebra
Instructor: Erhard Neher

Final Exam (April 2009) Time: 3 hours

FAMILY NAME (CAPITALS)	_____
FIRST NAME (CAPITALS)	_____
Signature	_____
Student number	_____

Please read these instructions carefully:

- Read each question carefully, and answer all questions in the space provided after each question. For questions 10–16 you may use the back of pages if necessary, but be sure to indicate to the marker that you have done so. There is an extra page at the end of the exam.
- Questions 1–9 are short answer questions, and no part marks will be given. You must show all the details for questions 10–16, and argue logically. Write legibly.
- Where it is possible to check your work, do so! Read each question carefully – you will save yourself time and unnecessary grief later on.
- **This is a closed book exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Do not detach the pages of the exam.

Good luck! Bonne chance !

Question	1–4	5–8	9–10	11–12	13–14	15–16	Total
Score							
Max. score	11	11	8	11	11	14	66

(1) (3 pts) For a homogeneous linear system answer the following questions

- (a) If the system has 6 equations and 7 variables and the rank of the coefficient matrix is 5, the general solution has how many parameters?

My answer: _____

- (b) If the system has more equations than variables, it has infinitely many solutions. Answer with T for true or F for false.

My answer: _____

- (c) If the system has only the trivial solution, the coefficient matrix is invertible. Answer with T for true or F for false.

My answer: _____

(2) (2 pts) If $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$, compute the first column of $C = AB$.

My answer: _____

- (3) (3 pts) Calculate the inverse of the 3×3 matrix: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

My answer: _____

- (4) (3 pts) Calculate the determinant of the following matrix: $A = \begin{bmatrix} -3 & -1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$.

My answer: _____

(5) (3 pts) Let V be a vector space of dimension 11. Answer the following questions with T for true and F for false.

(a) Every subset of 11 vectors of V is a basis of V .

My answer:_____

(b) Every spanning set of V can be extended to a basis of V .

My answer:_____

(c) V contains exactly 3 subspaces of dimension 3.

My answer:_____

(6) (3 pts) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^7$ be a linear map whose kernel has the basis $v = [1 \ 0 \ 1 \ 2]^T$. What is the dimension of the image of T ?

My answer:_____

- (7) (3 pts) If $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ is a linear map with $T(1 + x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(x + x^2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $T(1) = ?$

My answer: _____

- (8) (2 pts) Let $Z = [1 \ 2 \ 3]^T$. The map $T : \mathbb{R}^3 \rightarrow \mathbb{R}$, $T(X) = X \cdot Z$ is linear. Find its standard matrix.

My answer: _____

(9) (3 pts) For an $n \times n$ matrix A answer the questions (ii), (iii) and (iv) below. As an example to show you what type of answer is expected from you, I have given the answer to question (i),

(i) State a condition on the columns of A which is equivalent to the condition $\text{rank}(A) < n$.

Answer : The columns of A are linearly dependent.

(ii) State a condition on the determinant of A which is equivalent to the condition $\text{rank}(A) < n$.

Answer : _____

(iii) State a condition regarding the homogeneous linear system $AX = 0$ which is equivalent to the condition $\text{rank}(A) < n$.

Answer : _____

(iv) State a condition regarding invertibility of A which is equivalent to the condition $\text{rank}(A) < n$.

Answer : _____

(10) (5 pts) Let U be the subspace of \mathbb{R}^4 spanned by $\{v_1, v_2, v_3, v_4, v_5\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 11 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 13 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}.$$

Find a basis of U and determine the dimension of U .

(11) (a) (3 pts) Find all eigenvalues of the matrix A below together with their multiplicities:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) (1 pt) Based on the knowledge of the eigenvalues alone, decide if A is diagonalizable or not. Justify your answer!

(c) (1 pt) Based on the knowledge of the eigenvalues alone, decide if A is invertible or not. Justify your answer!

(12) (6 pts) The eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

are 0 and 1. (You do not need to show this.)

(a) (4 pts) For each eigenvalue of A find a basis of the corresponding eigenspace.

(b) (2 pts) Decide if A is diagonalizable or not. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Justify your answer.

(13) For the system of linear equations

$$\begin{array}{rcccccc} x & + & 2y & + & 3z & = & 1 \\ 2x & + & ay & - & 12z & = & -10 \\ -x & - & y & + & (a+2)z & = & 1 \end{array}$$

- (a) (5 pts) determine the values of a for which the system has
- (i) no solution,
 - (ii) infinitely many solutions,
 - (iii) a unique solution.
- (b) (2 pts) In case (ii) above describe give all solutions.

(14) (4 pts) Are the following sets linear independent? Give details!

(a) $\left\{ \frac{1}{x^2-1}, \frac{1}{x^2+2x-3}, \frac{1}{x^2+4x+3} \right\}$ in $\mathbb{F}[-1/2, 1/2]$, where $\mathbb{F}[-1/2, 1/2]$ is the vector spaces of functions $f : [-1/2, 1/2] \rightarrow \mathbb{R}$;

My answer: _____

(b) $\left\{ \begin{bmatrix} -7 & 13 \\ -9 & 47 \end{bmatrix}, \begin{bmatrix} 0 & 16 \\ 25 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix} \right\}$ in \mathbb{M}_{22} , where \mathbb{M}_{22} is the vector space of 2×2 matrices.

My answer: _____

- (15) (8 pts) Show that $U = \{p \in \mathbb{P}_3 : 3p(1) = p(0)\}$ is a subspace of \mathbb{P}_3 , find a basis and determine $\dim U$. (Recall that \mathbb{P}_3 is the vector space of polynomials of degree ≤ 3 .)
(bonus question: (2 bonus pts) Find a linear transformation $T : \mathbb{P}_3 \rightarrow \mathbb{R}$ such that U is the kernel of T .)

(16) (6 pts) Let U be the subspace of \mathbb{R}^4 spanned by $\{u_1, u_2\}$ where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

(a) (3 pts) Find a basis of U^\perp where U^\perp is the orthogonal complement of U :

$$\begin{aligned} U^\perp &= \{v \in \mathbb{R}^4 : v \text{ is orthogonal to every element of } U\} \\ &= \{v \in \mathbb{R}^4 : u \cdot v = 0 \text{ for every } u \in U\}. \end{aligned}$$

(b) (3 pts) Find the orthogonal projection of $[-3 \ -4 \ 1 \ 1]^T$ onto U . (Hint: The result will have integer coefficients.)

(extra page)