[5] 1. Determine if

$$\left\{ \frac{1}{x^2+x-6}, \frac{1}{x^2-x-2}, \frac{1}{x^2+4x+3} \right\}$$

is a linearly independent subset of the vector space $V = \mathbb{F}[0,1]$. If not, give a nontrivial linear combination which vanishes.

First observe $x^2+x-6 = (x+3)(x-2)$, $x^2-x-2 = (x-2)(x+1)$ and $x^2+4x+3 = (x+1)(x+3)$.

Suppose $\frac{a}{x^2+x-6} + \frac{b}{x^2-x-2} + \frac{c}{x^2+4x+3} = 0$. $(*)$

We can multiply this relation by $N = (x+3)(x-2)(x+1)$ ($\neq 0$ on $[0,1]$)

and obtain

$$a(x+1) + b(x+3) + c(x-2) = 0$$

After expanding, we get

$$(a+b+c)x + (a+3b-2c) = 0,$$

equivalently:

$$\begin{array}{l} a+b+c=0 \\ a+3b-2c=0 \end{array} \Leftrightarrow \begin{array}{l} a=-(b+c) \\ b+c=3b-2c \end{array} \Leftrightarrow \begin{array}{l} a=-(b+c) \\ 2b=3c \end{array}$$

The general solution is therefore

$$a = -\tfrac{5}{2}t$$
$$b = \tfrac{3}{2}t$$
$$c = t$$

Since there are infinitely many solutions, the 3 functions are linearly dependent. A nontrivial linear combination which vanishes is obtained for any $t \neq 0$, e.g. for $t=2$ we get

$$\frac{-5}{x^2+x-6} + \frac{3}{x^2-x-2} + \frac{2}{x^2+4x+3} = 0 \qquad (**)$$

Other approach: In $(*)$ substitute 3 different values for $x \in [0,1]$, e.g. $x=0$, $x=\frac{1}{2}$ and $x=1$. This gives a linear system of 3 eq for a, b, c. Then solve. One gets a nontrivial relation like $(**)$. One then verifies that it holds for all $x \in [0,1]$.

[5] **2.** (a) Show that $U = \{A \in \mathbb{M}_{33} : A^T = -A\}$ is a subspace of the vector space $V = \mathbb{M}_{33}$, find a basis of U and determine its dimension.

[4] (b) Find a linear map $T : \mathbb{M}_{33} \to \mathbb{M}_{33}$ such that $U = \ker(T)$. Use this to give a second proof that U is a subspace.

(a) Let $A = [a_{ij}]$. The condition $A^T = -A$ is equivalent to $a_{ij} = -a_{ji}$ for $1 \le i, j \le 3$. Therefore,

(2)

$$A \in U \iff A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = a_{12}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= a_{12} F_{12} + a_{13} F_{13} + a_{23} F_{23},$$

where $F_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $F_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

Since a_{12}, a_{13}, a_{23} above are arbitrary, we get

$$U = \operatorname{span}\{F_{12}, F_{13}, F_{23}\}.$$

In particular, as a span, U is a subspace of $\mathbb{M}_{33}$. We check that $\{F_{12}, F_{13}, F_{23}\}$ is l.i.:

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$$0 = aF_{12} + bF_{13} + cF_{23} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \text{ whence } a = 0 = b = c.$$

Therefore $\{F_{12}, F_{13}, F_{23}\}$ is a basis of U, and $\underbrace{\dim U = 3}_{(1)}$.

(b) Let $T : \mathbb{M}_{33} \to \mathbb{M}_{33}$ be the map $T(A) = A + A^T$. Then T is

(2) linear: $T(A+B) = A + B + (A+B)^T = A + A^T + B + B^T = T(A) + T(B)$, and $T(rA) = rA + (rA)^T = rA + rA^T = rT(A)$. We have

(1)

$$\ker(T) = \{A \in \mathbb{M}_{33} : T(A) = 0\} = \{A \in \mathbb{M}_{33} : A + A^T = 0\} =$$
$$= \{A \in \mathbb{M}_{33} : A^T = -A\} = U$$

(1) As the kernel of a linear map, U is a subspace

[6] **3.** A polynomial $p(x)$ is called *odd* if $p(x) = -p(-x)$. Show that the set U of odd polynomials in $\mathbb{P}_n$ is a subspace, find a basis of U and determine its dimension.

Let $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_n x^n$. Then

$p(-x) = a_0 - a_1x + a_2x^2 + \dots + a_n(-1)^n x^n$.

Hence $p(x) = -p(-x) \iff$

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = -a_0 + a_1x - a_2x^2 + \dots + (-1)^{n+1}a_nx^n$$

We distinguish two cases: n even and n odd.

(a) n even. Then $(-1)^{n+1}a_n = -a_n$, whence $p(x) = -p(x) \iff$

$2(a_0 + a_2x^2 + \dots + a_nx^n) = 0 \iff a_i = 0$ for all even i, $0 \le i \le n$.

Hence $p(x)$ is odd $\iff p(x) = a_1x + a_3x^3 + \dots + a_{n-1}x^{n-1}$, so $U = \operatorname{span}\{x, x^3, \dots, x^{n-1}\}$. The spanning set $\{x, x^3, \dots, x^{n-1}\}$ is linearly independent since it consists of polynomials of different degree. Hence $\{x, x^3, \dots, x^{n-1}\}$ is a basis of U, and therefore $\dim U = \frac{n}{2}$.

