1. Find a basis of the row and column space of A and determine the rank of A, where A is the matrix

$$\begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 2 & -5 & 6 & -3 & -2 \\ 0 & 5 & 0 & 15 & 10 \\ 2 & 6 & 6 & 18 & 8 \end{bmatrix}$$

We first find a row-echelon form of A:

(2)

$$\begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 2 & -5 & 6 & -3 & -2 \\ 0 & 5 & 0 & 15 & 10 \\ 2 & 6 & 6 & 18 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & -1 & 0 & -3 & -2 \\ 0 & 5 & 0 & 15 & 10 \\ 0 & 10 & 0 & 18 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: A'$$

(1) The non-zero rows of A' are a basis of the row-space of A:

$$\{[1,-2,3,0,0]^T,\ [0,1,0,3,2]^T,\ [0,0,0,1,1]^T\}$$

(1) The columns corresponding to the leading 1's of A' are a basis of the column space of A:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 15 \\ 18 \end{bmatrix} \right\}$$

(1) A has rank 3

2. Use the Gram-Schmidt algorithm to convert the basis

$$[1\ \ -1\ \ 1]^T,\ \ [1\ \ 0\ \ 1]^T,\ \ [1\ \ 1\ \ 2]^T$$

of $\mathbb{R}^3$ to an orthogonal basis.

(2)

$F_1 = X_1 = [1, -1, 1]^T$

$F_2 = X_2 - \frac{X_2 \cdot F_1}{F_1 \cdot F_1} F_1 = [1\ 0\ 1]^T - \frac{[1\ 0\ 1]^T \cdot [1, -1, 1]^T}{1+1+1} [1, -1, 1]^T$

$= [1\ 0\ 1]^T - \frac{2}{3}[1, -1, 1]^T = [\frac{1}{3}, \frac{2}{3}, \frac{1}{3}]^T = \frac{1}{3}[1, 2, 1]^T$

(2)

$F_3 = X_3 - \frac{X_3 \cdot F_2}{F_2 \cdot F_2} F_2 - \frac{X_3 \cdot F_1}{F_1 \cdot F_1} F_1 =$

$= [1, 1, 2]^T - \frac{\cancel{\frac{1}{3}}[1, 1, 2]^T \cdot [1, 2, 1]^T}{\cancel{\frac{1}{3}}(1+4+1)} \cancel{\frac{1}{3}}[1, 2, 1]^T - \frac{[1, 1, 2]^T \cdot [1, -1, 1]^T}{3}[1, -1, 1]^T$

$= [1, 1, 2]^T - \frac{5}{6}[1, 2, 1]^T - \frac{2}{3}[1, -1, 1]^T$

$= [1 - \frac{5}{6} - \frac{2}{3}, 1 - \frac{5}{3} + \frac{2}{3}, 2 - \frac{5}{6} - \frac{2}{3}]^T = [-\frac{1}{2}, 0, \frac{1}{2}]^T =$

$= \frac{1}{2}[-1, 0, 1]^T$

Hence, the Gram-Schmidt algorithm provides the orthogonal basis

$$\{[1, -1, 1]^T, \frac{1}{3}[1, 2, 1]^T, \frac{1}{2}[-1, 0, 1]^T\}$$

<u>Note</u> Since we are only interested in an orthogonal basis, we can multiply any basis vector by a non-zero scalar. This will yield another orthogonal basis. For example,

$$\{[1, -1, 1]^T, [1, 2, 1]^T, [-1, 0, 1]^T\}$$

is also an orthogonal basis

3. Write the vector X as a sum of a vector in U and in $U^\perp$:

$$X = [1 \ \ 1 \ \ 1]^T, \quad U = \mathrm{Span}\{[1 \ \ -1 \ \ 2]^T, [3 \ \ -1 \ \ 4]^T\}.$$

We first find an orthogonal basis of U, by applying the Gram-Schmidt algorithm:

(2)

$F_1 = X_1 = [1,-1,2]^T$

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{F_1 \cdot F_1} F_1 = [3,-1,4]^T - \frac{[3,-1,2]^T \cdot [1,-1,4]^T}{1+1+4} [1,-1,2]^T$$

$$= [3,-1,4]^T - \frac{3+1+8}{6} [1,-1,2]^T = [3,-1,4]^T - 2[1,-1,2]^T$$

$$= [1,1,0]^T$$

Thus, $\{[1,-1,2]^T, [1,1,0]^T\}$ is an orthogonal basis of U.