(b) n odd. As in (a) we get that $p(x)$ is odd $\iff$ $p(x) = a_1x + a_3x^3 + \dots + a_nx^n$, hence $U = \operatorname{span}\{x, x^2, \dots, x^n\}$. The spanning set $\{x, x^3, \dots, x^n\}$ is linearly independent since it consists of polynomials of different degree. Therefore, $\{x, x^3, x^5, \dots, x^n\}$ is a basis of U; we have $\dim U = \frac{n+1}{2}$.

[4] **4.** (a) Let U be a nonzero subspace of $\mathbb{R}^n$. Show that $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(X) = \text{proj}_U(X)$ is a linear transformation and determine $ker(T)$ and $im(T)$.

[3] (b) For $U = \text{Span}\{[1 \ -1 \ 1 \ 2]^T, [1 \ 1 \ 2 \ -1]^T\} \subset \mathbb{R}^4$ determine the standard matrix of the linear transformation T of (a).

(a) Let $\{F_1, \dots, F_k\}$ be an orthogonal basis of U. Then

$$T(X) = \text{proj}_U(X) = \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \dots + \frac{X \cdot F_k}{F_k \cdot F_k} F_k .$$

Hence $T(X+Y) = \frac{(X+Y)\cdot F_1}{F_1 \cdot F_1} F_1 + \dots + \frac{(X+Y)\cdot F_k}{F_k \cdot F_k} F_k =$

(2) $= \left(\frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \dots + \frac{X \cdot F_k}{F_k \cdot F_k} F_k\right) + \left(\frac{Y \cdot F_1}{F_1 \cdot F_1} F_1 + \dots + \frac{Y \cdot F_k}{F_k \cdot F_k}\right) F_k = T(X) + T(Y)$

and for $r \in \mathbb{R}$, $X \in \mathbb{R}^n$,

$$T(rX) = \frac{(rX)\cdot F_1}{F_1 \cdot F_1} F_1 + \dots + \frac{(rX)\cdot F_k}{F_k \cdot F_k} F_k = r\,\frac{X \cdot F_1}{F_1 \cdot F_1} + \dots + \frac{r X \cdot F_k}{F_k \cdot F_k} F_k =$$

$= r\,T(X)$.

Thus T is linear. Any $X \in \mathbb{R}^n$ can be decomposed as $X = \text{proj}_U(X) + (X - \text{proj}_U(X))$ where $\text{proj}_U(X) \in U$ and $X - \text{proj}_U(X) \in U^\perp$. Thus
(2) $\text{proj}_U(X) = \{0\} \Leftrightarrow X \in U^\perp$, thus $\ker(T) = U^\perp$. Also, $\text{proj}_U(X) \in U$, thus $\text{im}\, T \subset U$. Since $X = \text{proj}_U(X)$ for $X \in U$, it follows that $U = \text{im}\, T$.

(b) Note that $F_1 = [1, -1, 1, 2]^T$ and $F_2 = [1, 1, 2, -1]^T$ is an orthogonal basis of U. Hence for $X = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$ we get

$$T(X) = \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 = \frac{x_1 - x_2 + x_3 - 2x_4}{7} F_1 + \frac{x_1 + x_2 + 2x_3 - x_4}{7} F_2$$

$$= \frac{1}{7}\begin{bmatrix} 2x_1 + 3x_3 - 3x_4 \\ 2x_2 + x_3 + x_4 \\ 3x_1 + x_2 + 5x_3 - 4x_4 \\ x_1 - 3x_2 - 3x_4 \end{bmatrix}$$

The 4 columns of the standard matrix A are the images $T(E_i)$, $1 \le i \le 4$, where $E_1, \dots, E_4$ is the standard basis of $\mathbb{R}^4$. Hence

$$A = \frac{1}{7}\begin{bmatrix} 2 & 0 & 3 & -3 \\ 0 & 2 & 1 & 1 \\ 3 & 1 & 5 & -4 \\ 1 & -3 & 0 & -3 \end{bmatrix}$$

5. (4 extra points) Let V and W be vector spaces and let $T : V \to W$ be a linear map.

(a) Let $Z \subset W$ be a subspace of W. Show that $U = \{v \in V : T(v) \in Z\}$ is a subspace of V.

(b) Let $U_1 \subset V$ be a subspace of V. Show that $Z_1 = \{w \in W : w = T(u_1) \text{ for some } u_1 \in U_1\}$ is a subspace of W.

(a) We verify the 3 conditions of the subspace test.

- $0 \in U$ since $T(0) = 0 \in Z$
- Let $X, Y \in U$, i.e. $T(X) \in Z$ and $T(Y) \in Z$. Hence $T(X+Y) = T(X) + T(Y)$ $\in Z$ since Z is a subspace. But this means $X+Y \in U$.
- Let $X \in U$, $r \in \mathbb{R}$. By definition of U, we have $T(X) \in Z$. Then $T(rX) = rT(X) \in Z$ because Z is a subspace. Therefore $rX \in U$.

This proves that U is a subspace

(b) We proceed as in (a).

- $0_W \in Z_1$ since $0_W = T(0_V)$ and $0_V \in U_1$.
- Let $X, Y \in Z_1$, say $X = T(u_1)$ and $Y = T(v_1)$ for $u_1, v_1 \in U_1$. Then $X + Y = T(u_1) + T(v_1) = T(u_1 + v_1)$. But $u_1 + v_1 \in U_1$ because U_1 is a subspace. Hence $X + Y \in Z_1$ by definition of Z_1.
- Let $X \in Z_1$ and $r \in \mathbb{R}$. By definition of Z_1, we know $X = T(u_1)$ for some $u_1 \in U_1$. Now $rX = rT(u_1) = T(ru_1)$, where $ru_1 \in U_1$ because U_1 is a subspace. This proves $rX \in Z_1$.

Altogether, we have shown that Z_1 is a subspace.