We can now apply the projection formula (Lemma 1)

(2)

$$\mathrm{proj}_U(X) = \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 = \frac{[1,1,1]^T \cdot [1,-1,2]^T}{1+1+4} F_1 + \frac{[1,1,1]^T \cdot [1,1,0]^T}{1+1} F_2$$

$$= \frac{2}{6} [1,-1,2]^T + \frac{2}{2} [1,1,0]^T = \frac{1}{3} [4,2,2]^T$$

We obtain the component of X in $U^\perp$ as

(1)

$$X - \mathrm{proj}_U(X) = [1,1,1]^T - \frac{1}{3}[4,2,2]^T = \frac{1}{3}[-1,1,1]^T$$

Hence

$$X = \mathrm{proj}_U(X) + (X - \mathrm{proj}_U(X)) = \frac{1}{3}\begin{bmatrix}4\\2\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

where $\frac{1}{3}\begin{bmatrix}4\\2\\2\end{bmatrix} \in U$ and $\frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} \in U^\perp$.

4. Find the best approximation to a solution of the following linear system:

$$\begin{aligned} x &- y &= 3 \\ 2x &+ y &= -1 \\ x &+ 5y &= -4 \end{aligned}$$

The coefficient matrix is $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}$. Hence

$A^TA = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 27 \end{bmatrix}$, and

(2) $A^TB = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -24 \end{bmatrix}$. The normal equation

$(A^TA)Z = A^TB$ is therefore
$$\begin{aligned} 6z_1 + 6z_2 &= -3 \\ 6z_1 + 27z_2 &= -24 \end{aligned}$$

We find the solution using Gaussian elimination:

$$\begin{bmatrix} 6 & 6 & -3 \\ 6 & 27 & -24 \end{bmatrix} \sim \begin{bmatrix} 6 & 6 & -3 \\ 0 & 21 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \end{bmatrix}$$

(2) Hence there is a unique solution (clear since A^TA is invertible)

$$Z = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

which is the best approximation to a solution of the given linear system.

(1) The vector $AZ = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -9/2 \end{bmatrix}$ is as close to a solution as possible.

5. (3 extra points) (Continuation of question 5 of Assignment 2) Let U and V be subspaces of $\mathbb{R}^n$. Recall that then $U \cap V = \{x \in \mathbb{R}^n : x \in U \text{ and } x \in V\}$ and $U + V = \{x \in R^n : x = u + v \text{ for some } u \in U \text{ and } v \in V\}$ are subspaces of $\mathbb{R}^n$. Show that

$$\dim U + \dim V = \dim(U + V) + \dim(U \cap V).$$

(Hint: If $U \cap V \neq \{0\}$ let B be a basis of $U \cap V$, otherwise put $B = \emptyset$. By Theorem 3 of §4.3, one can extend B to a basis B_U of U and also to a basis B_V of V. Show that $B_U \cup B_V$ is a basis of $U + V$.)

We proceed as in the hint, and write

$B = \{X_1, \ldots X_k\}$ $\quad$ ($k = 0$ if $B = \emptyset$)

$B_U = \{X_1, \ldots, X_k, Y_1, \ldots, Y_l\}$

$B_V = \{X_1, \ldots, X_k, Z_1, \ldots Z_m\}$, hence

$B_U \cup B_V = \{X_1, \ldots, X_k, Y_1, \ldots, Y_k, Z_1, \ldots, Z_m\}$.

We show that

- $B_U \cup B_V$ is lin indep: $0 = \sum_{a=1}^{k} s_a X_a + \sum_{b=1}^{l} r_b Y_b + \sum_{c=1}^{m} p_c Z_c$

Hence

$$\sum_{a=1}^{k} s_a X_a + \sum_{b=1}^{k} r_b Y_b = \sum_{c=1}^{m} (-p_c) Z_c \in U \cap V$$

This vector is therefore a linear combination of $X_1, \ldots, X_k$. Since the coefficients are unique, it follows from the expression above that all $r_b = 0$ and all $p_c = 0$. But then $0 = \sum_{a=1}^{k} s_a X_a$ forces all $s_a = 0$. Thus $B_U \cup B_V$ is l.i.

- $B_U \cup B_V$ spans $U + V$: Indeed, let $x = u + v$ with $u \in U$ and $v \in V$. We can write $u = \sum_{a=1}^{k} s_a X_a + \sum_{b=1}^{l} r_b Y_b$ and $v = \sum_{a=1}^{k} s'_a X_a + \sum_{c=1}^{m} p_c Z_c$. Thus $x = u + v = \sum_{a=1}^{k} (s_a + s'_a) X_a + \sum_{b=1}^{l} r_b Y_b + \sum_{c=1}^{m} p_c Z_c$ is a linear combination of $B_U \cup B_V$.

We have now shown that $B_U \cup B_V$ is a basis of $U + V$, so $\dim(U + V) = k + l + m$, and the claim follows from

$$\dim U + \dim V = (k + l) + (k + m) = (k + l + m) + k$$
$$= \dim(U + V) + \dim(U \cap V).$$